A LOWER BOUND TO FINDING CONVEX HULLS

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Abstract.

Given a set S of n distinct points $\{(x_i, y_i) \mid 0 \leq i < n\}$, the convex hull problem is to determine the vertices of the convex hull H(S). All the known algorithms for solving this problem have a worst-case running time of cn log n or higher, and employ only quadratic tests, i.e., tests of the form $f(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}) : 0$ with f being any polynomial of degree not exceeding 2. In this paper, we show that any algorithm in the quadratic decision-tree model must make cn log n tests for some input.

Keywords: complexity, convex hull, decision tree, lower bound, quadratic decision-tree model, quadratic test.

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1. Introduction.

Let S be a set of n distinct points in the plane. The convex hull H(S) is the intersection of all convex sets which contain S. It is well known (see e.g. [2]) that H(S) is a convex polygon with all of its vertices in S. In fact, $H_0(S)$, the set of vertices of H(S), is exactly $\{\vec{p} \mid \vec{p} \in S, \vec{p}$ is not a convex combination*/ of the points in $S-\{\vec{p}\}\}$. We are interested in the following convex hull problem: Given a set S of n distinct points $\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{n-1}$ in the plane, determine the set of integers $V_S = \{i \mid \vec{r}_i \in H_0(S)\}$. In 1972, Graham [1] gave a $\Theta(n \log n)$ -time+/ algorithm for solving this problem. Since then, many other algorithms have been proposed (see Shamos [2] for some of them), all of which also have a worst-case running time cn log n or more. An interesting open question is whether better algorithms exist. The purpose of this paper is to show that, in the quadratic decision-tree model, any algorithm for the convex hull problem must use at least cn log n operations in the worst case.

We remark that if, in addition, the set V_S is also to be ordered as $i_1, i_2, \ldots, i_t$ so that $\vec{r}_{i_1}, \vec{r}_{i_2}, \ldots, \vec{r}_{i_t}$ are the vertices of H(S) in consecutive cyclic order, then the sorting of n numbers can be reduced

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*/ A point p is a convex combination of the pointes $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$ if there exist $\mu_i \geq 0$ $(i = 1,2,\ldots,m)$ such that $\sum_i \mu_i = 1$ and $\vec{p} = \sum_i \mu_i \vec{r}_i$.

+/ We use $\Theta(g(n))$ to denote any function f(n) with the property that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for some positive constants c_1, c_2 and for all sufficiently large n.

to essentially this problem (see [2]). The cn log n lower bound for sorting in rather general models is then immediately a lower bound to that version of the convex hull problem [2].

Note that we have restricted the input points to be all distinct. This enables us to avoid choosing among several possible definitions for V_S when some of the $\vec{r}_j$ may be identical. The lower bound derived in this paper of course remains true independent of the choice,

In the quadratic decision-tree model, algorithms are ternary decision trees employing quadratic tests, i.e., tests of the form " $f(z_1, z_2, \ldots, z_m) : 0$ " with f being any quadratic polynomial of the input numbers z_i (see Section 2 for more details). To the author's knowledge, all the known convex hull algorithms can be properly modeled as quadratic decision trees. For example, several algorithms (including that of Graham's [1]) use basically primitive operations of the following types (P1)-(P3). Let $\vec{r}_i = (x_i, y_i)$, $0 < i < n$, be the input points. (P1) linear test of the form " $\sum_i a_i x_i + \sum_i b_i y_i + c : 0$ "; (P2) generation of a new point $\vec{p} = \sum_i a_i \vec{r}_i$; (P3) for any existing points (input points or those generated by (P2)) $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$, a test "Is $\vec{p}_1$ lying to the left of, to the right of, or on the directed line from $\vec{p}_2$ to $\vec{p}_3$?" Mathematically, (P3) is expressed as " $\Delta(\vec{p}_1, \vec{p}_2, \vec{p}_3) : 0$ " where Δ is defined by

$$\Delta(\vec{p}_1, \vec{p}_2, \vec{p}_3) = \det \begin{pmatrix} p_{11} & p_{12} & 1 \\ p_{21} & p_{22} & 1 \\ p_{31} & p_{32} & 1 \end{pmatrix} ,$$

with $\vec{p}_i = (p_{i1}, p_{i2})$. As each $\vec{p}_i$ in (P3) is a linear combination of the input points, it is easy to verify that tests of the type (P1) or (P3) are quadratic tests. In these algorithms, operations of the type (P2) are used only occasionally to generate points <u>interior</u> to the convex hull by taking convex combinations of input points. Thus, the running time of the algorithms is properly accounted for if one only counts quadratic tests.

2. The Quadratic Decision-Tree Model.

Consider the convex hull problem for a set S of n distinct input points $\vec{r}_i = (x_i, y_i)$, $0 < i < n$. An algorithm T is a ternary decision tree, with each internal node containing a quadratic test $f(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}) : 0$ where f may be any polynomial of degree at most 2 . For any given input set S , the algorithm starts at the root, performing tests and branching accordingly until a leaf is reached, where the algorithm must be able to determine the set V . We denote by cost(T) the maximum number of tests made for any input. The complexity C(n) is defined to be the minimum of cost(T) for any such algorithm T .

The main result of this paper is the following theorem.

Theorem 1. There exists a constant $c > 0$ such that, for all $n \geq 3$, $C(n) \geq cn \log_2 n$.

3. Proof of Theorem 1.

Let $n \geq 3$ be an integer, and T any algorithm for the convex hull problem with 2n input points. We shall prove that there are at least n! distinct leaves in T . This will imply Theorem 1 by the following argument. The height of T , i.e., cost(T) , then must be at least $\log_3(n!)$. This proves $C(n) \geq \log_3((n/2)!) \geq \text{constant} \times n \log_2 n$ for all even $n \geq 6$, which implies*/ $C(n) \geq C(2\lfloor n/2 \rfloor) \geq \text{constant} \times n\log_2 n$ for all $n > 6$. Observing that $C(n) > 0$ for $n \in \{3,4,5\}$, we can obtain Theorem 1 by choosing c suitably.

The plan is as follows. Let σ be any permutation of $(n, n+1, \ldots, 2n-1)$, i.e., a one-to-one mapping from $\{n, n+1, \ldots, 2n-1\}$ onto itself. We shall associate with σ a leaf LEAF(σ) of T , and derive some constraints on the inputs that lead to LEAF(a) . We then show that LEAF(c) $\neq$ LEAF(σ') for any distinct σ , σ' . Thus, there are at least n! leaves.

3.1 Defining LEAF(a) .

For each $0 \leq j < n$, let

$$Q_j = \left\{ (x,y) \mid \left(x - \cos\frac{2\pi j}{n}\right)^2 + \left(y - \sin\frac{2\pi j}{n}\right)^2 < \epsilon_n \right\} ,$$

and

$$\Lambda_j = \{\lambda \mid \lambda \in (1/4, 3/4)\} ,$$

with $\epsilon_n > 0$ to be specified later.

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*/ C(n) is a non-decreasing function of n , since any algorithm for n+1 input points yields an algorithm of the same cost for n input points by setting $\vec{r}_n = \frac{1}{n}(\vec{r}_0 + \vec{r}_1 + \cdots + \vec{r}_{n-1})$.

Let us regard any input $S = \{\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{2n-1}\}$ as an element in E^{4n}, the 4n-dimensional Euclidean space, and write it either as $r = (\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{2n-1})$ or $r = (x_0, y_0, x_1, y_1, \ldots, x_{2n-1}, y_{2n-1})$. Define I_σ to be the set of inputs $r = (\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{2n-1})$ which satisfy $r_j \in Q_j$ and $\vec{r}_{\sigma(n+j)} = \lambda_j \vec{r}_j + (1-\lambda_j) \vec{r}_{(j+1) \bmod n}$ with $\lambda_j \in \Lambda_j$ for all $0 < j < n$. Informally, each input in I_σ has $\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{n-1}$ as the vertices of an approximate regular n-gon, and $\vec{r}_{\sigma(n+j)}$ on the line connecting $\vec{r}_j$ and $\vec{r}_{(j+1) \bmod n}$ for each $0 \leq j < n$ (see Figure 1). Note that $\Delta(\vec{r}_{\sigma(n+j)}, \vec{r}_{(j+1) \bmod n}, \vec{r}_j) = 0$ for all $r \in I_\sigma$, because of the elementary identity $\Delta(\mu\vec{p} + (1-\mu)\vec{p}\,', \vec{p}\,', \vec{p}) = 0$.

Choose an $\epsilon_n > 0$ so that the following properties are true for any $(\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{2n-1}) \in I_\sigma$.

Property I. All the 2n points $\vec{r}_j$ are distinct.

Property II. $H_0(S) = \{\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{n-1}\}$.

Property III. If a point p is a convex combination of $\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{n-1}$, then $\Delta(\vec{p}, \vec{r}_{(j+1) \bmod n}, \vec{r}_j) \leq 0$ for all $0 < j < n$.

Property IV. If $0 \leq \ell \neq j < n$, then $\Delta(\vec{r}_{\sigma(n+\ell)}, \vec{r}_{(j+1) \bmod n}, \vec{r}_j) \neq 0$.

It is intuitively*/ obvious that these properties are satisfied provided that $\epsilon_n > 0$ is small enough. A proof that such an ϵ_n exists will be given in the Appendix.

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*/ Keep in mind that the geometric interpretation of $\Delta(\vec{p}_1, \vec{p}_2, \vec{p}_3)$ is the signed "area" of the triangle $\vec{p}_1\vec{p}_2\vec{p}_3$, where the sign is determined by the orientation of $\vec{p}_1, \vec{p}_2, \vec{p}_3$ ("plus" if counterclockwise).

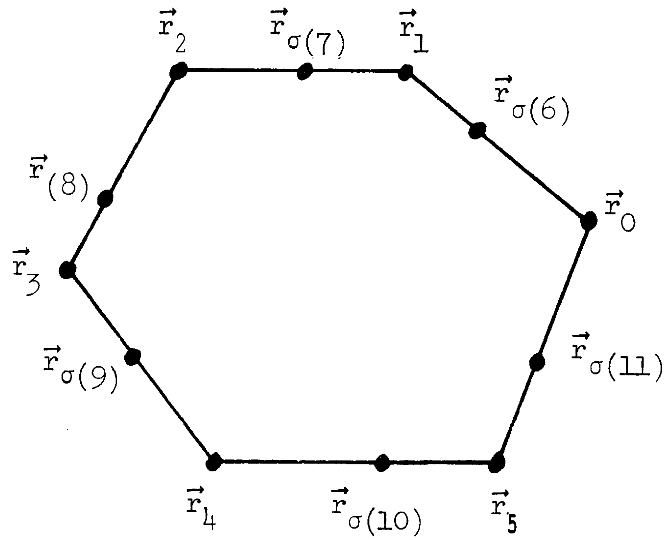

Figure 1. The configuration of $\vec{r}_i$ for an input $r = (\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{2n-1}) \in I_\sigma$.

Let $Q_\sigma = Q_0 \times Q_1 \times \cdots \times Q_{n-1} \times \Lambda_0 \times \Lambda_1 \times \cdots \times \Lambda_{n-1}$. One can regard any input $r \in I_\sigma$ as, alternatively, an element $q \in Q_\sigma$. This establishes a one-to-one correspondence between the elements of I, and the elements of Q_σ. For any function f on E^{4n}, let us denote by $f^{[\sigma]}$ the function on Q_σ induced by f, i.e.' $f^{[\sigma]}(x_0, y_0, \ldots, x_{n-1}, y_{n-1}, \ldots, \lambda_{n-1})$ is equal to $f(x_0, y_0, x_1, y_1, \ldots, x_{2n-1}, y_{2n-1})$ with

$x_{\sigma(n+j)} = \lambda_j x_j + (1-\lambda_j) x_{(j+1) \bmod n}$ and $y_{\sigma(n+j)} = \lambda_j y_j + (1-\lambda_j) y_{(j+1) \bmod n}$

for $0<j<n$. Clearly, if f is a polynomial, so is $f^{[\sigma]}$, in which case we shall regard $f^{[\sigma]}$ as defined over the whole E^{3n}.

Lemma 1. There exists a leaf ℓ_0 and a non-empty open set $Q' \subseteq Q_\sigma$ such that all inputs $q \in Q'$ will lead to ℓ_0.

Proof. For each leaf ℓ, denote by A(1) the set of inputs in Q_σ that lead to ℓ. Let L be the set of leaves ℓ with $A(R) \neq \emptyset$. Clearly,

$$Q_\sigma = \bigcup_{\ell \in L} A(\ell) .$$

Each A(1) can be written as $Q_\sigma \cap B(\ell)$, where

$$B(1) = \{q \mid q \in E^{3n};\ f^{[\sigma]}_{\ell,1}(q) = 0, \ldots, f^{[\sigma]}_{\ell,a_\ell}(q) = 0\ \ g^{[\sigma]}_{\ell,1}(q) > 0, \ldots, g^{[\sigma]}_{\ell,b_\ell}(q) > 0\} . \tag{1}$$

The functions $f^{[\sigma]}_{\ell,i}$, $g^{[\sigma]}_{\ell,j}$ are polynomials induced by the quadratic polynomials $f_{\ell,i}$, $g_{\ell,j}$ used at internal nodes along the path from the root to ℓ. Some of the constraints $f^{[\sigma]}_{\ell,i}(q) = 0$, $g^{[\sigma]}_{\ell,j}(q) > 0$ in (1) may be trivial in that they are satisfied by all $q \in E^{3n}$. We claim that, after removing the trivial constraints from the formulas in (1), there is some $B(\ell_0)$ $(\ell_0 \in L)$ that is defined only by inequalities. Otherwise, all $B(R)$ $(\ell \in L)$ would be of measure zero, implying that the open set $Q_\sigma = \bigcup_{\ell \in L} A(R) \subseteq \bigcup_{\ell \in L} B(L)$ is of measure zero, which is impossible. Clearly $A(\ell_0) = Q \cap B(\ell_0)$ is open and non-empty. The lemma follows by choosing $Q' = A(\ell_0)$. □

Let us choose a leaf ℓ_0 as in Lemma 1, call it LEAF(σ) and denote the open set Q' as Q'_σ . Now every input set $S = \{\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{2n-1}\}$ corresponding to an input in Q'_σ has $V_S = \{0,1,\ldots,n-1\}$ (by Property II). It follows that all the input sets $S = \{\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{2n-1}\}$ that lead to LEAF(σ) have $V_S = \{0,1,\ldots,n-1\}$.

3.2 Constraints on the Inputs Leading to LEAF(σ).

Let the set of constraints on the inputs leading to LEAF(σ) be

$$f_i(x_0, y_0, x_1, y_1, \ldots, x_{2n-1}, y_{2n-1}) = 0 \qquad 1 \le i \le a \;, \tag{2}$$

and

$$g_j(x_0, y_0, x_1, y_1, \ldots, x_{2n-1}, y_{2n-1}) > 0 \qquad 1 \le j \le b \;. \tag{3}$$

By the definition of Q'_σ , we have, for each $1 \le i \le a$,

$$f_i^{[\sigma]}(q) = 0 \qquad \text{for all } q \in Q'_\sigma \;.$$

The next lemma then implies that each f_i can be written as a linear combination of $\Delta(\vec{r}_{\sigma(n+j)}, \vec{r}_{(j+1) \bmod n}, \vec{r}_j)$, $0 \le j < n$. To simplify notations, we shall write $\Delta[r,\sigma,j]$ for $\Delta(\vec{r}_{\sigma(n+j)}, \vec{r}_{(j+1) \bmod n}, \vec{r}_j)$ from now on. Keep in mind that $\Delta[r,\sigma,j] = 0$ for all $r \in I_\sigma$.

Lemma 2. Let $f(x_0, y_0, x_1, y_1, \ldots, x_{2n-1}, y_{2n-1})$ be a polynomial of at most degree 2 . If $f^{[\sigma]}(q) = 0$ for all $q \in Q'_\sigma$, then $f = \sum_{0<j<n} \xi_j \Delta[r,\sigma,j]$ for some constants ξ_j .

Proof. Write

$$f = \sum_{0 \le i \le j < n} a_{ij}^{(1)} x_{\sigma(n+i)} x_{\sigma(n+j)} + \sum_{0 \le i,j < n} a_{ij}^{(2)} x_{\sigma(n+i)} y_{\sigma(n+j)}$$

$$+ \sum_{0 \le i \le j < n} a_{ij}^{(3)} y_{\sigma(n+i)} y_{\sigma(n+j)} + \sum_{0 < j < n} b_j^{(1)} x_{\sigma(n+j)} s_j(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

$$+ \sum_{0 \le j < n} b_j^{(2)} y_{\sigma(n+j)} t_j(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

$$d(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \quad , \tag{4}$$

where s_j , t_j are linear functions and d a polynomial of degree at most 2 .

As $f^{[\sigma]}(q) = 0$ for all $q \in Q'_\sigma$, we have

$$\frac{\partial^2 f^{[\sigma]}}{\partial \lambda_i \partial \lambda_j} = 0 \tag{5}$$

for all. $0 \le i \le j < n$. We can also calculate from (4) to obtain

$$\frac{\partial^2 f^{[\sigma]}}{\partial \lambda_i \partial \lambda_j} = \begin{cases} a_{ij}^{(1)}(x_i - x_{i+1})(x_j - x_{j+1}) + a_{ij}^{(2)}(x_i - x_{i+1})(y_j - y_{j+1}) \\ \quad + a_{ji}^{(2)}(x_j - x_{j+1})(y_i - y_{i+1}) + a_{ij}^{(3)}(y_i - y_{i+1})(y_j - y_{j+1}) & \text{if } i < j , \\ \\ 2a_{ii}^{(1)}(x_i - x_{i+1})^2 + 2a_{ii}^{(2)}(x_i - x_{i+1})(y_i - y_{i+1}) \\ \quad + 2a_{ii}^{(3)}(y_i - y_{i+1})^2 & \text{if } i = j , \end{cases} \tag{6}$$

where we agree that $x_n = x_0$, $y_n = y_0$ in the above equation. It is easy to see that, for (5) and (6) to be consistent for all $q \in Q'_\sigma$, one must have

$$a_{ij}^{(k)} = 0 \qquad \text{for all } i, j, k . \tag{7}$$

Similarly, for all $q \in Q'_\sigma$, one has from (4) and (7)

$$0 = \frac{\partial f^{[\sigma]}}{\partial \lambda_j} = b_j^{(1)}(x_j - x_{(j+1) \bmod n}) s_j(x_0, \ldots, y_{n-1})$$
$$+ b_j^{(2)}(y_i - y_{(j+1) \bmod n}) t_j(x_0, \ldots, y_{n-1}) ,$$

for each $0 \leq j < n$. This implies the existence of constants ξ_j such that

$$b_j^{(1)} s_j (x_0, \ldots, y_{n-1}) = \xi_j (y_{(j+1) \bmod n} - y_j) ,$$

and (8)

$$b_j^{(2)} t_j (x_0, \ldots, y_{n-1}) = -\xi_j (x_{(j+1) \bmod n} - x_j) .$$

Formulas (4), (7) and (8) lead to

$$f = \sum_{0 < j < n} \xi_j [x_{\sigma(n+j)} (y_{(j+1) \bmod n} - y_j) - y_{\sigma(n+j)} (x_{(j+1) \bmod n} - x_j)]$$
$$+ d(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

$$= \sum_{0 \leq j < n} \xi_j \Delta[r, \sigma, j] + d_0(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) ,$$

where $d_0(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) = d(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$
$$\sum_{0 \leq j < n} (x_{(j+1) \bmod n} y_j - y_{(j+1) \bmod n} x_j) .$$

Since $f^{[\sigma]}(q) = 0$ and $\Delta[r, \sigma, j] = 0$ for all $q \in Q'_\sigma$, we must have $d_0(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ identically zero. This proves the lemma. □

We now state without proof an elementary fact, to be used in the proof of Lemma 3 as well as in the Appendix. Let $\vec{p}, \vec{p}', \vec{p}'', \vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$ be Points in the plane.

<u>Fact 1</u>. If $P = \sum_{1 \le i \le n} \lambda_i \vec{p}_i$ with $\sum_{1 \le i \le n} \lambda_i = 1$, then

$$\Delta(\vec{p}, \vec{p}', \vec{p}'') = \sum_{1 \le i \le n} \lambda_i \Delta(\vec{p}_i, \vec{p}', \vec{p}'') \quad .$$

<u>Lemma 3</u>. For any input $r = (\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{2n\ 1}) \in E^{4n}$ that leads to $\text{LEAF}(\sigma)$, $\Delta[r,\sigma,j] = 0$ for all $0 \le j < n$.

<u>Proof.</u> By Lemma 2, the constraints (2) and (3) on inputs leading to $\text{LEAF}(\sigma)$ can be written as

$$\sum_{0<j<n} \xi_{ij} \Delta[r,\sigma,j] = 0 \qquad i = 1,2,\ldots,a \ , \tag{9}$$

and

$$g_i(r) > 0 \qquad i = 1,2,\ldots,b \ , \tag{10}$$

where ξ_{ij} are constants.

If the rank of the a by n matrix (ξ_{ij}) is n , then the constraints in (9) force all $\Delta[r,\sigma,j] = 0$, and the lemma is true. We thus assume that the rank of (ξ_{ij}) is less than n , in which case there exist a non-empty $J \subset \{0,1,\ldots,n-1\}$ and constants η_{ij} such that (9) is equivalent to

$$\Delta[r,\sigma,i] = \sum_{j \in J} \eta_{ij} \Delta[r, \sigma, j] \quad \text{for } i \in \{0,1,\ldots,n-1\}-J \ . \tag{9'}$$

Let $r = (\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{2n-1})$ be any input in I_σ . We shall construct an input $r' = (\vec{r}'_0, \vec{r}'_1, \ldots, \vec{r}'_{2n-1})$ so that constraints (9), (10) are satisfied but $H_0(S') \neq \{r'_0, r'_1, \ldots, r'_{n-1}\}$ where $S' = \{\vec{r}'_0, \vec{r}'_1, \ldots, \vec{r}'_{2n-1}\}$. Thus the input set S' leads to LEAF(a) but $V_{S'} \neq \{0,1,\ldots,n-1\}$, which is a contradiction. The proof of the lemma is then complete.

For each $0 \leq j < n$, let $\vec{e}_j$ be a point so that

$$\Delta(\vec{e}_j, \vec{r}_{(j+1) \bmod n}, \vec{r}_j) > 0 \ . \tag{11}$$

Let $\delta > 0$ be a small number to be specified. Define

$$\beta_i = \begin{cases} \delta / \Delta(\vec{e}_i, \vec{r}_{(i+1) \bmod n}, \vec{r}_i) & \text{if } i \in J , \\ \\ (\delta \sum_{j \in J} \eta_{ij}) / \Delta(\vec{e}_i, \vec{r}_{(i+1) \bmod n}, \vec{r}_i) & \text{if } i \in \{0,1,\ldots,n-1\} - J . \end{cases} \tag{12}$$

Let $r' = (\vec{r}'_0, \vec{r}'_1,$ [illegible] $_1)$, where

$$\begin{cases} \vec{r}'_i = \vec{r}_i & \text{for } 0 \leq i < n , \\ \vec{r}'_{\sigma(n+j)} = (1-\beta_j)\vec{r}_{\sigma(n+j)} + \beta_j \vec{e}_j & \text{for } 0 < j < n . \end{cases} \tag{13}$$

Choose $\delta > 0$ small enough so that all $\vec{r}'_i$ are distinct and that all the inequalities in (10) are satisfied for r' . To show that all constraints in (9) are satisfied, we need only check that all. equations in (9)' are true for r' . For each $0 < i < n$, we have

$$\Delta[r',\sigma,i] = \Delta((1-\beta_i)\vec{r}_{\sigma(n+i)} + \beta_i\vec{e}_i , \vec{r}_{(i+1) \bmod n}, \vec{r}_i)$$

$$= (1-\beta_i)\Delta[r,\sigma,i] + \beta_i\Delta(\vec{e}_i , \vec{r}_{(i+1) \bmod n}, \vec{r}_i)$$

$$= \beta_i\Delta(\vec{e}_i , \vec{r}_{(i+1) \bmod n}, \vec{r}_i) \quad , \tag{14}$$

where we have used Fact 1 and the equalities $\Delta[r,\sigma,i] = 0$. For $j \in J$, this gives $\Delta[r',\sigma,j] = \delta$. For $i \in \{0,1,\ldots,n-1\}-J$, we have from (14) and (12)

$$\Delta[r',\sigma,i] = \delta \sum_{j \in J} \eta_{ij}$$

$$= \sum_{j \in J} \eta_{ij}\Delta[r',\sigma,j] \quad .$$

This proves that r' satisfies (9)'.

We have proved that r' , defined by (13), satisfies constraints (9) and (10). To finish the proof of Lemma 3, it remains to show that $H_0(S') \neq \{\vec{r}'_0,\vec{r}'_1,\ldots,\vec{r}'_{n-1}\}$. Let $j \in J$. If $H_0(S') = \{\vec{r}'_0,\vec{r}'_1,\ldots,\vec{r}'_{n-1}\}$, then $\vec{r}'_{\sigma(n+j)}$ must be a convex combination of $\vec{r}'_0, \vec{r}'_1, \ldots, \vec{r}'_{n-1}$, or equivalently, a convex combination of $\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{n-1}$ (since $\vec{r}'_i = \vec{r}_i$ for $0 \leq i < n$). By Property III, this implies

$$\Delta(\vec{r}'_{\sigma(n+j)} , \vec{r}_{(j+1) \bmod n}, \vec{r}_j) \leq 0 \ .$$

But, repeating the derivation of (14), we obtain

$$\Delta(\vec{r}'_{\sigma(n+j)}, \vec{r}_{(j+1) \bmod n}, \vec{r}_j) = \delta > 0 \ ,$$

which is a contradiction. This proves $H_0(S') \neq \{\vec{r}'_0,\vec{r}'_1,\ldots,\vec{r}'_{n-1}\}$ and the lemma. □

3.3 Completing the Proof.

Lemma 4. If $\sigma \neq \sigma'$, then LEAF(a) $\neq$ LEAF(σ') .

Proof. Choose $0 < \ell \neq j < n$ such that $\sigma(n+j) = \sigma'(n+\ell)$. Let $(\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{2n-1}) \in I_{\sigma'}$ be any input leading to LEAF(σ') . If LEAF(a) = LEAF(@) , then by Lemma 3, we have

$$\Delta(\vec{r}_{\sigma(n+j)}, \vec{r}_{(j+1) \bmod n}, \vec{r}_j) = 0 ,$$

i.e.,

$$\Delta(\vec{r}_{\sigma'(n+\ell)}, \vec{r}_{(j+1) \bmod n}, \vec{r}_j) = 0 .$$

But this contradicts Property IV for I,, , □

We have demonstrated the existence of n! leaves in the algorithm T , This completes the proof of Theorem 1.

4. Remarks.

We have proved a cn log n lower bound for the convex hull problem in a reasonably general model, which includes all the known algorithms. This seems to be a rare instance, in which a non-trivial lower bound is obtained by exploiting the properties of quadratic tests explicitly. We remark that quadratic or high order tests are needed to solve the convex hull problem. In fact, there can not be any decision tree algorithm for finding convex hulls that only use linear tests, This can be seen from the fact that the set of equality constraints at LEAF(a) must not be linear equations (according to Lemma 2).

It remains an open problem whether decision trees of height o(n log n) exist when high order polynomials are permitted in the tests. We conjecture not, even if no restriction is put on the maximum degree of the polynomials allowed.

Appendix. A Proof for the Existence of ϵ_n .

In this appendix, we will prove that there exists an $\epsilon_n > 0$ such that Properties I-IV are true for all $(\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{2n-1}) \in I_\sigma$. See Section 3.1 for terminology. Remember that $n > 3$.

Let $\vec{r}_j^{(0)} = \left(\cos \frac{2\pi j}{n}, \sin \frac{2\pi j}{n}\right)$ for $0 \leq j < n$. For any vector $\vec{v} = (w,u)$, let $\|\vec{v}\| = \sqrt{w^2 + u^2}$. We need to show the existence of an $\epsilon_n > 0$ such that Properties I-IV are true for all $(\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{2n-1})$ satisfying the following conditions:

$$\|\vec{r}_i - \vec{r}_i^{(0)}\|^2 < \epsilon_n \qquad \text{for } 0 \leq i < n , \tag{A1}$$

and, for each $0 \leq j < n$,

$$\vec{r}_{\sigma(n+j)} = \lambda_j \vec{r}_j + (1-\lambda_j)\vec{r}_{(j+1) \bmod n} \qquad \text{for some } \lambda_j \in (1/4, 3/4) . \tag{A2}$$

We need the following fact.

<u>Fact 2.</u> Let $0 \leq i,j < n$ and $i \notin \{j, (j+1) \bmod n\}$. Then $\Delta(\vec{r}_i^{(0)}, \vec{r}_{(j+1) \bmod n}^{(0)}, \vec{r}_j^{(0)}) < 0$.

<u>Proof.</u> Let $\theta = 2\pi/n$. Then

$$\Delta(\vec{r}_i^{(0)}, \vec{r}_{(j+1) \bmod n}^{(0)}, \vec{r}_j^{(0)}) = \det \begin{pmatrix} \cos i\theta & \sin i\theta & 1 \\ \cos (j+1)\theta & \sin (j+1)\theta & 1 \\ \cos j\theta & \sin j\theta & 1 \end{pmatrix}$$

$$= \det \left[\begin{pmatrix} \cos i\theta & \sin i\theta & 1 \\ \cos (j+1)\theta & \sin(j+1)\theta & 1 \\ \cos j\theta & \sin j\theta & 1 \end{pmatrix} \begin{pmatrix} \cos j\theta & -\sin j\theta & 0 \\ \sin j\theta & \cos j\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$= \det \begin{pmatrix} \cos (i-j)\theta & \sin (i-j)\theta & 1 \\ \cos \theta & \sin \theta & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= ((\sin \theta)(\cos (i-j)\theta - 1)) - (\sin (i-j)\theta)(\cos \theta - 1)$$

$$= \left(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right)\left(-2\left(\sin \frac{i-j}{2}\theta\right)^2\right)$$

$$+ 2 \sin\left(\frac{i-j}{2}\theta\right) \cos\left(\frac{i-j}{2}\theta\right) 2 \sin^2 \frac{\theta}{2}$$

$$= 4\left(\sin \frac{\theta}{2}\right)^2 \left(\sin \frac{i-j}{2}\theta\right)^2 \left(- \cot \frac{\theta}{2} + \cot\left(\frac{i-j}{2}\theta\right)\right) .$$

Using the properties of cot and the facts $0 < i,j < n$, $i \notin \{j, (j+1) \bmod n\}$, one can show that

$$\cot\left(\frac{i-j}{2}\theta\right) = \cot\left(\frac{(i-j)\pi}{n}\right)$$

$$\leq \cot \frac{2\pi}{n}$$

$$< \cot \frac{\pi}{n}$$

$$= \cot \frac{\theta}{2} .$$

Note also that $\left(\sin \frac{\theta}{2}\right)^2 \left(\sin\left(\frac{i-j}{2}\theta\right)\right)^2 > 0$. Fact 2 follows easily. □

Observing Fact 2 and the continuity property of the function Δ , we can choose a sufficiently small $\epsilon_n > 0$ such that, if $\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{n-1}$ satisfy (A1), then the following conditions are true.

(i) all $\vec{r}_i$ $(0 \le i < n)$ are distinct,

(ii) for $0 \le i,j < n$ and $i \notin \{j, (j+1) \bmod n\}$, $\Delta(\vec{r}_i, \vec{r}_{(j+1) \bmod n}, \vec{r}_j) < 0$.

We will now prove that, for this choice of ϵ_n, any $(\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{2n-1})$ that satisfies (A1) and (A2) must have Properties I-IV. We shall freely use Fact 1 (in Section 3.2) in the ensuing arguments.

Property IV. Let $0 \le \ell \ne j < n$. Then,

$$\Delta(\vec{r}_{\sigma(n+\ell)}, \vec{r}_{(j+1) \bmod n}, \vec{r}_j) = \lambda_\ell \Delta(\vec{r}_\ell, \vec{r}_{(j+1) \bmod n}, \vec{r}_j)$$

$$+ (1-\lambda_\ell)\Delta(\vec{r}_{(\ell+1) \bmod n}, \vec{r}_{(j+1) \bmod n}, \vec{r}_j)$$

$$< 0 ,$$

because of condition (ii), the fact $n \ge 3$, and the fact both λ_ℓ, $(1-\lambda_\ell)$ are positive. This verifies Property IV.

Property I. The points $\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{n-1}$ are distinct by condition (i). Property IV, together with the equalities $\Delta(\vec{r}_{\sigma(n+j)}, \vec{r}_{(j+1) \bmod n}, \vec{r}_j) = 0$ for $0 \le j < n$, ensures that the points $\vec{r}_n, \vec{r}_{n+1}, \ldots, \vec{r}_{2n-1}$ are all distinct. That, for each $0 \le j < n$, $\vec{r}_{\sigma(n+j)}$ is distinct from all $\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{n-1}$, follows from the fact $\vec{r}_{\sigma(n+j)} \ne \vec{r}_j$, $\vec{r}_{(j+1) \bmod n}$ and Property IV. This verifies Property I.

Property II. Because of (A2), one has $H_0(S) \subseteq \{\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{n-1}\}$. We now claim that each $\vec{r}_i \in H_0(S)$, $0 \le i < n$, Otherwise, we can write

$$\vec{r}_i = \mu_{i+1}\vec{r}_{(i+1)\bmod n} + \sum_{\substack{0 \le j < n \\ j \ne i,\ (i+1) \bmod n}} \mu_j \vec{r}_j$$

for some $\mu_j \geq 0$ and $\sum_j \mu_j = 1$. As $\vec{r}_i \neq \vec{r}_{(i+1) \bmod n}$ by condition (i), at least some $\mu_j > 0$ with $j \neq i, (i+1) \bmod n$. This, together with condition (ii), leads to

$$\Delta(\vec{r}_i, \vec{r}_{(i+1)\bmod n}, \vec{r}_i) = \sum_{\substack{0 \le j < n \\ j \ne i,\ (i+1) \bmod n}} \mu_j \Delta(\vec{r}_j, \vec{r}_{(i+1)\bmod n}, \vec{r}_i)$$

$$< 0 \quad .$$

But this is impossible as $\Delta(\vec{r}_i, \vec{r}_{(i+1)\bmod n}, \vec{r}_i)$ should be 0 . Thus, every $\vec{r}_i$ must be in $H_0(S)$. This proves $H_0(S) = \{\vec{r}_0, \vec{r}_1,$ [illegible] **,&) .

<u>Property III</u>. Let $\vec{p} = \sum_{0<i<n} \mu_i \vec{r}_i$ with $\mu_i > 0$ and $\sum_i \mu_i = 1$.

We have, using conditions (ii),

$$\Delta(\vec{p}, \vec{r}_{(j+1)\bmod n}, \vec{r}_j) = \sum_{0 \le i < n} \mu_i \Delta(\vec{r}_i, \vec{r}_{(j+1)\bmod n}, \vec{r}_j)$$

$$= \sum_{\substack{0 \le i < n \\ \text{if } j,\ (j+1) \bmod n}} \mu_i \Delta(\vec{r}_i, \vec{r}_{(j+1)\bmod n}, \vec{r}_j)$$

$$\leq 0 \quad ,$$

for each $0 < j < n$, This verifies Property III.

We have thus verified Properties I-IV for all $(\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{2n-1})$ that satisfy (A1) and (A2). This completes the proof for the existence of an $\epsilon_n > 0$ with the desired property.

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