

A SYMMETRIC CHAIN DECOMPOSITION OF $L(4,n)$

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Abstract.

$L(m,n)$ is the set of integer m -tuples (a_1, \dots, a_m) with $0 \leq a_1 \leq \dots \leq a_m \leq n$, ordered by $a \leq b$ when $a_i \leq b_i$ for all i . R. Stanley conjectured that $L(m,n)$ is a symmetric chain order for all (m,n) . We verify this by construction for $m = 4$.

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$L(m, n)$ is defined as the lattice formed by order ideals in the direct product of two chains with m and n elements, respectively. Equivalently, it is the collection of integer sequences $a = (a_1, \dots, a_m)$ satisfying $0 \leq a_1 \leq \dots \leq a_m \leq n$, with ordering $a \leq b$ when $a_i \leq b_i$ for all i . The correspondence is simple. If the chain elements are $x_1 < \dots < x_m$ and $y_1 < \dots < y_n$, then the number of elements paired with x_i in the ideal corresponding to a is $n - a_i$. In other words, the antichain generating the ideal is $\{(x_1, y_{n-a_1}), \dots, (x_m, y_{n-a_m})\}$.

Clearly, the rank of element a is $\sum a_i$, the rank of the entire lattice is mn , and the cardinality of the lattice is $\binom{m+n}{m}$. For any element a , we define its conjugate $a^* = (n - a_m, \dots, n - a_1)$. Note that $a^{**} = a$. The ranks of an element and its conjugate sum to mn , so the sizes of the ranks are symmetric about the middle. Using complex algebraic methods, R. Stanley [3] proved the sizes of the ranks are also unimodal. These are necessary conditions for a stronger property he conjectured also holds. He conjectured that $L(m, n)$ is a symmetric chain order. A symmetric chain order is one whose elements can be partitioned into chains which are saturated (skip no ranks) and symmetric about the middle rank. The conjecture is clear when $m = 1$ or $m = 2$. Lindström [2] provided an inductive construction to verify it for $m = 3$. Here we give a construction somewhat different from his which verifies the conjecture when $m = 4$.

Let $S(m, n)$, the "shell" of $L(m, n)$, be those elements which begin with 0 or end with n . When these are removed from $L(m, n)$ the remainder is isomorphic to $L(m, n-2)$. The conjecture holds trivially when $n = 1$, and $L(m; 0)$ can be defined as having a single element.

So, providing a symmetric chain decomposition of $S(m,n)$ proves the conjecture by induction. We use this approach here for $L(4,n)$. Unfortunately, when m is odd and n is even the rank sizes in $S(m,n)$ are not unimodal. So, for that case Lindström was forced to strip off two shells for his induction. For $m = 4$ this difficulty does not arise. It is possible that Lindström's construction generalizes for odd m and this does so for even m . When m and n both exceed 2 , $L(m,n)$ is not an LYM-order, so Griggs' sufficient conditions for a symmetric chain order [1] cannot be applied.

Theorem. $L(4,n)$ is a symmetric chain order.

It suffices to give a symmetric chain decomposition of $S(4,n)$. The chains will be of two types, C_{ij} and D_{ij} for suitable values of i and j . The chains are clearly saturated, so two steps will complete the proof.

- (1) No element appears in more than one chain.
- (2) The number of elements in the construction is the size of $S(m,n)$.

Each chain is composed of six segments, with the top element of one segment and the bottom element of the next identical. Throughout a given segment only one position in the integer sequence changes. Table 1 explicitly defines the chains and gives the ranks where the changes between segments occur.

Segments must have length at least 0 . That is, top and bottom elements may be identical, but the top element must not have rank below the bottom element. Examining the lengths of segments and ensuring that

rank	C_{ij}	segment	D_{ij}	rank
$4n-6i-2j$	$n-3i-j, n-2i-j, n-i, n)$	$(n-3i-j-2, n-2i-j-1, n-i, n)$	$4n-6i-2j-3$	
	\vdots	6	\vdots	
$4n-6i-3j$	$n-3i-j, n-2i-j, n-i-j, n)$	$(j+1, n-2i-j-1, n-i, n)$	$3n-3i$	
	\vdots	5	\vdots	
$3n-3i-2j$	$0, n-2i-j, n-i-j, n)$	$(j+1, i+j+1, n-i, n)$	$2n+2j+2$	
	\vdots	4	\vdots	
$3n-3i-3j$	$(0, n-2i-j, n-i-j, n-j)$	$(j+1, i+j+1, 2i+j+1, n)$	$n+3i+3j+3$	
	\vdots	3	\vdots	
$2n-2j$	$(0, i, n-i-j, n-j)$	$(0, i+j+1, 2i+j+1, n)$	$n+3i+2j+2$	
	\vdots	2	\vdots	
$n+3i$	$(0, i, 2i+j, n-j)$	$(0, i+j+1, 2i+j+1, 3i+j+1)$	$6i+3j+3$	
	\vdots	1	\vdots	
$6i+2j$	$(0, \bar{z}, 2i+j, 3i+j)$	$(\bar{c}, i+\bar{x}, 2i+j+1, 3i+j+1)$	$6i+2j+3$	

Table 1

we have legal elements at the bottom of C_{ij} and the top of D_{ij} yields necessary conditions on i and j . We claim the desired decomposition is obtained by taking all chains for which these necessary conditions are satisfied.

$$S(4, n) = \{C_{ij} : 3i+2j \leq n, i \geq 0, j \geq 0\} \cup \{D_{ij} : 3i+2j \leq n-3, i \geq 0, j \geq 0\}$$

Figure 1 gives $S(4, 7)$ explicitly as an example.

7777											
6777											
5777	6677										
4777	6667										
3777	5667	5577									
2777	4667	5567									
1777	3667	5557	4477								
0777	2667	4557	4467	1677							
0677	1667	3557	4457	1577	2477						
0577	0667	2557	4447	1477	2377	3377					
0477	0666	1557	3447	1377	2277	3357	0567				
0377	0566	0557	2447	1277	2267	3347	0467	1457			
0277	0466	0556	1447	1177	2257	3337	0367	0457	2347		
0177	0366	0555	0447	1167	2247	2337	0267	0456	1347	1257	
0077	0266	0455	0446	1157	2237	1337	0167	0356	0347	1247	0257
0067	0166	0355	0445	1147	2227	0337	0157	0256	0346	1237	0247
0057	0066	0255	0444	J-J-37	1227	0336	0147	0156	0345	0237	0246
0047	0056	0155	0344	1127	0227	0335	0137	0146	0245	0236	
0037	0046	0055	0244	1117	0276	0334	0127	0136	0145	0235	
0027	0036	0045	0144	0117	0225	0333	0126	0135		0234	
0017	0026	0035	0044	0116	0224	0233	0125	0134			
0007	0016	0025	0034	0115	0223	0133	0124				
0006	0015	0024	0033	0114	0222		0123				
0005	0014	0023		0113	0122						
0004	0013	0022		0112							
0003	0012			0111							
0002	0011										
0001											
0000											
C_{00}	C_{01}	C_{02}	C_{03}	D_{00}	D_{01}	D_{02}	C_{10}	C_{11}	C_{12}	D_{10}	C_{20}

Figure 1. $S(4, 7)$

Outline of Proof. To show the elements are all distinct, we express the D-chains in terms of the C-chains and then restrict our attention to the C-chains. Let C_{ij}^r be the element of C_{ij} of rank r , similarly for D_{ij}^r . We claim that chain $D_{i,j-1}$ is the conjugate of chain $C_{i,j}$ when the top and bottoms elements of the latter are removed. That is, $(D_{i,j-1}^r)^* = C_{i,j}^{4n-r}$. It suffices to perform the conjugation on the transition elements between segments of $D_{i,j-1}$. They become the transition elements of $C_{i,j}$. Note the top and bottom elements of $C_{i,j}$ are unaffected and are conjugates of each other. Whenever $D_{i,j-1}$ exists, C_{ij} exists. The affected C_{ij} are those where $j > 0$ and $3i+2j < n$.

Distinctness now reduces to showing:

- (1a) The elements of $\cup \{C_{ij}\}$ are all distinct.
- (1b) The chains C_{i0} and $C_{i,(n-3i)/2}$ are self-conjugate.
- (1c) There are no conjugate pairs among the elements of $\cup \{C_{ij}\}$, where $0 < j < (n-3i)/2$, other than the tops and bottoms of chains.

(1b) is seen immediately by conjugating the transition elements in those chains. The other two statements require eliminating a large number of easy cases.

To show we have the correct number of elements, we proceed by induction. Simple counting verifies it for small n . In general, the size of $S(m,n)$ is $|L(m,n)| - |L(m,n-2)|$. So,

$$|S(4,n)| = \binom{n+4}{4} - \binom{n+2}{4} = \frac{(n+1)(n+2)(2n+3)}{6}$$

This is the sum of a familiar sequence. Indeed,

$$|S(4,n)| - |S(4,n-1)| = (n+1)^2 .$$

Now we examine the changes in the construction between $n-1$ and n ,
 For all values of i and j such that $C_{ij}.$ or $D_{ij}.$ exists in the
 construction for $n-1$, a similarly indexed chain exists in the construction
 for n . Subtracting ranks, the number of elements in $C_{ij}.$ is
 $4(n-3i-j)+1$, and the number in $D_{ij}.$ is $4(n-3i-j)-5$. Each of these
 chains has 4 more elements than the similarly indexed chain in $S(4,n-1)$,
 if that chain exists. We will see there is a C_{ij} for every element of
 the middle rank which begins with 0 and a D_{ij} for every such element
 whose first position is not zero.

The chains which arise newly when n is reached are those $C_{ij}.$ for
 which $3i+2j = n$ and those $D_{ij}.$ for which $3i+2j = n-3$. For each value
 of i from 0 up to $\lfloor n/3 \rfloor$ or $\lfloor n/3 \rfloor - 1$, depending on parities, there
 will be one new $C_{ij}.$ or D_{ij} but not both.

Verifying that the construction picks up the proper number of elements
 reduces to:

(2a) Computing (and multiplying by 4) the number of chains in the
 construction for $S(4,n-1)$ -- that is, the sum of the number
 of solutions to $3i+2j < n-1$ and $3i+2j \leq n-4$.

(2b) Computing the total number of elements in new chains.

(2c) Verifying the sum of new elements in (2a) and (2b) is $(n+1)^2$.

(2b) breaks into cases depending on the parity of n , and (2a) does the
 same with the parity of $\lfloor n/3 \rfloor$, so (2c) requires six cases, depending
 on the congruence class of n modulo 6 .

Details of Step 1. If (la) does not hold, suppose $a = C_{ij}^r = C_{kl}^r$. We
 have a number of cases to consider, depending on which segment contains a

in each of the two chains. Let $p_{C_{ij}}$ denote segment p in C_{ij} . Equating the descriptions of the segments in Table 1 give us a number of linear relationships between i , j , k , and ℓ . If \underline{a} comes from $p_{C_{ij}}$ and $p_{C_{kl}}$, equating the positions which do not change in that segment implies $i = k$ and $j = \ell$ in all six cases, by straightforward subtraction of equalities.

By symmetry we may assume \underline{a} occurs in a lower numbered segment in C_{ij} than in C_{kl} . We allow the transition elements between segments to belong to either segment. So, if \underline{a} is in $p_{C_{ij}}$ and $q_{C_{kl}}$, we may assume \underline{a} is not the top element of $p_{C_{ij}}$ nor the bottom element of $q_{C_{kl}}$, else we have a case with smaller $q-p$. In particular, the rank of the top element in $p_{C_{ij}}$ must be strictly greater than the rank of the bottom element in $q_{C_{kl}}$.

Suppose $q = p+1$. This comparison of ranks yields a strict inequality when a particular linear function is applied to (i, j) and to (k, ℓ) . Whenever $q = p+1$ two positions in the elements remain constant from the bottom of segment p to the top of segment q . This expresses two positions of \underline{a} as identical linear functions of (i, j) and (k, ℓ) . In all five cases, we readily get the same linear function we obtained by considering ranks, but with equality this time.

If the first position of \underline{a} is nonzero, \underline{a} can occur only in segments 5 or 6. If it is zero, \underline{a} occurs in segment 4 or below. This eliminates all but three of the cases which might have $C_{ij}^r = C_{kl}^r$ with $(i, j) \neq (k, \ell)$. The remainder we handle individually.

If \underline{a} is in ${}^2C_{ij}$ and ${}^4C_{kl}$, positions 2 and 3 require $i = n-2k-\ell$ and $n-i-j > n-k-\ell$. Adding these gives $n-j > 2n-3k-2\ell \geq n$.

Next suppose \underline{a} is in ${}^1C_{ij}$ and ${}^3C_{kl}$. Equality of the last three positions requires $k < i$, $n-k-l = 2i+j$, and $n-l \geq 3i+j$. Substituting for k and $n-l$ in the equation gives $2i+j < 2i+j$. Finally, suppose \underline{a} is in ${}^1C_{ij}$ and ${}^4C_{kl}$. Comparing the top of ${}^1C_{ij}$ with the bottom of ${}^4C_{kl}$ yields $n-3i > 3n-3k-3l \geq n-3k+l$ or $i > k$. On the other hand, the middle two positions of \underline{a} remain constant in both sections, so $i = n-2k-l$ and $2i+j = n-k-l$. Subtraction gives $i+j = k$ or $i \leq k$.

(lc) also breaks into cases depending on the segments. We assume $\underline{a} = C_{ij}^r = (C_{kl}^{4n-r})^*$, with $0 < j < (n-3i)/2$ and $0 < l < (n-3k)/2$. Here the arguments do not group together as cleanly. One element of such a conjugate pair occurs at least as high as the middle rank in one chain. Call this chain C_{ij} . For ease of comparison, we have recorded C_{ij} and C_{kl}^* in Table 2. Since $3n-3i-3j < 2n$, \underline{a} lies in segment 4, 5, or 6 of C_{ij} . Since $n+3k+2l < 2n$, \underline{a} lies in segment 3, 4, 5, or 6 of C_{kl}^* . Assume $\underline{a} \in ({}^pC_{ij} \cap {}^qC_{kl}^*)$.

We first notice $p = 4$ is impossible, as it would imply $l \leq 0$. We handle the remaining cases individually. Again we equate corresponding positions in \underline{a} . The requirements on j and l figure prominently. For example, $i+j \leq k$ and $i \geq k+l$ give us a contradiction, as do $n-3i-j \leq l$ and $n-3k-l \leq j$.

$p = 6$, $q = 6$. $a_2 \Rightarrow 2i+j = 2k+l$. $a_3 \Rightarrow i \leq k$. $a_1 \Rightarrow 3i+j \geq 3k+l$. Subtracting a_2 implies $i \geq k$. So $(i, j) = (k, l)$, and this is the case where the top and bottom of the chain are conjugate.

$p = 5$, $q = 5$. $a_3 \Rightarrow i+j = k$. $a_2 \Rightarrow 2i+j \geq 2k+l$. Subtracting a_3 implies $i \geq k+l$.

$p = 6$, $q = 5$. $a_3 \Rightarrow k \geq i$. $a_1 \Rightarrow n-3i-j = l$. Substituting for i gives $n-3k-l \leq j$. As mentioned earlier, this is a contradiction since both $3i+2j$ and $3k+2l$ must be less than n .

rank	c_{ij}	segment	c_{kl}^*	rank
$4n-6i-2j$	$(n-3i-j, n-2i-j, n-i, n)$	$(n-3k-l, n-2k-l, n-k, n)$	$4n-6k-2l$	
	\vdots	6	\vdots	
$4n-68-3j$	$(n-3i-j, n-2i-j, n-i-j, n)$	$(l, n-2k-l, n-k, n)$	$3n-3k$	
	\vdots	5	\vdots	
$3n-3i-2j$	$(0, n-2i-j, n-i-j, n)$	$(l, k+l, n-k, n)$	$2n+2l$	
	\vdots	4	\vdots	
$3n-3i-3j$	$(0, n-2i-j, n-i-j, n-j)$	$(l, k+l, 2k+l, n)$	$n+3k+3l$	
	\vdots	3	\vdots	
$2n-2j$	$(0, i, n-i-j, n-j)$	$(0, k+l, 2k+l, n)$	$n+3k+2l$	
	\vdots	2	\vdots	
$n+3i$	$(0, i, 2i+j, n-j)$	$(0, k+l, 2k+l, 3k+l)$	$6k+3l$	
	\vdots	1	\vdots	
$6i+2j$	$(0, i, \sigma i+j, 3i+j)$	$(0, k, 2k+l, 3k+l)$	$6k+2l$	

Table 2.

$p = 5, q = 6$. $a_3 \Rightarrow i+j = k$. $a_2 \Rightarrow 2i+j = 2k+l$. Subtracting a_3 implies $i = k+l$, so $j = l = 0$.

$p = 6, q = 4$. $a_1 \Rightarrow n-3i-j = l$. $a_2 \Rightarrow n-2i-j = k+l$. Subtracting a_1 gives $i = k$. Substituting in a_2 yields $n-3k-l = j$, giving the same contradiction as in $(p, q) = (6, 5)$.

$p = 5, q = 4$. $a_1 \Rightarrow n-3i-j > l$ (equality returns us to the previous case). $a_2 \Rightarrow n-2i-j = k+l$. Subtracting a_1 gives $i < k$. $a_3 \Rightarrow n-i-j \geq 2k+l$. Subtracting a_2 gives $i \geq k$.

$p = 6, q = 3$. Let $p-q$ be smaller, the requirement on ranks is $4n-3i-3j < n+3k+3l$, so $n-2i-j < k+l$. But $a_2 \Rightarrow n-2i-j = k+l$.

$p = 5, q = 3$. $a_2 \Rightarrow n-2i-j = k+l$. $a_3 \Rightarrow n-i-j = 2k+l$.

Subtracting a_1 yields $i = k$. Substituting this in the two previous equations gives the familiar contradiction $n-3i-j = l$ and $n-3k-l = j$.

This completes the proof of (1).

Details of Step 2. We begin with (2a). The top element of segment 4 in C_{ij} has rank $3n-3i-2j \geq 2n$, so every C_{ij} has a 0 in the first position of its middle rank element. The bottom rank of segment 3 in D_{ij} is $n+3i+2j+2 < 2n-1$, so D_{ij} has a positive first position in its middle rank element. The non-decreasing sequences of length 4 which start with 0, end in k , and sum to $2n$ run from $(0, 2n-2k, k, k)$ to $(0, \lfloor (2n-k)/2 \rfloor, \lceil (2n-k)/2 \rceil, k)$ when $n \geq k > \lceil 2n/3 \rceil$. So, we want the number of C_{ij} 's to be $\sum_{\lceil 2n/3 \rceil \leq k \leq n} k - \lceil (2n-k)/2 \rceil + 1$. Similarly, the elements covered by D_{ij} 's run from $(k, k, n-2k, n)$ to

$(k, \lfloor (n-k)/2 \rfloor, \lceil (n-k)/2 \rceil, n)$ for $1 \leq k \leq \lfloor n/3 \rfloor$, for a total of

$$\sum_{1 \leq k \leq \lfloor n/3 \rfloor} \lfloor (n-k)/2 \rfloor - k + 1.$$

On the other hand, the number of solutions to $3i+2j \leq n$ is

$$\sum_{0 \leq i \leq \lfloor n/3 \rfloor} 1 + \lfloor (n-3i)/2 \rfloor \quad \text{and to } 3i+2j \leq n-3 \text{ is}$$

$$\sum_{0 \leq i \leq \lfloor n/3 \rfloor - 1} 1 + \lfloor (n-3i-3)/2 \rfloor. \quad \text{These turn into the desired}$$

summations when i is set to $n-k$ in the first case and $k-1$ in the second.

We wish to combine the summations. Separating the $i = 0$ term from the first and adjusting the index in the second, the total number $f(n)$ of chains becomes

$$f(n) = 1 + \lfloor n/2 \rfloor + 2 \sum_{1 \leq i \leq \lfloor n/3 \rfloor} (1 + \lfloor (n-3i)/2 \rfloor).$$

To compute the summation, we pair terms for consecutive values of i . If $\lfloor n/3 \rfloor$ is odd, we separate $i = \lfloor n/3 \rfloor$. Adding the terms for $i = 2k-1$ and $i = 2k$ gives $2 + \lfloor (n-6k+3)/2 \rfloor + \lfloor (n-6k)/2 \rfloor = n+3-6k$. There are $\lfloor n/6 \rfloor$ pairs altogether, and $\sum_{1 \leq k \leq \lfloor n/6 \rfloor} (n+3-6k) = (n+3)\lfloor n/6 \rfloor - 3\lfloor n/6 \rfloor \lfloor (n+6)/6 \rfloor$. When $\lfloor n/3 \rfloor$ is odd, the term $1 + \lfloor (n-3\lfloor n/3 \rfloor)/2 \rfloor$ remains. This is 1 if $n \equiv 3, 4 \pmod{6}$, but 2 if $n \equiv 5 \pmod{6}$.

Summarizing, if $n \equiv r \pmod{6}$, $0 \leq r \leq 5$, then the total number of chains is

$$f(n) = \lfloor n/2 \rfloor + 2(n+3)\lfloor n/6 \rfloor \cdot 6\lfloor n/6 \rfloor \lfloor (n+6)/6 \rfloor + \begin{cases} 1 & ; r = 0, 1, 2 \\ 3 & ; r = 3, 4 \\ 5 & ; r = 5 \end{cases}$$

$$= \lfloor n/2 \rfloor + (n+3)(n-r)/3 - (n-r)(n-r+6)/6 + \begin{cases} 1 & ; r = 0, 1, 2 \\ 3 & ; r = 3, 4 \\ 5 & ; r = 5 \end{cases}$$

Next we consider (2b). If n is even, a new chain C_{ij} occurs for even values of i with $0 \leq i \leq \lfloor n/3 \rfloor$, and a new D_{ij} for odd values of i with $1 \leq i \leq \lfloor n/3 \rfloor - 1$. Similarly, when n is odd we have a new D_{ij} for even i with $1 \leq i \leq \lfloor n/3 \rfloor - 1$ and a new C_{ij} for odd i with $1 \leq i \leq \lfloor n/3 \rfloor$.

To sum the number of elements in these chains, we can again pair consecutive terms. For the total number $g(n)$ of these elements, we have

$$g(n) = \begin{cases} |C_{0,n/2}| + \sum_{1 \leq k \leq \lfloor n/6 \rfloor} |D_{2k-1, (n-6k)/2}| + |C_{2k, (n-6k)/2}| & ; n \text{ even} \\ \sum_{0 \leq k \leq \lfloor (n-3)/6 \rfloor} |D_{2k, (n-6k-3)/2}| + |C_{2k+1, (n-6k-3)/2}| & ; n \text{ odd} \end{cases}$$

Since $|C_{ij}| = 4(n-3i-j)+1$ and $|D_{ij}| = 4(n-3i-j)-5$, this quickly becomes

$$g(n) = \begin{cases} 1 + 2n + \sum_{1 \leq k \leq \lfloor n/6 \rfloor} 4(n-6k) + 8 & ; n \text{ even} \\ \sum_{0 \leq k \leq \lfloor (n-3)/6 \rfloor} 4(n-6k) - 4 & ; n \text{ odd} \end{cases}$$

$$= \begin{cases} 1 + 2n + 4(n+2) \lfloor n/6 \rfloor - 12 \lfloor n/6 \rfloor \lfloor (n+6)/6 \rfloor & ; n \text{ even} \\ 4(n-1) \lfloor (n-3)/6 \rfloor - 12 \lfloor (n-3)/6 \rfloor \lfloor (n+3)/6 \rfloor & ; n \text{ odd} \end{cases}$$

$$= \begin{cases} 1 + 2n + 2(n+2)(n-r)/3 - (n-r)(n-r+6)/3 & ; r = 0, 2, 4 \\ 2(n-1)(n-r+6)/3 - (n-r)(n-r+6)/3 & ; r = 3, 5 \\ 2((n-1)^2/3 - (n-7)(n-1)/3 & ; r = 1 \end{cases}$$

For (2c), we need only compute $4f(n-1) + g(n)$, which becomes simple algebraic manipulation when we consider a particular congruence class of n modulo 6 . Beginning with $r = 1$, we easily obtain expressions like

$$r = 1: 4 + (n-1)(n+3)$$

$$r = 4: 4n + 9 + (n+2)(n-4)$$

$$r = 2: 2n + 5 + (n-2)(n+2)$$

$$r = 5: 2n + 10 + (n-5)(n+5)/3 + 2(n-1)(n+1)/3$$

$$r = 3: 4 + (n-1)(n+3)$$

$$r = 0: 4n + 17 + 2(n-6)(n+4)/3 + n(n-2)/3$$

all of which reduce to $(n+1)^2$.

This completes the proof.

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