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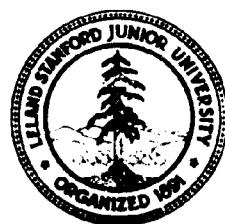
**TWO RESULTS CONCERNING MULTICOLORING**

by

**V. Chvatal, M. R. Garey and D. S. Johnson**

**STAN-CS-76-582**  
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ABSTRACT

The  $m$ -chromatic number  $x_m(G)$  of a graph  $G = (V, E)$  is the least integer  $k$  such that there exists a mapping  $f: V \rightarrow \{S \subseteq \{1, 2, \dots, k\}: |S| = m\}$  having the property that  $f(u) \cap f(v) = \emptyset$  whenever  $\{u, v\} \in E$ . This is a generalization of the standard notion of chromatic number and arises in connection with mobile telephone frequency assignments.

Answering a question of Lovász, our first result shows that for any  $m \geq 1$  and any  $\epsilon > 0$ , there exists a graph  $G$  for which  $x_{m+1}(G)/x_m(G) > 2-\epsilon$ . This shows that the known bound of 2 for all  $m$  and  $G$  is essentially best possible. Our second result shows that the least integer  $m_0$  for which  $x_{m_0}(G)/m_0 =$

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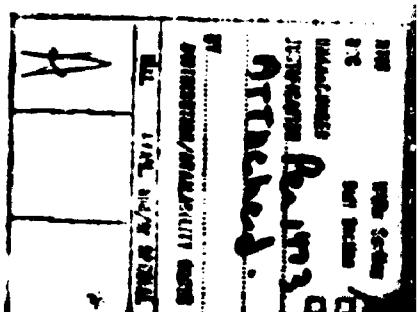
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### I. INTRODUCTION

The following generalization of the standard notion of graph coloring has been of recent interest [1,3,4,6,7,8]. A multicoloring of a graph  $G = (V, E)$  is a function  $f$  defined on  $V$  whose values are sets (of "colors") satisfying  $f(u) \cap f(v) = \emptyset$  whenever  $\{u, v\} \in E$ . For positive integers  $k, m$ , a  $(k, m)$ -coloring of  $G = (V, E)$  is a multicoloring  $f$  of  $G$  such that  $|f(v)| = m$  for each  $v \in V$  and  $|\cup_{v \in V} f(v)| = k$ . The  $m$ -chromatic number  $x_m(G)$  is the least integer  $k$  such that there exists a  $(k, m)$ -coloring of  $G$ . (This last definition differs from that of [6,7] by a factor of  $m$ .) Notice that for  $m = 1$  these definitions correspond to the usual graph coloring notions. The purpose of this note is to resolve two questions about multicoloring conveyed to us by P. Erdős [2].

The first question deals with the relationship between  $x_m(G)$  and  $x_{m+1}(G)$ . It is not difficult to see that

$$x_{m+1}(G) \leq x_m(G) + x_1(G) \leq 2 \cdot x_m(G)$$



with equality possible in the right-hand inequality only for  $m = 1$ . Lovász asked [2] whether, for each value of  $m$ , there exist graphs  $G$  such that  $\chi_{m+1}(G) > (2-\epsilon)\chi_m(G)$ . We shall answer this question in the affirmative.

The graphs we shall use are defined as follows: for positive integers  $n \geq 2m$ , the graph  $G_m^n$  has vertex set consisting of all  $m$ -element subsets of  $\{1, 2, \dots, n\}$  and has an edge between two such vertices exactly when their intersection is empty. It is easy to see that  $\chi_m(G_m^n) \leq n$  merely by considering the multicoloring provided by the definition of  $G_m^n$  and, in fact, it is proved in [7,8] that  $\chi_m(G_m^n) = n$ . Thus, to answer the question of Lovász, it suffices to prove the following theorem:

Theorem 1. For each  $m \geq 2$ , there exists a constant  $c$  such that for all sufficiently large  $n$

$$\chi_{m+1}(G_m^n) \geq 2n - c.$$

In order to prove Theorem 1, we require the following lemma, which is an immediate consequence of a special case of Theorem 3 in [5].

Lemma 1. For fixed  $m \geq 2$  and  $n$  sufficiently large, there exists a constant  $a_0$  such that the number of  $m$ -element subsets of  $\{1, 2, \dots, n\}$  which can be chosen so that no two are disjoint but there is no element common to all is at most  $a_0 n^{m-2}$ .

Proof of Theorem 1. Fix  $m$ . We merely need to show that, for all sufficiently large  $n$ ,

$$x_{m+1}(G_m^n) - x_{m+1}(G_m^{n-1}) \geq 2$$

and the result will follow by induction. So suppose we have a  $(k, m+1)$ -coloring of  $G_m^n$  such that  $k = x_{m+1}(G_m^n)$ , where  $n$  is any integer sufficiently large that the conclusion of Lemma 1 holds and such that  $\binom{n-1}{m-1} > m a_0 n^{m-2}$ , where  $a_0$  is the constant of Lemma 1. We first claim that there must be at least  $n+1$  colors which each appear on more than  $a_0 n^{m-2}$  vertices.

Suppose there are  $n$  or fewer colors which each appear on more than  $a_0 n^{m-2}$  vertices. By Lemma 1, each such color can appear on at most  $\binom{n-1}{m-1} > a_0 n^{m-2}$  vertices since they must all share a common element. Thus, since each of the  $\binom{n}{m}$  vertices receives exactly  $m+1$  colors, we must have

$$\begin{aligned} (m+1) \binom{n}{m} &\leq n \binom{n-1}{m-1} + (k-n)a_0 n^{m-2} \\ &\leq m \binom{n}{m} + (k-n)a_0 n^{m-2} \end{aligned}$$

or, rewriting,

$$\binom{n}{m} \leq (k-n)a_0 n^{m-2}$$

Since

$$\begin{aligned} k = x_{m+1}(G_m^n) &\leq x_m(G_m^n) + x(G_m^n) \\ &\leq 2x_m(G_m^n) = 2n \end{aligned}$$

it follows that we must have

$$\binom{n}{m} \leq a_0^{n^{m-1}}$$

However this is a contradiction, since  $n$  was chosen sufficiently large that  $\binom{n}{m} = \frac{n}{m} \binom{n-1}{m-1} > \frac{n}{m} m a_0^{n^{m-2}} = a_0^{n^{m-1}}$ , and the claim follows.

Thus there are at least  $n+1$  colors which each appear on more than  $a_0^{n^{m-2}}$  vertices. The set of vertices on which any color  $i$  appears must form a collection of pairwise-intersecting  $m$ -element subsets of  $\{1, 2, \dots, n\}$ , by definition of  $G_m^n$ . Thus, by Lemma 1, whenever color  $i$  appears on more than  $a_0^{n^{m-2}}$  vertices, all those vertices must contain some common element  $e_i$ . Since there are more than  $n$  such colors, we must have  $e_i = e_j$  for some  $i$  and  $j$ . If we delete from  $G_m^n$  all the vertices containing  $e_i = e_j$ , we obtain a copy of  $G_m^{n-1}$  and a  $(k-2, m+1)$ -coloring of it, since colors  $i$  and  $j$  have disappeared. Therefore

$$x_{m+1}(G_m^{n-1}) \leq k-2 = x_{m+1}(G_m^n) - 2$$

and the theorem is proved.  $\blacksquare$

The second question involves what we call the ultimate multichromatic number  $\chi^*(G)$  defined by

$$\chi^*(G) = \inf_m x_m(G)/m.$$

It is proved in [1,7] that the value of  $\chi^*(G)$  is always achieved for some finite  $m$ . One easy way to see this is to formulate the problem of determining  $\chi^*(G)$  as a linear programming problem (as done in [4]): Let  $v_1, v_2, \dots, v_n$  be an ordering of the vertices of  $G$  and let  $S_1, S_2, \dots, S_\ell$  be an ordering of the independent sets of  $G$ . Define  $x_{ij}$  to be 1 whenever  $v_i \in S_j$  and 0 otherwise. Then the value of  $\chi^*(G)$  is given by

$$\chi^*(G) = \min \sum_{j=1}^{\ell} r_j$$

subject to:  $r_j \geq 0, 1 \leq j \leq \ell;$

$$\sum_{j=1}^{\ell} x_{ij} r_j = 1, 1 \leq i \leq n.$$

One can show easily, using Hadamard's Theorem, that no basis matrix for this problem can have determinant exceeding  $n^{n/2}$  and this is an upper bound on the value of  $m$  required.

This upper bound however seems ridiculously large. Erdős asked [2] (as did the authors, independently) whether  $\chi^*(G)$  could always be achieved for an  $m$  not exceeding the number of vertices of  $G$ . We answer this in the negative, constructing graphs for which extremely large values of  $m$  are necessary to achieve  $\chi^*(G)$ .

Let  $C_p$  denote the graph which is a cycle on  $p$  vertices. The join  $G_1+G_2$  of two graphs  $G_1$  and  $G_2$ , having disjoint vertex sets, consists of all edges and vertices in the two given graphs along with all edges joining a vertex from  $G_1$  to a vertex from  $G_2$ . We use the following two lemmas in our construction:

Lemma 2. [4,7] For all integers  $p \geq 1$ ,

$$\chi^*(C_{2p+1}) = 2 + (1/p).$$

Lemma 3. [7] For all graphs  $G_1$  and  $G_2$ ,

$$\chi^*(G_1+G_2) = \chi^*(G_1) + \chi^*(G_2).$$

Let  $p_i$  denote the  $i^{\text{th}}$  prime and define the graph  $G(i)$  to be  $C_{2p_1+1} + C_{2p_2+1} + \dots + C_{2p_i+1}$ . The number of vertices  $n$  of  $G(i)$  is given by

$$n = i + 2 \sum_{j=1}^i p_j.$$

Applying Lemmas 2 and 3, we obtain

$$\chi^*(G(i)) = 2i + \sum_{j=1}^i (1/p_j).$$

Since  $\chi_m(G(i))$  must always be an integer, it follows that the least value of  $m$  for which  $\chi^*(G(i)) = \chi_m(G(i))/m$  can be no less than  $\prod_{j=1}^i p_j$  (and in fact that value of  $m$  will work). Using the Prime Number Theorem and expressing this lower bound in

terms of  $n$ , we obtain the asymptotic lower bound of

$$e^{\sqrt{(n \log n)/2}}.$$

Thus, though this is still quite far from the upper bound of

$$n^{n/2} = e^{(n \log n)/2}$$

we see that extremely large values of  $m$  can be required in order to achieve  $\chi^*(G)$ .

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