GRAPH 2-ISOMORPHISM IS NP-COMPLETE

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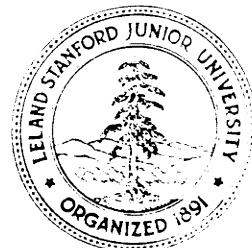
Graph 2-Isomorphism Is NP-Complete

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Abstract.

Two graphs G and G' are said to be k-isomorphic if their edge sets can be partitioned into $E(G) = E_1 \cup E_2 \cup \ldots \cup E_k$ and $E(G') = E_1' \cup E_2' \cup \ldots \cup E_k'$ such that as graphs, E_i and E_1' are isomorphic for $1 \leq i \leq k$. In this note we show that it is NP-complete to decide whether two graphs are 2-isomorphic.

Given two graphs */ $G = (V, E)$ and $G' (V', E')$ with the same number

of edges, by a k-isomorphism of G and G' we mean a partition of

$E = E_1 \cup E_2 \cup \ldots \cup E_k$ and a partition of $E' = E_1' \cup E_2' \cup \ldots \cup E_k'$ such

that as graphs, E_i and E_i' are isomorphic for $1 \leq i \leq k$. Let

$U(G, G')$ be the minimum value of k for which a k-isomorphism of G

and G' exists.   (See [1] for a study of k-isomorphism of graphs.)

In general, the determination of whether $U(G, G') \leq k$ for two

graphs G , G' , and positive integer k is an NP-complete problem.

For, it clearly belongs to NP; and if we take G' to be a star graph

(with the same number of edges as G), then $U(G, G')$ is simply the

minimum size of a vertex cover for G , a well-known NP-complete

-problem [4].   The question "Is $U(G, G') = 1$?" is the familiar graph

isomorphism problem, which is not known to be NP-complete or not [2], [4].

In this note we show that graph-2-isomorphism (G2I), i.e., to decide

whether $U(G, G') \leq 2$, is an NP-complete -problem.

We will use a transformation from the following problem, which is

known to be NP-complete [2].

Exact Cover by 3-Sets (X3C).

Instance: Set $X = \{1, 2, \ldots, n\}$ and a family $\mathscr{A} = \{A_i\}$ of 3-element

                subsets of X .

Question: Does $\mathscr{A}$ contain an exact cover for X , i.e., a subfamily

                $\mathscr{A}' \subseteq \mathscr{A}$ such that every element of X occurs in exactly

                one member of $\mathscr{A}'$?

Theorem.   X3C is polynomially transformable to G2I. Therefore, the

graph 2-isomorphism problem is NP-complete.

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*/ We follow [3] for the terminology on graphs.

<u>Proof.</u>   Given an instance of X3C, we may assume without loss of generality that $n = 3m \geq 6$, $|\mathscr{B}| = m+\ell \geq m$, and $X \subseteq \bigcup_{i=1}^{m+\ell} A_i$. We shall construct a pair of graphs G and H corresponding to $(X,\mathscr{B})$, as shown in Figure 1.

Graph G contains a connected component TS_i corresponding to each A_i in $\mathscr{B}$. If $A_i = \{p,q,r\}$, then TS_i is a triangle, with, additionally, three stars of size $p+1$, $q+1$ and $r+1$ attached to the vertices of the triangle.   We will denote $\bigcup_{i=1}^{m+\ell} TS_i$ by TS .   In addition to TS , graph G contains a connected component M , which is a complete graph on n vertices with m disjoint triangles removed.

Graph H is the disjoint union of four subgraphs KS , N , T , and S . In KS , we have a complete graph on n   vertices $\{v_1, v_2, \ldots, v_n\}$, together with an i-star attached to each v_i .   The complete graph of KS will be referred to as K_n henceforth.   Subgraph N consists of n disjoint edges, and T consists of ℓ disjoint triangles.   Finally, S consists of 3ℓ disjoint stars, one of size $p+1$ for each p that occurs in the multiset $\left(\bigcup_{i=1}^{m+\ell} A_i \right) - X$.

Clearly,   G and H can be constructed from $(X,\mathscr{B})$ in polynomial time.   Since G and H are not isomorphic,   U(G,H) is at least 2 . We now show that U(G,H) < 2 if and only if $\mathscr{B}$ contains an exact cover for X.

<u>Lemma 1.</u> $U(G,H) \leq 2$ if $\mathscr{B}$ contains an exact cover for X .

<u>Proof of Lemma 1.</u>   Without loss of generality, assume that $\{A_1, A_2, \ldots, A_m\}$ forms an exact cover for X .   We decompose G and H in two steps as follows.

G

H

TS_1

p+1

r+1   q+1

KS

v_n v_1
  v_2
    v_3

K_n

N

I   I   . . .

n

$TS_{m+\ell}$

T

ℓ

S

3ℓ

$M = K_n - mK_3$

(dotted lines are the missing edges)
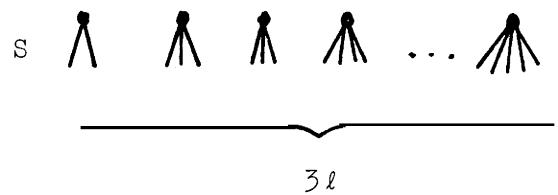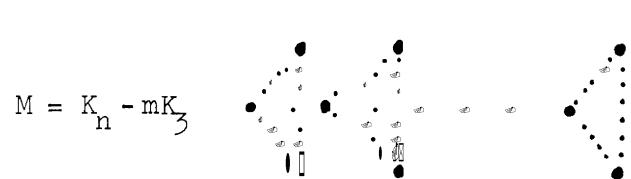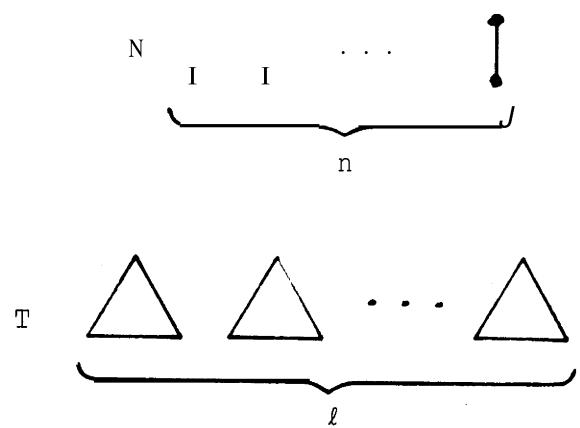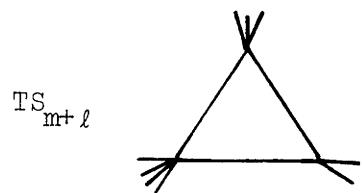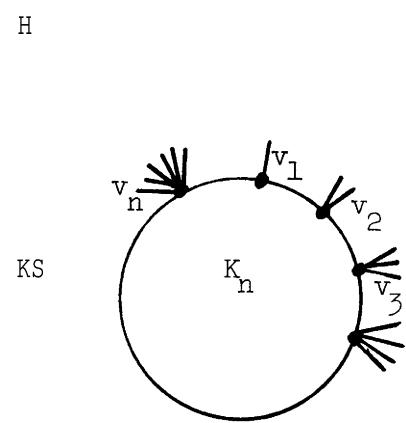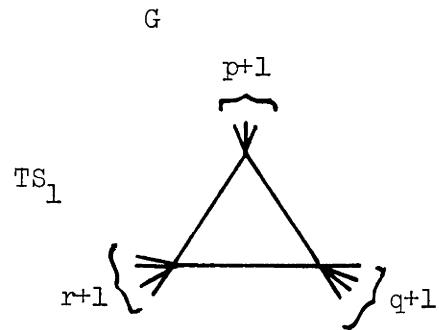
Figure 1.   The graphs G and H ,

Step 1.   Suppose $A_i = \{p, q, r\}$, where $1 \leq i \leq m$.   In the corresponding

TS_i , we take a subgraph consisting of the triangle together with stars

of size p , q , r , and map it into the triangle in KS incident at

$\{v_p, v_q, v_r\}$ with the matching stars.   At the same time map the triangles

of TS_j , for $m+1 \leq j \leq m+\ell$, onto the ℓ triangles of T .

Step 2.   The subgraph that is left in G consists of n = 3m isolated

edges from $\bigcup\limits_{i=1}^{m} TS_i$, which are isomorphic to N ; 3ℓ stars from

$\bigcup\limits_{j=m+1}^{m+\ell} TS_j$, isomorphic to S ; and subgraph M , which is isomorphic

to the remainder of KS . 3

    We mention in -passing that actually U(G,H) $\leq$ 3 for the graphs   (G, H)

constructed from any $(X, \mathcal{B})$.   For we can first map all $m+\ell$ triangles of

TS into K_n and T ; next map the $3(m+\ell)$ stars of TS into KS and S ;

what is left then in both graphs is isomorphic to $M \cup N$. The rest of this

note is devoted to proving the converse of Lemma 1.

Lemma 2.   U(G,H) $\leq$ 2 only if $\mathcal{B}$ contains an exact cover for X .

    We first introduce some notations. Under the assumption U(G,H) $\leq$ 2 ,

let E(G) = $G^{(1)} \cup G^{(2)}$,   E(H) = $H^{(1)} \cup H^{(2)}$ be fixed -partitions of the

edge sets, with isomorphism mappings $\varphi_1: G^{(1)}$ 3 $H^{(1)}$ and $\varphi_2: G^{(2)} \to H^{(2)}$.

For any subgraph F of G (or H), we use $F^{(i)}$ to denote $F \cap G^{(i)}$

(or $F \cap H^{(i)}$, respectively); also, let $\langle F^{(i)} \rangle$ be the isomorphic image

of $F^{(i)}$ under $\varphi^{(i)}$ (or $(\varphi^{(i)})^{-1}$, respectively). For a graph

F = (V,E) , we use e(F) to denote $|E|$. Define vertex cover to

be the minimum size of a subset V' $\subseteq$ V such that for every edge $(u,v) \in E$,

at least one of u and v belongs to V' . The following facts will be useful.

Fact A. If $F \subseteq K_n$ and vertexcover $\leq a$, then $K_n - F$ contains an (n-a) -clique.

Fact B. Let F be a connected component in G . Any edge of H that is incident with a vertex of $(F^{(1)})$ but not contained in $\langle F^{(1)} \rangle$ must belong to $H^{(2)}$.

Proof of Lemma 2. First, we show that any 2-isomorphism of G and H must decompose KS into M and a collection of triangles with stars. Indeed, since KS has more edges than M , we must have either $\langle KS^{(1)} \rangle \cap TS \neq \emptyset$ or $\langle KS^{(2)} \rangle \cap TS \neq \emptyset$. Assume it is the former.

Proposition. Under the assumption that U(G,H) = 2 and $(KS^{(1)}) \cap TS \neq \emptyset$, we must have $\langle KS^{(2)} \rangle = \langle K_n^{(2)} \rangle = M$.

Proof of Proposition. Let TS_i be such that $\langle KS^{(1)} \rangle \cap TS_i \neq \emptyset$. Consider the image of $TS_i^{(1)}$ in H . Let $\{v_{i_1}, v_{i_2}, \bullet \boxtimes \bullet v_{i_h}\}$ be the vertices of K_n that are incident with $\langle TS_i^{(1)} \rangle$.

Fact C. (i) $\langle TS_{i}^{(1)} \rangle \cap K_n$ contains at most h edges.

(ii) $\langle TS^{(1)} \rangle \cap K_n$ contains $\leq$ n edges; equality holds only if $\langle M^{(1)} \rangle \cap K_n = \emptyset$.

Proof. (i) is true since TS_i with one edge removed is a tree.

(ii) follows from (i) immediately. □

<u>Fact D.</u> $2 \le h \le n-2$.

<u>Proof.</u>   (a) Suppose $h \ge n-1$. Then since K_n has no edges disjoint from $\{v_{i_1}, v_{i_2}, \cdots v_{i_h}\}$, we must have $K_n^{(1)} \subseteq \langle TS_i^{(1)} \rangle$. This implies that $\text{vertexcover}(K_n^{(1)}) \le \text{vertexcover}(TS_i) = 3$. By Fact A, $K_n^{(2)}$ must contain a $(n-3)$-clique. Since G does not contain a $(n-3)$ -clique when $n \ge 6$, this is impossible.

(b) Next suppose $h = 1$. Then by Fact B, an $(n-1)$ -star R must be contained in $K_n^{(2)}$. Since the maximum degree of a vertex in M is $n-3$, we must have $R \subseteq \langle TS_j^{(2)} \rangle$ for some j . But then $\langle TS_j^{(2)} \rangle$ is incident with n vertices of K_n , and the same argument as given in (a), with step 1 and step 2 interchanged, shows that this is impossible, This proves Fact D. $\square$

<u>Fact E.</u> $(KS^{(2)}) = \langle K_n^{(2)} \rangle \subseteq M$.

<u>Proof.</u>   Given $2 < h < n-2$, and that an $h \times (n-h)$ bipartite graph Y must be contained in $K_n^{(2)}$ because of Fact B, it is easy to see that Y must lie in $\langle M^{(2)} \rangle$, thus $\langle M^{(2)} \rangle$ is incident with all n vertices $\{v_1, v_2, \ldots, v_n\}$. It follows that $\langle KS^{(2)} \rangle = \langle K_n^{(2)} \rangle \subset M$. $\square$

To finish the proof of the Proposition, note that by Fact E, the edges of K_n are divided into those in $\langle TS^{(1)} \rangle \cap K_n$ and those in $(\langle M^{(1)} \rangle \cup \langle M^{(2)} \rangle) \cap K_n$. This is possible only if the latter contains $e(M) = \binom{n}{2} - n$ edges and the former contains n edges, because of Fact C (ii). But then, $\langle M^{(1)} \rangle \cap K_n = \emptyset$ by Fact C, which implies that $e(\langle M^{(2)} \rangle \cap K_n) = \binom{n}{2} - n$, and hence $\langle K_n^{(2)} \rangle = M$. This proves the Proposition. $\square$

We can now complete the proof of Lemma 2. It follows from the Proposition that $KS^{(2)}$ is the isomorphic image of M, while $KS^{(1)}$ consists of m disjoint triangles, each attached with three stars. Without loss of generality, write $\langle KS^{(1)} \rangle = TS_1^{(1)} \cup TS_2^{(1)} \cup \ldots \cup TS_m^{(1)}$ where for $1 \leq i \leq m$, $TS_i^{(1)}$ is a subgraph of TS_i and moreover, they are triangles with stars of size $\{p', q', r'\}$ and $\{p+1, q+1, r+1\}$ respectively, with $p' \leq p$, $q' \leq q$ and $r' \leq r$.

If $\mathscr{J}$ does not contain an exact cover for X, then we will not have $p' = p$, $q' = q$, $r' = r$ in $TS_i^{(1)}$ and TX_i for all $1 \leq i \leq m$. Hence $TS_1^{(2)} \cup TS_2^{(2)} \cup \ldots \cup TS_m^{(2)}$ will contain fewer than n isolated edges. This makes it necessary, because of the subgraph N in H, for $TS_{m+1} \cup TS_{m+2} \ldots \cup TS_{m+\ell}$ to yield $\delta \geq 1$ isolated edges in either step 1 or 2. Assume without loss of generality that TS_{m+1} contributes an isolated edge (u,v) in step 1. We examine two cases.

Case 1. Suppose (u,v) is in the triangle of TS_{m+1}. Then $TS_{m+1}^{(2)}$ contains a path of length 4, which does not exist in $N \cup T \cup S$ of H (Figure 2(a), 2(b)).

Case 2. Suppose (u,v) is in one of the stars of TS_{m+1}. Then in $TS_{m+1}^{(2)}$, u is a vertex of degree ≥ 3, and hence must be

mapped by φ_2 into a star of S . This implies that $TS_{m+1}^{(1)}$ contains a path of length ≥ 3, which again does not exist in $N \cup T \cup$ S (Figure 2(c), 2(d)).

Thus we can have U(G,H) = 2 only if $\mathcal{J}$ contains an exact cover for X , and this completes the proof of Lemma 2 and the Theorem. $\square$


(a)

TS_{m+1}

(b)

A subgraph of $TS_{m+1}^{(2)}$

(c)

TS_{m+1}

(d)

A subgraph of $TS_{m+1}^{(1)}$
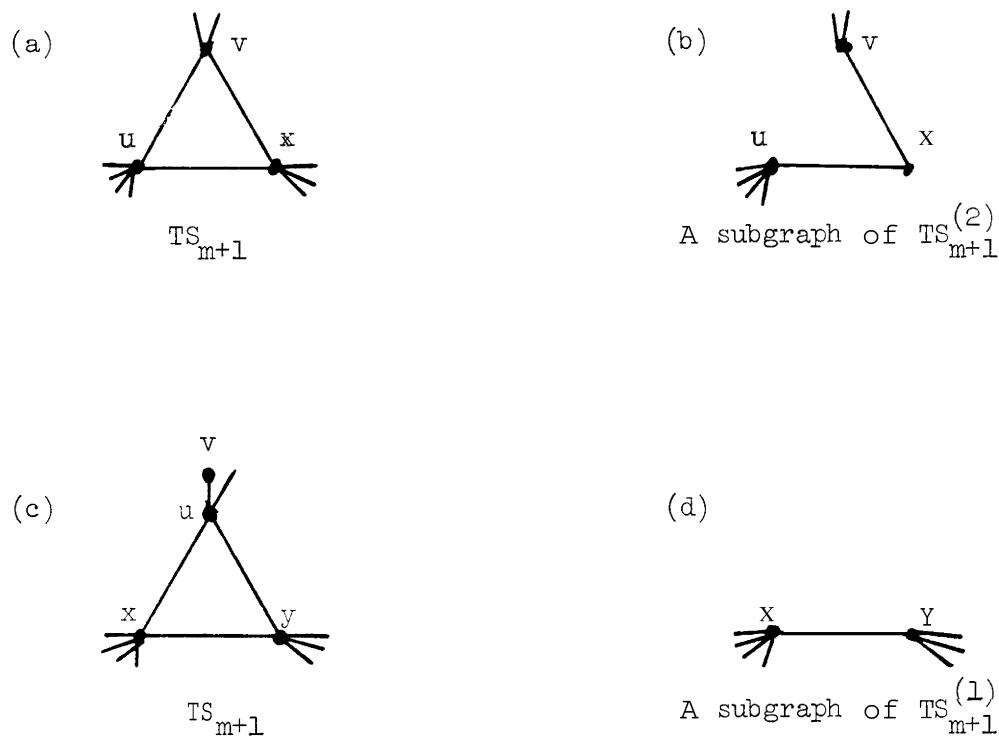
Figure2

We wish to point out that in our construction, it is necessary to employ a different representation for elements of X in TS than in KS (such as using $(p+1)$ -stars versus p-stars for $p \in X$). The following example shows that, for instance, if just p-stars were used in both G' and H' , then one could have $U(G',H') = 2$ even though $\mathcal{A}$ does not contain an exact cover for X .

<u>Example.</u>   Let $X = \{1,2,\ldots,6\}$ and $\mathcal{A} = \{A_1 = \{1,2,5\}, A_2 = \{4,5,6\},$ $A_3 = \{2,3,4\}\}$. (See Figure 3. We use R_p to denote a p-star.) One can first map two of the edges of R_5 in TS_1 into the R_5 of s ; the triangle of TX_3 into T ; and M into K_n . The remaining subgraphs of G' and H' are then isomorphic. Such unwanted phenomena cannot be remedied simply by choosing other representations, say, using p^2 -stars for $p \in X$, in both G' and H' .

G'

TS_1

R_1
R_2 R_5

TS_2

R_4
R_5 R_6

TS_3

R_2
R_3 R_4

M

H'

KS

T

S

R_2 R_4 R_5
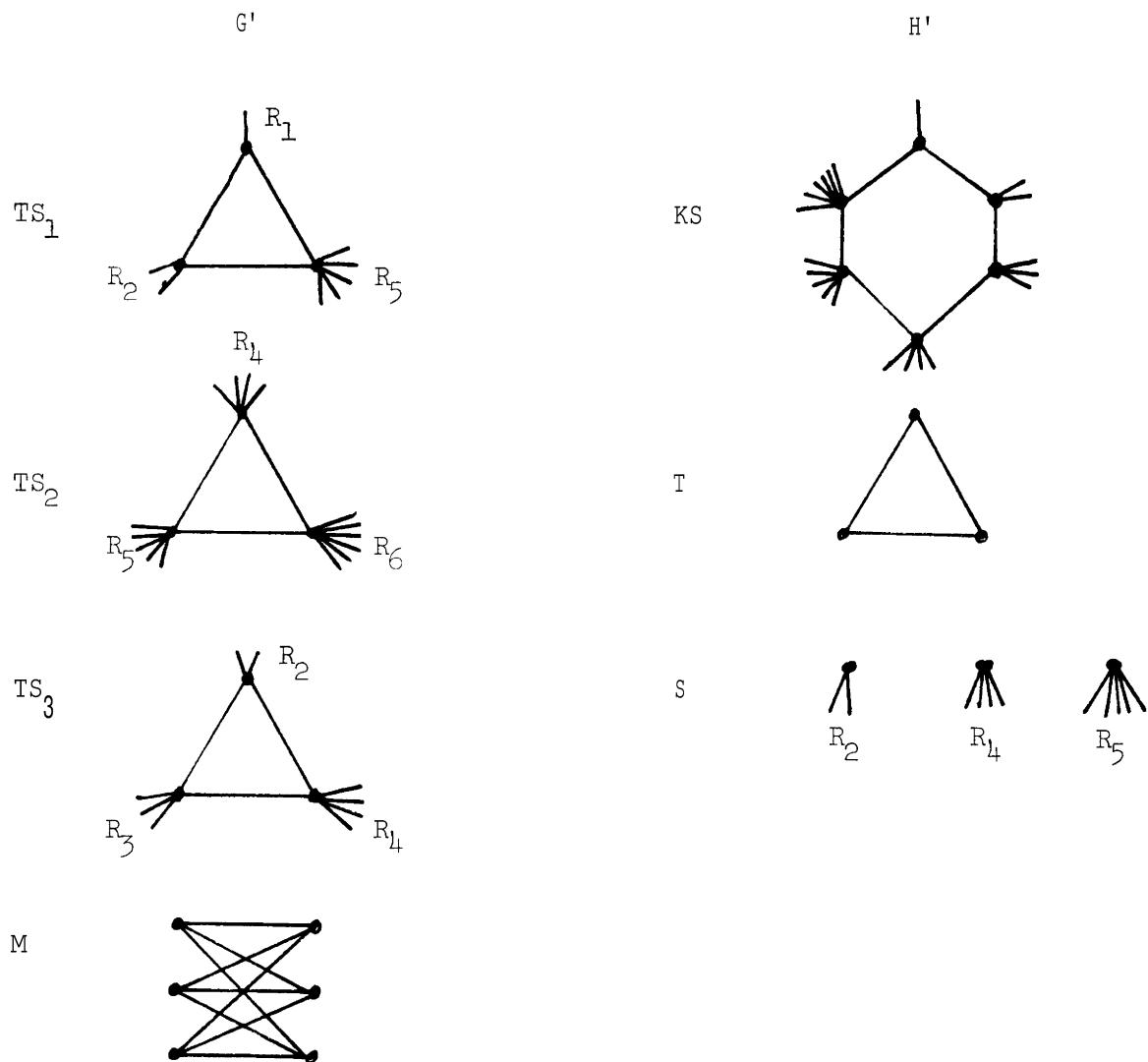
Figure 3.   An example with $U(G',H') = 2$.

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