A CLASS OF SOLUTIONS TO THE GOSSIP PROBLEM

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Abstract

We characterize and count optimal solutions to the gossip problem in which no one hears his own information. That is, we consider graphs with n vertices where the edges have a linear ordering such that an increasing path exists from each vertex to every other, but there is no increasing path from any vertex to itself. Such graphs exist only when n is even, in which case the fewest number of edges is 2n-4, as in the original gossip problem. We characterize optimal solutions of this sort (NOHO-graphs) using a correspondence with a set of permutations and binary sequences, This correspondence enables us to count these solutions and several subclasses of' solutions. The numbers of solutions in each class are simple powers of 2 and 3, with exponents determined by n. We also show constructively that NOHO-graphs are planar and Hamiltonian, and we mention applications to related problems.

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There are two kinds of people who
   blow through life like a breeze;
And one kind is gossipers, and the
   other kind is gossipees,
--Ogden Nash

Gossip is mischievous, light and easy
to raise, but grievous to bear and
hard to get rid of. No gossip ever
dies away entirely, if many people
voice it; it too is a kind of divinity.
--Hesiod

1. Introduction

The "gossip problem" has the unusual distinction of being solved four times within a year, Proposed by Boyd and popularized by Erdös, it considers a group of n people, each possessing a distinct item of information, Telephone calls are arranged between two people at a time, in which they exchange all the information they know. (It is also called the "telephone problem.") We seek the minimum number of calls required to transmit all the information to everyone. For $n \geq 4$, it is $2n-4$. This was proved by Bumby and Spencer(unpublished), Baker and Shostak[1], Tijdeman[12], and Hajnal, Milner, and Szemerédi[7]. These proofs were all different and fairly short.

Ways were quickly found to generalize the problem. The calling scheme can be represented by a graph whose edges are linearly ordered to represent the order of calls. We require an "increasing path" from each vertex to every other. Edges may be repeated in the ordering, in which case they are counted twice, representing repeated calls.

Moving from graphs to hypergraphs, we can ask the same question when the medium of transmission is "conference calls" of a fixed size k. The minimum number here was discovered by Lebensold[10]. It is on the order of $2(n-1)/(k-1)$, with a number of technical adjustments. Bermond[2] recently rederived the result with a shorter proof.

Thus far we have considered complete graphs, Suppose the "allowable" calls are restricted to some subgraph. For example, we don't wish to assign sworn enemies to talk to each other, This problem was considered by Harary and Schwenk[8], and also by Golumbic[6]. As long as the graph is connected, we can transmit the information in 2n-3 calls using a spanning tree, with the calls ordered to and then from some root. If the graph contains a 4-cycle, we can still achieve 2n-4. Here we use the 4-cycle and edges which grow tree-like to the remaining vertices. It is easy to find a suitable ordering. It is conjectured that if the graph does not contain a 4-cycle, then 2n-3 edges are required.

Instead of ordinary graphs, we could consider directed graphs , representing one-directional transfers of information. This is the "telegraph problem." Harary and Schwenk[8] and Golumbic[6] have shown that if the digraph of allowed edges is strongly connected, then the minimum number of messages for complete transmission is 2n-2. Golumbic also examines how many messages are required to tranmit whatever can be transmitted when the digraph is not strongly connected.

Another variation asks for the minimum time of transmission, where each vertex can participate in at most one call per time unit. Knodel[9] solved this for complete graphs, and Schmitt[11] for complete hypergraphs. Cockayne, Hedetniemi, and Slater[3] consider this in terms of individual vertices. Entringer and Slater[5] consider time of transmission in complete digraphs.

The behavior of all these minima is logarithmic in the number of vertices, adjusted by constant terms depending on residue classes of n.

Cot[4] discusses ways to vary the problem. We consider here not a generalization of the situation, but a restriction of the allowable calling schemes. We consider calling schemes that transmit all information, with the additional requirement that no one ever hears his own information. That is, no one speaks to anyone who knows his original tidbit, In the graphical formulation, with an ordering on the edges, this means we can find no pathwhichleaves a vertex, continually "increases", and returns to it, We determine when such solutions exist and how many edges they require, and we characterize and count the optimal ones.

We show that calling schemes completing all transmissions and satisfying NOHO ("no one hears his own information") exist only when n is even. We call such such a salution with fewest edges (on n vertices) a NOHO-graph. NOHO-graphs have 2n-4 edges, the usual gossip result. Particular examples include C_4 (the 4-cycle) and any regular graph of degree 3 on 8 vertices having no triangles. The latter set we call Q*, since it includes the cube. We characterize other NOHO-graphs by two permutations and two binary sequences. Each of the four describes the placement of approximately n/2-1 edges in the graph. We show that any two of the four suffice to determine the other two and hence the entire graph. We use this to count the number of realizable quadruples determining NOHO-graphs on n vertices, (Realizable quadruples, or simply "solutions," are those **sets** of sequences which correspond to NOHO-graphs.) Letting $p=(n-4)/2$, this number is 3^{p-1} *for* $n\geq 6$, n even. NOHO-graphs which are not symmetric are counted twice in this; that is, they correspond to two realizable quadruples. We later count the number of symmetric solutions, so the number of NOHO-graphs is retrievable.

We also define an operation of **"concatenation,"** which puts two solutions together two form a larger solution, This yields a concept of an "irreducible" solution as one which admits no concatenation from smaller solutions. We show the number of solutions on n vertices concatenated from k irreducible parts is $\binom{p-1}{k-1}2^{p-k}$. We also determine the number of symmetric solutions concatenated from k irreducible parts. In particular, the number of irreducible solutions is 2^{p-1}, the number of symmetric solutions is $3^{\lfloor p/2 \rfloor}$, and the number of symmetric irreducible solutions is $2^{\lfloor p/2 \rfloor}$. Ignoring the special graphs C_4 and Q^* and eliminating the double-counting, the number of NOHO-graphs is $(3^{p-1}+3^{\lfloor p/2 \rfloor})/2$.

Additional results include constructive proofs that NOHO-graphs are planar and Hamiltonian and applications to related gossip questions. In the next section, we outline the steps of the proofs toward these goals.

2. Summary of Proofs and Results

The original argument used by Baker and Shostak[1] begins by showing that the smallest graph which could transmit all information in fewer than 2n-4 edges would have to satisfy NOHO. They use NOHO to discuss the "first edges" and "last edges" of the graph and consider the components of the subgraph obtained by deleting those edges, They obtain a contradiction by showing that not all transmissions can be completed. In our preliminary details, we parallel this argument, In a graph satisfying NOHO, the set of edges which correspond to first calls made by some vertex and the set of edges which correspond to last calls made by some vertex each forms a complete matching in the graph, As a corollary, we see that NOHO-graphs must have an even number of vertices.

We consider, for each vertex x, a tree O(x) of edges used to pass its information elsewhere and a tree I(x) carrying information to it. Characterizing the edges which appear in the intersection of the trees, we determine the number c(x) which appear in neither. c(x) turns out to be two less than the degree of the vertex. Now we consider the graph M(G) obtained by deleting the first edges and last edges. Considering where edges of O(x) and I(x) can appear in it and bounding the "useless" edges by c(x), we obtain the major result of section 3. For a NOHO-graph G, M(G) consists of exactly four components which are all trees. Along the way we exhibit such solutions with 2n-4 edges, The contradiction obtained by Baker and Shostak does not arise because these graphs have enough edges,

In section 4 we consider the case where G has no vertex of degree 2. The trees of M(G) must each contain an edge, and examination of cases shows they must all consist of single edges, This requires G to be a 3-regular graph on 8 vertices, and NOHO prohibits triangles, All such graphs admit an edge-ordering which transmits all information, so they are NOHO-graphs.

Returning in section 5 to graphs with vertices of degree 2, we find C_4, which works. If $n>4$, then M(G) consists of two isolated vertices and two caterpillars on $n/2-1$ vertices each, (A caterpillar is a tree with a path hitting every edge,) This enables us to label the vertices of the graph $\{x_j^i\}$, where $i \in \{1,2\}$, $j \in \{0,1,...,n/2-1\}$, according to the order in which information from the isolated vertices x_0^i travels along the caterpillars, The placement of edges in the caterpillars can be described by binary sequeces, where the j^{th} element describes how x_{j+1}^i is joined to the earlier vertices.

To completely characterize the graph, we must describe how the first edges and last edges may be added. To satisfy NOHO a first edge or last edge must always join x_j^i and $x_{j'}^{i'}$, with $i \neq i'$. So, the placement of these edges can be described by permutations, where the j^{th} element of the permutation is k if x_k^2 is the first (respectively, last) neighbor of x_j^1.

In section 6 we derive necessary conditions for pairs of these integer sequences to be realizable by NOHO-graphs. One condition imposes inequalities relating elements of the two permutations. Another restricts where 1% occur in the binary sequences in terms of where reversions occur in the first-edge permutation, The reversions of that permutation are explicitly characterized, (A reversion is a maximal contiguous subsequence of a permutation where the first element is the least.) The characterization is equivalent to forbidding subsequences of length three (in a permutation) whose last element is the largest. All these conditions follow from requiring NOHO, transmission of all information, and the characterization of the graph in terms of the caterpillars. Other conditions follow from the same basic reasons when the graph is reflected, which consists of relabeling the vertices of the graph so the two caterpillars are switched. The sequences for the reflected graph are easily obtained from the original sequences.

Having derived enough necessary conditions, we can show (section 7) that any pair of sequences satisfying the appropriate ones uniquely determines the remaining pair, Furthermore, the resulting quadruple is realizable, so the conditions are sufficient. Therefore, we need only count realizable pairs (P,S), where P is the first-edge permutation and S is the sequence determining the first caterpillar. There are $\binom{p-1}{r-1}$ such permutations whth r reversions (where $p=(n-4)/2$), and 2^{r-1} realizable binary sequences for each of those, so a simple application of the binomial theorem gives 3^{p-1} realizable quadruples.

In section 8 we consider symmetric NOHO-graphs. When the operation of reflection yields the same sequences as before, the graph is symmetric, Otherwise, two quadruples determine the same graph. To count the number-of symmetric **NOHO-graphs,** we first countthenumber of symmetric realizable first-edge permutation& A simple fact about the number of entries in a permutation enables us to construct such permutations step by **step,** where at each step we have two options and determine two elements with our choice. Then we count the number of symmetric **NOHO-graphs** associated with it by counting the number of **last-**edge permutations which can be pa&red with it. For the choice made at each step in constructing the first permutation, making it *one* way results in two options at a corresponding stage of the second construction, while making it the other way leaves only one. Boiling all this down, we have another simple application of the binomial theorem to obtain altogether $3^{\lfloor p/2 \rfloor}$ symmetric NOHO-graphs.

Section 9 treats concatenation, Concatenation creates a NOHO-graph from two smaller ones by identifying two vertices and merging the edge-orderings in a natural **way.** -Also, one vertex of degree two is deleted from each. So, the resulting graph has four fewer vertices than the union of the original two graphs. This is one reason to define $p=(n-4)/2$; that quantity adds **directly** under concatenation, With adjustments for the deleted and identified vertices, the **"top"** caterpillars, **"bottom"** caterpillars, first edges, and last edges of the two small graphs are united to form those respective sets in the new graph. The orderings are merged to make **infor-**

mation flow properly along the caterpillars.

In section 10 we examine . irreducible NOHO-graphs—those which cannot be formed by concatenation. We show there is a unique decomposition of any NOHO-graph as a concatenation of irreducible ones, This follows because the "least refinement" (in terms of compositions of integers) of two such decompositions is also a decomposition, and would lead to a decomposition of one of the original irreducible pieces, Now, using concatenation and the number of compositions of p into k parts, an induction shows there are $\binom{p-1}{k-1}2^{p-k}$ realizable quadruples formed from k irreducible parts, This holds for k=1 also, since precisely that many remain when the others are subtracted from the total. When we require symmetry also, the number with k parts remains an ugly summation, but the proof is similar. In the special case of symmetric irreducible solutions, the summation can be computed, and the number of these is $2^{\lfloor p/2 \rfloor}$.

In section11 we show that NOHO-graphs (except Q*) have two properties that are frequently investigated; they are Hamiltonian and planar. Uniting the first edges and last edges of the graph forms a Hamiltonian circuit. This is proved by dividing it into two paths which are shown to meet at their endpoints and be simple, disjoint, and exhaustive. For planarity, we take those two paths and draw one inside and one outside of the "Hamiltonian caterpillar" formed by M(G). This accounts for all the edges. Showing the no crossings exist completes the proof,

Finally, section 12 presents applications to a few related gossip **questions.** We note that every **NOHO-graph** contains a 4-cycle and that **NOHO-graphs** other **than** C_4 and **Q*** contain duplicated transmissions, A generalization of the gossip problem is proposed, and some trivial special cases of it are solved.

3. Preliminary Results

To facilitate comprehension; we attempt certain rules of notation. In general, the following apply. Upper case letters indicate graphs or graph-valued functions, except that P through T usually denote integer sequences, Where upper case letters refer to sets of some sort, lower case letters refer to elements, except for the elements of a sequence, which are simply subscripted. a through e denote integer-valued functions. **f,g,h** are vertex-valued functions, **i,j,k,l** are indices or utility integers, n,m,p are fixed integers with a particular relationship. q,r,s,t are utility integers, and finally, u through z denote vertices of a graph.

We deal with undirected graphs G which have n vertices and e(G) **edges.** Let V(G) be the vertex set, E(Q) the edge set. $|S|$ denotes the cardinality of a set S. The <u>edges</u> of a graph are unordered pairs chosen, with possible repetition, from the Cartesian product $V(G)\times V(G)$. (x,y) denotes the edge with x and y as endpoints. d(x) denotes the <u>degree</u> of vertex x, which is the number of edges to which it belongs. A <u>**regular**</u> graph of degree k, or a k-regular graph, is one where each vertex has degree k, A path of length k from v_0 to vk is an ordered sequence of vertices $(v_0,v_1,\ldots,v_k)$, where $(v_i,v_{i+1}) \in E(G)$ and v_i are distinct, except possibly $v_0=v_k$. If $v_0=v_k$ the path is a cycle. A graph is connected if it has a path from each vertex to every other, A tree is a connected graph for which $e(G)=n-1$; equivalently, a connected graph with no cycles. A spanning tree of a graph is a subgraph which is a tree on all n vertices, A caterpillar is a tree with a path that covers (contains one vertex of) every edge. [Alternatively, it is a tree not containing Y as a subgraph, where Y is obtained from the complete bipartite $K_{1,3}$ by subdividing each edge with a new vertex.] Caterpillars have also been called "hairy paths."

For a graph G whose edges are linearly ordered, we adopt the following notation. We put $(x,y) < (u,v)$ if (x,y) is less than (u,v) in that ordering. Similarly for othernotations of order. F(G) denotes the set of first edges of G. A first edge is the least edge incident to some vertex. Similarly L(G) denotes the set of last edges of G, any of which is the greatest edge incident to some vertex. Let M(G) be the graph obtained from G by deleting the edges of F(G) and L(G), and let C(x) be the connected component of M(G) containing x.

For any vertex x, let f(x) be its first neighbor, namely the vertex adjacent to it via the least incident edge. Similarly, h(x) denotes its last neighbor, adjacent via the greatest incident edge. We use $x \to y$ to replace the words "an increasing path from x to y," meaning a path from x to y where each **successive** edge **is** greater than the prsvioua one,

Henceforth, whenever we refer to a graph, we assume its edges are associated with a linear ordering, If for every **x**, there is not **x→x**, we say "no one hears his own information," or the graph <u>satisfies NOHO</u>.

REMARK (1). A graph satisfying NOHO has no loops, repeated edges, or triangles.

Proof: The first two are immediate. If there is a triangle, the edges obey some order, and the vertex at the intersection of the least and greatest edges violates NOHO. ▯

Expanding on this argument, we obtain

LEMMA (2). In a graph satisfying NOHO the first edges and the last edges each form a disjoint matching.

Proof: Suppose F(G) is not a matching, so there exists y=f(x), z=f(y), with $z \neq x$. Then $(y,z)<(x,y)$. Since y=f(x), (x,y) is no greater than the least edge in **x→z**. If **they** are equal, replacing (x,y) by (z,y) at the beginning of the path creates **z→z**. If they **are** not equal,, adding (z,y) and (y,x) at the beginning of **x→z** again produces **z→z**. So, NOHO requires x=f(y), and F(G) is a matching.

Similarly for L(G). If y=h(x), z=h(y), and $z \neq x$, we require $(y,z)>(x,y)$ and (x,y) no less than the greatest edge in **z→x**. This time the end of **z→x** can be adjusted to produce **z→z**. ▯

COROLLARY (3). Graphs satisfying NOHO exist only .on even numbers of vertices,

Proof: Complete matchings exist, ▯

If $x \rightarrow y$ exists for all $x \neq y$, then we say the graph "solves the gossip problem," From previous results [1,10,12], we know such a graph on n vertices has at least 2n-4 edges, If a graph on n vertices solves the gossip problem, satisfies NOHO, and has the fewest edges among all such graphs, we call it a NOHO-graph,

LEMMA (4). NOHO-graphs have 2n-4 edges, for $n \geq 4$, n even.

Proof: A NOHO-graph solves the gossipproblem, so requires at least 2n-4 edges, We exhibit such a graph with that many edges. Let D_n be a graph on vertices $\{x_j^i: i=1,2; j=0,1,...,n/2-1\}$. We write $x_{n/2}^1 = x_0^2$, $x_{n/2}^2 = x_0^1$. Let $F(D_n) = \{(x_i^2, x_{n/2+1-i}^1): i=1,2,...,n/2\}$ and $L(D_n) = \{(x_i^1, x_{n/2-1-i}^2): i=0,1,...,n/2-1\}$. The intermediate edges of D_n are $\{(x_j^i, x_{j+1}^i): i=1,2; j=1,...,n/2-2\}$, ordered by $(x_{j-1}^i, x_j^i) < (x_j^i, x_{j+1}^i)$. Any linear ordering compatible with this partial ordering is acceptable, Easy inspection shows that D_n solves the gossip problem and satifies NOHO, and it has 2n-4 edges. ▯

Figure 1 illustrates D_{14}. Whenever we draw a NOHO-graph, first edges will be dotted and last edges dashed.

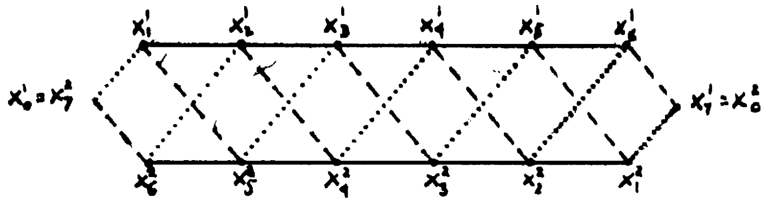

Figure 1, D_{14}, a NOHO-graph

COROLLARY (5). For a NOHO-graph G, M(G) has at least four components,

Proof: Recall M(G)=G-(F(G)∪L(G)). By (2), e(F(G))=e(L(G))=n/2, and they share no edges (1). So, (4) implies e(M(G))=n-4. With n vertices, this means it must have at least 4 components. ▯

A graph solving the gossip problem is connected, so the following conceptsare meaningful. For any vertex x, let O(x) be the "spanning tree of useful edges transmitting information from x," or simply the <u>out-tree</u> from x. It can be defined uniquely and recursively as follows, Begin with x. At each step add the least edge incident to but not contained in the tree that i) does not create a cycle and ii) becomes the greatest -edge of an increasing path from x along the tree, After n-l steps the result is O(x). Thr tree must exist, since x→y exists for all y≠x. Similarly, I(x) denotes the <u>in-tree</u> to x. It is defined recursively and uniquely like O(x) by adding at each step the greatest non-cyclic edge which is the least edge of an increasing path to x along the tree. Again, I(x) exists, since y→x exists for all y≠x. Let c(x) be the number of edges <u>useless</u> to x. Deleting them leaves increasing paths for x to and from every other vertex. We have $c(x) = e(G) - e(O(x) \cup I(x))$. Now we can characterize the edges lying both in O(x) and in I(x).

LEMMA (6). If G solves the gossip problem and staisfies NOHO, then $(y,z) \in (O(x) \cap I(x))$ if and only if (y,z) is incident to x,

Proof: Suppose $(y,z) \in (O(x) \cap I(x))$. Then (y,z) is the greatest edge of some increasing path starting from x and the least edge of some increasing path ending at x, Joining the two paths and dropping (y,z) if they connect to it at the same endpoint, we have $x \rightarrow x$, unless (y,z) was the only edge in both paths, in which case it is incident to x.

Conversely, suppose $(x,y) \notin O(x)$. Then there exists $x \rightarrow y$ in O(x) disjoint from (x,y). To avoid having $x \rightarrow x$, (x,y) must be less than the greatest edge in that path. But then, according to the construction for O(x), at the time when that edge was added (x,y) was also available, and we would have chosen it instead, Similarly, we cannot have $(x,y) \notin I(x)$ unless we have $x \rightarrow x$. ▯

COROLLARY (7). In a NOHO-graph, $c(x)=d(x)-2$ for any vertex x,

Proof: $c(x) = 2n-4-e(O(x) \cup I(x)) = 2n-4-(n-1)-(n-1)+e(O(x) \cap I(x))$ $= d(x)-2$, since by (6) $e(O(x) \cap (I(x)) = d(x)$. ▯

Vertices in a NOHO-graph always have degree at least 2, so $c(x)=d(x)-2$ makes sense,

The next lemma investigates how the edges of $O(x)$ and $I(x)$

are distributed. Recall that C(x) is the component of M(G) containing x. We claim that edges of M(G) not in C(x) or C(f(x)) are useless for carrying information out of x, and those not in C(x) or C(h(x)) are useless for bringing it in. In other words,

LEMMA (8). If G solves the gossip problem and satisfies NOHO, then for any vertex x, $(M(G) \cap O(x)) \subset (C(x) \cup C(f(x))$ and $(M(G) \cap I(x)) \subset (C(x) \cup C(h(x))$, so $e(M(G)) - e(C(x) \cup C(f(x)) \cup C(h(x))) \leq c(x)$.

Proof: First consider O(x), No edge of M(G) not in C(x) or C(f(x)) can belong to an increasing path beginning at x. The path would have to enter that component via a first edge or a last edge. No first edge other than (x,f(x)) exists on any increasing path from x, and any path which uses a last edge cannot continue increasing thereafter. Applying similar reasoning to I(x), no edge of M(G) not in C(x) or C(h(x)) can belong to an increasing path leading to x. Therefore, the number of edges of M(G) not in $C(x) \cup C(f(x)) \cup C(h(x))$, all of which are useless to x, is at most c(x). ▯

The "excess edges" counted in (8) can be fewer than c(x) if one of the components of M(G) is not a tree or if some edge in F(G) or L(G) is useless to x. As we see next, the former cannot occur in a NOHO-graph.

LEMMA (9). For a NOHO-graph G, M(G) consists of exactly four components, all of which are trees.

Proof: By (6), M(G) has at least four components. In showing it has at most four and they are trees, we consider two cases,

Case I, Every vertex of G has degree at least 3. This means M(G) has no isolated vertices, and each component has at least one edge. G must have at least 8 vertices of degree exactly 3, else the sum of all degrees will exceed 4n-8, which is twice the number of edges, By (7), a vertex x of degree 3 has c(x)=1. By (8), M(G) has at most one edge not in $C(x) \cup C(f(x)) \cup C(h(x))$, so there can be at most one other component. If any component were not a tree it would have at least as many edges as vertices. Then the remaining three components would have together at least four more vertices than edges. As before such a situation requires at least four components,

Case II. G has some vertex x of degree 2. C(x) is an isolated vertex in M(G). By (7), c(x)=0. Since $M(G) \cap O(x)$ and $M(G) \cap I(x)$ can have no cycles, (8) then implies C(f(x)) and C(h(x)) are trees and all other components are isolated vertices Two trees have two more vertices than edges. Since M(G) has n-4 edges, the two components have n-2 vertices, leaving x and one other isolated vertex for a total of four components, ▯

REMARK (10). For any x in a NOHO-graph G, M(G) contains at least n/2-2 edges of O(x) and of I(x).

Proof: At most one edge of O(x) lies in F(G) and at most n/2 in L(G), while I(x) has at most one edge in L(G) and n/2 in F(G), ▯

The remaining lemma in this section becomes useful when we show later that for a NOHO-graph every tree in M(G) is of the type in its hypothesis, This lemma applies to all graphs, because if G does not solve the gossip problem we can still define O(x) and I(x) with the same construction, and simply grow the trees as far as possible. They may not span.

LEMMA (11). A tree lying in both O(x) and I(y) for some x and y is a caterpillar with an increasing path touching every edge.

Proof: Let (v_0,v_1) be the least edge in the tree, and let $(v_0,v_1,\ldots,v_k)=V$ be the longest increasing path in the tree. Suppose the assertion is false, and the tree contains an edge (w,z) with neither w nor z in v_i. Since the tree is connected, there must be some path that joins V to this edge, say $U=(v_j,u_1,u_2,\ldots,u_r,w,z)$. Each edge is in O(x) and must lie on an increasing path from x. Consider (v_j,u_1). If the increasing path containing it does not include (v_{j-1},v_j), there would be two increasing paths to v j, impossible in O(x). If it does, then (v_{j-1},v_j) (v_j,u_1).

Applying this argument to each successive edge of U, we find that $(v_0,v_1,\ldots,v_j,u_1,\ldots,u_r,w,z)$ is an increasing path. Similarly, each edge is in I(y), and must lie on an increasing path to y. V is part of such a path, Since I(y) is a tree, an argument like that above yields (u_1,v_j) (v_j,v_{j+1}) . Applying the argument to each successive edge of U, we find that $(z,w,u_r,\ldots,u_1,v_j,\ldots,vk)$ is also an increasing path. This can happen only if (w,z) is the only edge in U, So, every edge of the tree is incident to a single increasing path, If it is not on the path, it occurs between the neighboring edges of the path in the edge ordering, ▯

4. Q*, the "Generalized Cube"

The remainder of the characterization of NOHO-graphs varies greatly depending on whether the graph has a vertex of degree 2. In this section we consider the case where it does not.

Let Q* be the set of 8-vertex j-regular graphs with no triangles, Q* contains the cube. We have

THEOREM (12). A NOHO-graph with no vertex of degree two may be any graph in Q*, but no other.

Proof: By (9), M(G) consists of four non-trivial trees. Thus $n \geq 8$. If $n=8$, then M(G) consits of four single edges. So G admits a factorization into disjoint matchings F(G), M(G), and L(G), and by (1) it must lie in Q*. We claim any graph in Q* can be suitably edge-ordered.

Suppose $G \in Q^*$. We will assign first neigbors, last neighbors, and "middle neighbors" (denoted g(x)) to satisfy all the required conditions, Consider the passage of information out from x. It can reach f(x),g(x),h(x),g(f(x)),h(f(x)),h(g(x)), and h(g(f(x))). To reach all vertices, these must all be distinct, (This implies there is no duplication of transmission in these solutions. See (40).) So, we find a spanning tree with two neighboring vertices of' degree 3, each of whose other neighbors have degrees 2 and 1. For a graph in Q*, this is always possible, since it has no triangles. Place the central edge in F(G), the end edges in L(G), and the remainder in M(G). Information can come to x from h(x), g(x), f(x), g(h(x)), f(h(x)), f(g(x)), and f(g(h(x))) along a similar tree. Five edges remain unassigned in G. This tree will use four of them, adding three edges to F(G) and one to M(G), Again, for a graph in Q* it is possible to- find the additional tree, The remaining edge is assigned to M(G),

In choosing and labeling this second tree we must take care to preserve the matching property of F, L, and M and to avoid completing a circuit with two edges of M and one each of F and L, Such a circuit would result in duplicated transmission between two other vertices. Having labeled these trees to satisfy vertex x and these latter conditions, detailed checking shows that all other information is also transmitted and NOHO is satisfed.

Suppose $n>8$ and G is a NOHO-graph. We will produce a contradiction. Let x be an end-vertex of one of the trees in M(G), $d(x)=3$, so $c(x)=1$ (7). (8) shows that at least one of the remaining components is entirely useless to x and must be a single edge. Applying the same argument to an endpoint of that edge, we obtain a second isolated edge in M(G),

Let (x_1^1,x_2^1) and (x_1^2,x_2^2) be such single edges. By (10), $C(f(x_j^i))$ contains increasing paths from $f(x_j^i)$ to at least $n/2-3$

other vertices, and $C(h(x^i_j))$ contains increasing paths to $h(x^i_j))$ from at least $n/2-3$ other vertices, Since $c(x^i_j)=1$, $f(x^i_j)$ and $h(x^i_j)$ must lie in different components, each of which contains half the remaining vertices. When $n>8$ these components contain more than two vertices, and all their edges must be useful to x^i_j. In particular, $C(f(x^i_j))\subset O(x^i_j)$ and $C(h(x^i_j))\subset I(x^i_j)$.

Suppose $f(x^i_j)$ and $f(x^{i'}_{j'})$ lie in the same component of M(G). That component is a tree of increasing paths out of each of those vertices, so they must be joined by the least edge in that component. Therefore, it is not possible for three such vertices to lie in the same component, Similarly, no three of $\{h(x^i_j)\}$ lie in the same component. Each of the "large" components contains two each from $\{f(x^i_j)\}$ and $\{h(x^i_j)\}$, so by (11) they must both be caterpillars,

Let (v,w) be the least edge in one of the caterpillars, so $v=f(x^i_j)$, $w=f(x^{i'}_{j'})$. Let $y=h(x^i_j)$, $z=h(x^{i'}_{j'})$. y and z lie in the other caterpillar, For v and w both to be "roots" of the caterpillar, one of them must be an endpoint, say v. Now $d(v)=3$, $c(v)=1$. $f(v)=x^i_j$ lies in a single-edge component; the other such component must be the edge useless to v. Therefore, the other caterpillar must be a tree of increasing paths into $h(v)$. -However,, it already does that for y and z, also. y, z, and h(v) are distinct, since their last neighbors are distinct, but we saw in the last paragraph that three distinct vertices could not all play this role, This gives us the final contradiction that eliminates the possibility $n>8$. ▯

Figure 2 gives several examples of NOHO-graphs in Q*, including the cube. The usual conventions are observed for drawing edges in F, M, and L.

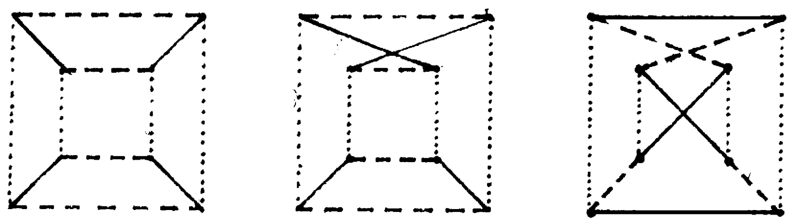

Figure 2, Some graphs in Q*

5. NOHO-graphs as Quadruples of Sequences

We now embark on a journey to narrow down and finally characterize NOHO-graphs having a vertex of degree 2, Henceforth when we refer to NOHO-graphs we generally ignore Q*. We already know by (9) that the "middle edges" of such a graph form four componenents, at least two of which are isolated vertices. Proceeding from there, this section describes the edges of a NOHO-graph with four integer sequences, The first edges and last edges are described by permutations, and the middle edges by two binary sequences.

We begin by taking a closer look at the components of M(G).

LEMMA (13). If a NOHO-graph with a vertex of degree two has adjacent vertices of degree two, then it is a 4-cycle. If $n>4$, then it has exactly two non-adjacent vertices of degree two, and the remaining components of M(G) are caterpillars on $n/2-1$ vertices,

Proof: Suppose G has adjacent vertices $\{x,y\}$ of degree 2. (x,y) may lie in $F(G)$ or in $L(G)$. Suppose $(x,y)\in F(G)$ and consider $O(x)$. $O(x)$ contains $(x,h(x))$, (x,y), and $(y,h(y))$, but after hitting these edges in $L(G)$ there can be no further increasing paths in $O(x)$. $h(x)\neq h(y)$ by (1) or (2), so G contains exactly 4 vertices and must have an edge in $F(G)$ joining $h(x)$ and $h(y)$. If $(x,y)\in L(G)$, then considering $I(x)$ leads to the same conclusion.

Now suppose $n>4$. 'By (9) there are two vertices of degree **two,** and the remaining two components may be two trees or a tree and isolated vertex, Suppose the latter, so we have $\{x_1,x_2,x_3\}$ isolated in $M(G)$. By the above they must be non-adjacent in G. Consider the increasing paths by which information is exchanged among them, Let z_i be the last vertex before $x_i \to x_j$ and $x_i \to x_k$ permanently diverge edgewise, That is, we have increasing paths $(x_i,\ldots,y_i,z_i,u_{ij},\ldots,x_j)$ and $(x_i,\ldots,y_i,z_i,u_{ik},\ldots,x_k)$, where $u_{ij}\neq u_{ik}$. z_i is different from x_i, since all increasing paths from x_i to non-adjacent vertices must pass through $f(x_i)$. So, the edge $(y_i,z_i)\in O(x_i)$ is well-defined. **Simmilarly,** let vi be the first vertex where $x_j \to x_i$ and $x_k \to x_i$ share an edge. We have increasing paths $(x_j,\ldots,t_{ji}, v_i, w_i,\ldots,x_i)$ and $(x_k,\ldots,t_{ki},v_i,w_i,\ldots,x_i)$. Again, v_i is different from x_i since all paths from non-adjacent vertices pass through h(Xi) when $d(x_i)=2$, SO the edge $(v_i,w_i)\in I(x_i)$ is well-defined,

In fact, the paths from x_i to x_j are all unique, so that

z_i and v_j lie on a single increasing path from x_i to v_j. suppose there are two increasing paths from vertex r to vertex s, where $d(r)=2$. Since O(r) is a tree containing the edges incident to r (6), some other edge in the paths is useless or lies in I(r). The former is forbidden by (7) since $c(x)=0$, while the latter creates $r \rightarrow r$. The same conclusion follows from considering I(s) if $d(s)=2$.

Now, consider the ordering of zi and y_j on $x_i \rightarrow x_j$. We have three cases:' Case I, v_j strictly precedes z_i on the path, i.e. $(v_j,w_j) \leq (y_i,z_i)$. Then for the remaining vertex v_k there exists $v_k \rightarrow v_k$ via $(x_k,\ldots,t_{kj},v_j,\ldots,z_i,u_{ik},\ldots,x_k)$.

Case II. z_i strictly precedes y_j on the path, i.e. $(z_i,u_{ij}) \leq (t_{ij},v_j)$. If $(z_i,u_{ij}) \in I(x_k)$, then (z_i,u_{ij}) lies on $x_i \rightarrow x_k$ and z_i was not the furthest shared vertex from x_i, or $I(x_k)$ is not a tree. If $(z_i,u_{ij}) \in O(x_k)$, then (z_i,u_{ij}) lies on $x_k \rightarrow x_j$ and v_j was not the first shared vertex on the way to x_j, or $O(x_k)$ is not a tree. But (z_i,u_{ij}) Cannot be useless to xk since $c(x_k)=0$.

Case III. Neither of these possibilities can occur for any pair (i,j), so we must have $v_1=z_1=v_2=z_2=v_3=z_3$. To avoid $x_i \rightarrow x_i$ we must have $(v_i,w_i)<(y_i,z_i)$ for all i, but to maintain the other paths we need $(y_i,z_i)<(v_j,w_j)$ for $i \neq j$. But $(v_i,w_i) <(y_i,z_i)<(v_j,w_j)<(y_j,z_j)<(v_i,w_i)$ is impossible.

So, there must be exactly two isolated vertices x_1 and x_2 in M(G), and the two remaining components are non-trivial trees. $f(x_i)$ and $h(x_i)$ appear in different components, since $c(x_i)=0$.

By (10) each of these components contains exactly $n/2-2$ or half of the edges in M(G), and $C(f(x_i))\subset O(x_i)$, $C(h(x_i))\subset I(x_i)$. In order to have $x_i\rightarrow x_j$, $f(x_i)$ and $h(x_j)$ must appear in the same component, Now we can apply (11) and conclude that the two non-trivial components of M(G) are caterpillars on n/2-1 vertices each. $\square$

To facilitate the subsequent discussion, we introduce some additional notation, Henceforth fix $m=n/2-1$. Label the vertices of G $\{x^i_j:\ i=1,2;\ j=0,1,\dots,m\}$. Let x^i_0 be the vertices of degree 2, and $x^i_1=f(x^i_0)$. Let C^i be the caterpillars of M(G). The vertices of C^i get the labels x^i_j, where $j=1,2,\dots,m$ and x^i_j is the j^{th} to receive the information originating from x^i_0. We may refer to x^j_0 as x^i_{m+1}.

Since C^i is a caterpillar of increasing paths from x^i_0 to x^i_m, the following properties are obvious.

REMARK (14). Let C^i be defined as above. Then

i) C^i contains $x^i_j\rightarrow x^i_k$ whenever $j<k$.

ii) x^i_k neighbors exactly one x^i_j with $j<k$.

iii) If x^i_k neighbors any x^i_r with $r>k$, it neighbors every x^i_j with $k<j\le r$.

iv) $x^i_k\rightarrow x^i_j$ within C^i with $j<k$ requires $(x^i_k,x^i_j)\in E(G)$.

Suppose we have a caterpillar C with a fixed initial and final vertex, and an ordering of edges to make it a tree of increasing paths both out of the former and into the latter.

We claim C can be uniquely described by a forward sequence R(C) or a backward sequence R'(C) of zeroes and ones. The length of these sequences is one less than the number of edges in C, We will not use the backward sequence, We merely note it exists, arises from considering the edges in reverse order, and refers to a different ordering of the vertices,

To obtain R(C), proceed as follows, Begin with the least edge and a null sequence for R(C), Call the initial vertex the "active" vertex (x_1^i in the caterpillar Ci) and its neighbor the "current" vertex, When the next smallest edge is added to the caterpillar, adding also a vertex, the new vertex becomes the current vertex. The label "active" stays where it is if the new edge is incident to it. If the new edge is incident to the former current vertex, then that vertex becomes the active vertex. In the former case, append a 0 to R(C) as generated so far, In the latter case append a 1.

As each edge is added to the tree in order, it can only be incident to the active vertex or the current vertex, This follows-because the caterpillar must remain a tree of increasing paths toward the final vertex, At any stage the tree is one of increasing paths toward both the active and current vertices.

All 2^r binary sequences of length r describe caterpillars in this way and correspond one-to-one with caterpilars on r+1 edges and r+2 vertices, where the initial vertex and order of edges is specified, The initial vertex must be specified to distinguish between sequences that differ only in the first place.

If we add the edge $(h(x_0^j),x_0^j)$ to C^i, we still have a caterpillar, since this is a last edge. It has paths from x_1^i and to x_0^j. This is the caterpillar of interest. Note that $h(x_0^j)$ need not be x_m^i. Let S(G) be the associated sequence $R(C^1 \cup (h(x_0^2),x_0^2))$, and let T(G) be the associated sequence $R(C^2 \cup (h(x_0^1),x_0^1))$, but written backwards. When we discuss irreducibility and concatenation in section 8 it will become clear why T(G) is written backwards,

From S and T we can reconstruct M(G) and know the first and last neighbors of x_0^i. To complete the characterization of G we need to know which pairs of sequences (S,T) can be associated with a NOHO-graph and how the edges of F(G) and L(G) can be placed to complete the graph,

No vertex in C^i can have a first or last neighbor in C^i. By (14.i), having such an edge in F(G) or L(G) would violate NOHO. So, the edges in F(G) and L(G) can be described by permutations P(G) and Q(G), where $P_i=j$ means $f(x_i^1)=x_j^2$, and $Q_i=j$ means $h(x_i^1)=x_j^2$. (Whenever R is a sequence of integers, we denote its i^{th} element by R_i.)

S and T have m-l elements; P and Q as described have m elements. P is a permutation of $\{2,3,...,m+1\}$ which begins with m+1, since $x_0^i=x_{m+1}^j=f(x_1^i)$. Q is a permutation of $\{0,1,...,m\}$ with some element deleted. The deleted element is j, where $h(x_0^1)=x_j^2$. Note that 0 is never deleted. We will see that 0 appears in Q at the same position as 2 in P, so that P and Q could be compressed to m-l pieces of information. However, bookkeeping and proofs will be easier if we leave them as is, To align the useful information properly, we say that the elements of S and T as generated above appear in positions 2 through m. S_i indicates what happens when C^1 reaches x^1_{i+1}, and T_i indicates what happens when C^2 grows to reach x^2_{m-i+3}.

We can summarize the construction of these sequences and the properties required of them in the last few pages by the following remark;

REMARK (15). The quadruple (P,Q,S,T) defined above completely specifies a graph. Such a graph has the properties ascribed to NOHO-graphs in (2) through (14).

If (P,Q,S,T) = (P(G),Q(G),S(G),T(G)) for some NOHO-graph G, we call the quadruple realizable, We have not yet determined what is required of (P,Q,S,T) to transmit all information and to satisfy NOHO. For example, although any S or T except the zero sequence can appear in realizable quadruples, it is not true that every permutation P or Q defined above appears in a realizable quadruple, nor is it true that every pair (S,T) is realizable. In the next section we determine necessary conditions for realizability.

6. Necessary Conditions for Realizability

We will derive a number of necessary conditions for pairs from (P,Q,S,T) to be realizable.

LEMMA (16). For a NOHO-graph G, the pair (P(G),Q(G)) satisfies

i) $P_i > Q_i$ for all $i=1,2,\ldots,m$.

ii) If $P_i = Q_j$, then $i>j$.

iii) P_2 is the element missing from Q, and $Q_j=0$ iff $P_j=2$. Equivalently, $f(x_2^i)=h(x_0^i)$.

Proof: Consider (i). $P_1=m+1$, which is greater than any element of Q. For some k from 1 to m, $Q_k=0$, which is less than any element of P. For $i\neq 1$, $i\neq k$, $f(x_i^1)$ and $h(x_i^1)$ lie in C^2. If $P_i<Q_i$, (14.i) guarantees $x_{P_i}^2 \rightarrow x_{Q_i}^2$ in C^2. Now we can add $(x_i^1, x_{P_i}^2)$ to the beginning and $(x_{Q_i}^2, x_i^1)$ to the end to obtain $x_i^1 \rightarrow x_i^1$.

For (ii), we argue similarly, If $P_i=k=Q_j$ with $i<j$, then we can add (x_k^2, x_i^1) at the beginning of $x_i^1 \rightarrow x_j^1$ and (x_j^1, x_k^2) at its end to obtain $x_k^2 \rightarrow x_k^2$.

Finally, consider P_2. By (ii), if it appears in Q it must be Q_1. Then $f(x_2^1)=h(x_1^1)$. The caterpillar C^i always contains the edge (x_1^i, x_2^i), so we have a triangle. Similarly, if $P_k=2$ but $Q_k\neq 0$, (i) says $Q_k=1$. Now $f(x_2^2)=h(x_1^2)$, and again we *have* a triangle. ▯

If P or Q is not strictly decreasing, certain edges must appear in the graph, LEMMA (17). For a NOHO-graph G, P(G) and Q(G) satisfy

i) If $P_i<P_j$ with $i<j$, then E(G) contains $\{(x_i^1, x_j^1), (x_{P_i}^2, x_{P_j}^2)\}$.

ii) If $Q_i<Q_j$ with $i<j$, then E(G) contains at least one of $\{(x_i^1, x_j^1), (x_{Q_i}^2, x_{Q_j}^2)\}$.

Proof: Consider any increasing pair in P. Suppose $P_i=r$ and $P_j=s$, where $i<j$ and $r<s$. If (x_i^1, x_j^1) is not an edge, then (14.iv) implies information from x_j^1 could reach x_i^1 only via the other caterpillar. So, we use $(x_j^1, x_s^2)\in F(G)$, continue to x_t^2 in C^2 where $t\geq s$ or (x_s^2, x_t^2) is an edge, and finish with $(x_t^2, x_i^1)\in L(G)$. $t>r$ would imply $Q_i>P_i$, violating (16.i). therefore (x_t^2, x_s^2) must be an edge, with $t<r<s$. By (14.iii), (x_t^2, x_r^2) is also an edge, but this creates a triangle with x_i^1.

Now suppose (x_r^2, x_s^2) is not an edge, By a similar chain of reasoning that switches the roles of C^1 and C^2, completing $x_s^2\rightarrow x_r^2$ will contradict (16.ii) or (1).

Finally, suppose $Q_i<Q_j$ with $i<j$, but (x_i^1, x_j^1) is not an edge. We use (14.iv) again to require $x_{P_j}^2 \rightarrow x_{Q_i}^2$ in C^2 for $x_j^1\rightarrow x_i^1$. By (16.i) $Q_i<Q_j<P_j$, so (14.iv) requires $(x_{P_j}^2, x_{Q_i}^2)$ as an edge to complete that path, Now (14.iii) says $(x_{Q_i}^2, x_{Q_j}^2)$ must also be an edge, $\square$

We define a <u>reversion</u> in a permutation to be a maximal consecutive subsequence of the per-mutation where the first element is the least. The reversions of a permutation partition it into segments, In a NOHO-graph, the reversions of P(G) have a very special form,

LEMMA (18). If G is a NOHO-graph, then P(G) has the following form,

i) Every reversion of P is a single element or has the form (r,s,s-1,...,r+1) with s-r+1 elements,

ii) Equivalently, P has no subsequence of length 3 whose last element is largest.

Proof: First we show equivalence, By definition, the first elements of reversions form a decreasing subsequence, else the reversions would not be maximal. If reversions are as in (i), any increasing subsequence must lie entirely within a single reversion, The form described in (i) prohibit6 two increasing pairs with the same second element.

Conversely, assume (ii), Suppose a reversion has more than one element and we drop the first element r, This must leave a decreasing subsequence beginning with s, since any increasing pair would violate (ii) with r. Suppose there is some element t, $r<t<s$, that does not appear in this reversion. Its appearance before r violates (ii) with r and s, and its appearance in a later reversion violates (ii) with r and the first element of that reversion.

That (ii) holds for realizable P follows immediately from (17.i), (14.iii), and (1). They provide a contradiction if some such subsequence is assumed to exist.

REMARK (19). A permutation P satisfying (18) is uniquely determined by choosing a subset of Indices from $\{3,\ldots,m\}$ at which reversions will begin in P, in addition to the reversions beginning at P_1 and P_2. Hence, there are 2^{m-2} such permutations.

Note the equivalence of (18.i) and (18.ii) is independent of realizability, We will see that the necessary conditions (16) and (18) together are sufficient. Also, it is easy to see that for any P satisfying (18) there is at least one Q satisfying (16).

Next, we derive a condition for the pair (P,S).

LEMMA (20). If G is a NOHO-graph, then P(G) and S(G) satisfy the following,

i) Suppose P_j begins a reversion in P(G), P_k begins the next reversion, and $k\geq j+2$. Then $S_j=1$, and if $k>j+2$ then $S_{j+1}=\ldots=S_{k-2}=0$.

ii) If $P_t=2$, beginning the last reversion in P(G), then $S_t=1$ and any succeeding elements of S(G) are 0,

Proof. If Pj begins a reversion of length at least two, every succeeding element of the reversion forms an increasing pair with P_j. By (17.i), $\{(x_j^1,x_i^1)\colon i=j+1,\ldots,k-1\}\subset E(G)$. Si indicates what happens when C^1 grows to meet x_{i+1}^1. Considering the edges we have just shown to exist, x_{j+1}^1 is joined to the then-current vertex, and succeeding x_i^1 are joined to the active vertex, So $S_j=1$ and succeeding S_i are 0, if $k>j+2$. S_{k-1} tells what happens when the vertex beginning the next reversion is added to the tree, so it is unrestricted,

Now consider the last reversion in P(G), which begins with $P_t=2$. By (16.iii) $Q_t=0$ and (x_t^1, x_{m+1}^1) is an edge, Applying (14.iii) to the caterpillar $C^1\cup\{(x_t^1,x_0^2)\}$, we deduce that $\{(x_t^1,x_i^1): i=t+1,\dots,m+1\}$ are all edges, since $t\leq m$. As above we conclude $S_t=1$ and any succeeding Si are 0. ▯

REMARK (21). For each P satisfying (18), the number of sequences S satisfying (20) is 2^{r-1}, where r is the number of reversions after P_1.

Proof: An element of S is unrestricted if and only if its position (S_{k-1} in (20)) corresponds to the last element of a reversion in P other than the last reversion. ▯

Define $(P'(G),Q'(G),S'(G),T'(G))$ as follows. Set $P'_i=j$ if $P_j=i$. Extend Q so that $Q_0=k$ where $x_k^2=h(x_0^1)$, then set $Q'_i=j$ if $Q_j=i$. Set $S'_i=T_{m+2-i}$, and set $S'_i=S_{m+2\ i}$. We call (P',Q',S',T') the <u>reflection</u> of (P,Q,S,T). A little "reflection" shows

REMARK (22). The reflection of a realizable quadruple is also realizable, in fact by the same graph.

Proof: Considering (P',Q',S',T') instead of (P,Q,S,T) is equivalent to interchanging the roles of C^1 and C^2 and looks at the graph upside down, ▯

If G is a NOHO-graph, we define the reverse graph K(G) as the graph with the same vertices and edges as G, but with (x,y) (u,v) in K(G) if and only if (x,y) (u,v) in G. All increasing paths of G are increasing in the opposite direction in K(G) and vice versa, so X(G) is clearly a NOHO-graph. Note that the vertices need to be relabeled with C^1 and C^2 to o tain the defining sequences for K(G). The "hairs" of the caterpillar swing around as the wind blows from the other direction.

By reflecting and reversing, we obtain additional necessary conditions.

REMARK (23). If G is a NOHO-graph, then

i) (P(G),T(G)) is such that (P'(G),S'(G)) satisfies (20),(18).

ii) (Q(G),S(G)) is such that (P(K(G)),S(K(G)) satisfies (20),(18).

iii) (Q(G),T(G)) is such that (P'(K(G)),S'(K(G)) satisfies (20),(18).

(16), (20), and (23) are necessary conditions for any pair from {P,Q,S,T} except (S,T) to be realizable. There are appropriate conditions for (S,T), but we have no simple expression for them, We will soon see that when paired with (18) each of these conditions is sufficient,

7. The Number of Realizable Quadruples

Besides showing the sufficiency of the previous conditions, we will show that any pair from {P,Q,S,T} satisfying them is realized by a unique NOHO-graph. To prove this, we need a lemma that will enable us to generate one sequence in {P(G), Q(G), S(G)} when we know the other two, By reflection we can apply it to {P, Q', S'} to obtain similar results for {P, Q, T}.

S(G) is a binary sequence indexed from 2 through m, On its index set we can define a function b that points to the previous 1 in the sequence. Let b(i) be the greatest positive integer such that $j<i$ and $S_j=1$, if such exists. If there is no such integer, set b(i)=1. Then we have

LEMMA (24). For a NOHO-graph G, P(G), Q(G), and S(G) are related by

i) $S_i=1$ if and only if $P_{i+1}=Q_{b(i)}$.

ii) $S_i=0$ if and only if $P_{i+1}=Q_i$.

Proof: In one direction the lemma is trivial, Recall the construction of S from active and current vertices. $S_i=0$ if and only if $(x^1_{b(i)}, x^1_{i+1})$ is an edge, and $S_i=1$ if and only if (x^1_i, x^1_{i+1}) is an edge, So, if $P_{i+1}=Q_i$, then choosing $S_i=1$ creates a triangle, while if $P_{i+1}=Q_{b(i)}$ then $S_i=0$ creates a triangle,

We prove the other direction by induction. For the basis step, b(2)=1, and by (16.ii,iii) we always have $P_3=Q_2$ or $P_3=Q_1$. If $S_2=0$, then choosing $P_3=Q_1$ creates a triangle, while if $S_2=1$ then $P_3=Q_2$ creates a triangle.

Now we prove the lemma for k, assuming it holds for all $2\le i<k$. By (16.ii,iii) we know that $P_{k+1}=Q_j=r$ for some j with

$j \leq k$. Suppose $j=k$. Then if $S_k=0$ we are finished, while if $S_k=1$ we have a triangle, Suppose j-$b(k)$, Now if $S_k=1$ we are finished, and if $S_k=0$ we have a triangle, So, if the lemma fails we may assume $j<k$, $j \neq b(k)$. If $S_j=0$, then by induction we have $P_{i+1}=Q_i=r$, which contradicts P being a permuation, So assume $S_j=1$, in which case $j \leq b(k)$ by the definition of b, We assumed $j \neq b(k)$, so let t be the least integer greater than j such that $S_t=1$. $j=b(t)$, and $t<k$ since $S_{b(k)}=1$, so we have $j<t \leq b(k)<k$. Applying induction, $P_{t+1}=Q_{b(t)}=Q_j=r$, which again contradicts P being a permutation, $\square$

Now we proceed to the main results. Henceforth, fix $p=(n-4)/2=m-1$.

THEOREM (25). Any pair from (P,Q,S,T) which satisfies the corresponding necessary conditions for realizibility in (16), (18), (20), (23) is realized by a unique NOHO-graph.

Proof: First we show how to uniquely generate the remaining sequences from any pair satisfying the necessary conditions, Then we show the resulting quadruple is realizable.

Suppose the two known sequences lie in {P,Q,S}. We generate S from (P,Q) satisfying (16),(18) so as to satisfy (24). Initialize $k=1$. Then for $i=2,3,\ldots,m$ in order, if $P_{i+1}=Q_k$, set $S_i=1$ and reset $k=i$. If $P_{i+1}=Q_i$, set $S_i=0$ and leave k unchanged, This is well-defined for (P,Q) satisfying (16). P_2 disappearing leaves one index "free." As we proceed in P, the only previous elements of Q which have not been encountered in P are Q_k and Q_i.

We claim the resulting (P,S) satisfies (20). It is easy to show the requirement for when Si must be 1 holds. Otherwise, we have $P_{i+1}=Q_i$ when Pi starts a reversion and is less than P_{i+1}, violating (16.i). For the other requirement, consider the first time Si is set to 1 by $P_{i+1}=Q_k$ with i+1 in the midst of a reversion, k is the previous 1, so it is the starting position-of the reversion, Thus $P_{i+1}>P_k$, and we violate (16.i) again,

Next we generate Q from (P,S) satisfying (18),(20) so as to satisfy (24). Set $Q_i=0$ if $P_i=2$. If k is the least integer such that $S_k=1$, set $Q_2=P_{k+1}$. (If S has no ones, $x_0^2=h(f(x_0^1))$. With (16.iii), this contradicts $n>4$.) For all other i, if $S_i=0$ set $Q_i=P_{i+1}$, while if $S_i=1$ set $Q_{b(i)}=P_{i+1}$. Again, this is well-defined, The Qi skipped by the first option are those with $s_i=1$, so that subsequence is just shifted within itself from P to Q. P_2 disappearing makes room for the shift, and 0 under $P_i=2$ fills the hole left at the end, since that's where the last 1 occurs in S.

We claim the resulting (P,Q) satisfies (16). (16.ii,iii) are obvious by construction, so assume some $P_j<Q_j$. The algorithm sets $Q_j=P_i$ for some $i>j$, so by (18) P_j must begin a reversion containing P_i. By (20) $S_j=1$, so Q_j is set the next time a 1 is encountered in S, i.e. at Si_1 with $i-1>j$. (20) then implies Pi must be in a later reversion than P_j.

For the remaining cases, we give less detail. To generate P from (Q,S), set P_1=m+1, and fet P_2 be the element in {1,...,m} missing from Q. For all other i, if S_{i-1}=0 set $P_i=Q_{i-1}$, while if S_{i-1}=1 set $P_i=Q_{b(i-1)}$. This is well-defined for (Q,S) satisfying (23), since the only elements of Q not placed in P at the i^{th} stage are Q_{i-1} and $Q_{b(i-1)}$. The resulting (P,Q) clearly satisfies (16.ii,ii)) and can be shown to satisfy (16.1) and (18). By the construction, they also satisfy (24).

To generate T from (P,Q,S), form (P',Q') and use the first algorithm above to get S'. Then T=(S')'.

To generate the unknown sequences knowing T and one of {P,Q}, reflect them and apply the above algorithms for S and one of {P,Q}. This generates T' and the unknown element of {P',Q'}, and reflecting again gives the desired quadruple.

This leaves the case of generating (P,Q) knowing (S,T). Set P_1=m+1, P_2=j where T_{m+2-j} is the first 1 in T, and P_j=2 where S_j is the last 1 in S. These requirements follow from (16.iii), since those elements of S and T determine $h(x_0^i)$. The remaining elements of P and Q can be uniquely generated by refusing to violate (17), (24), or (1). We omit the details of this algorithm.

By (24), etc., the unknown sequences can only be as generated above. We have shown uniqueness, now we show sufficiency. No matter what pair we started out with, we have shown that for the generated quadruple all the necessary conditions are satisfied, We must show that increasing paths exist between all ordered pairs of vertices and NOHO is satisfied.

As noted in (14.i), $x_j^i \to x_k^i$ with $j<k$ exists, Next we show $x_j^1 \to x_k^2$ exists, If $P_j=s$ with $s\le k$ or $Q_r=k$ with $r\ge j$, we are done by (14.i) again. Suppose both of these possibilities fail and $P_t=k$. If $t<j$, then (P_t,P_j) form an increasing subsequence of P, The condition (20) on (P,S) was determined so that G would satisfy (17.i). So, (x_s^2,x_k^2) is an edge of G, and (s_j^1,x_s^2,x_k^2) is the desired path. Suppose instead $t>j$, and apply (24). Since $r<j<t$, we have $t\neq r+1$ but $P_t=Q_r$, so we must have $r=b(t-1)$ and $S_{t-1}=1$. So, $(x_r^1,x_{t-1}^1)\in E(G)$. By (1&iii), (x_r^1,x_j^1) is also an edge, making (x_j^1,x_r^1,x_k^2) the desired path.

We must also have $x_j^1 \to x_r^1$, even if $r<j$, Let $s=P_j$ and $k=Q_r$. If (x_j^1,x_r^1) or (x_s^2,x_k^2) is an edge or if $s\le k$, then we are done, In considering $x_j^1 \to x_k^2$ above, we showed that if $r<j$ and $s>k$ we must have (x_j^1, x_r^1) or (x_s^2,x_k^2) as an edge.

That paths $x_j^2 \to x_k^1$ and $x_j^2 \to x_r^2$ also exist follows from reflection and the preceding two paragraphs,

As constructed, G trivially satisfies NOHO. $v\to v$ cannot occur using the edges in a single tree, so it must cross to f(v) and return from h(v), Suppose $f(v)=x_j^i$ and $h(v)=x_k^i$. Completing the path requires (x_j^i,x_k^i) to be an edge or $j\le k$, The former never occurs because we've constructed a graph with no triangles, and the latter never occurs because (P,Q) satisfies (16). So, the graph determined by the generated quadruple is a NOHC-graph. ▯

THEOREM (26). The number of realizable quadruples is 3^{p-1}, where $p=(n-4)/2$, n even, $n\geq 6$.

Proof: By (25), pairs (P,S) determine the rest of the quadruple, so we count those. As noted in (21), a realizable P has 2^{r-1} realizable S associated with it satisfying (20), where r is the number of reversions atter P_1. By (19), there are $\binom{m-2}{r-1}$ such realizable P. Using the binomial theorem, the total number of realizable quadruples is $\sum_{r=1}^{m-1}\binom{m-2}{r-1}2^{r-1} = 3^{p-1}$. ▯

Figure 3 exhibits the quadruples and associated graphs for n=6 and n=8.

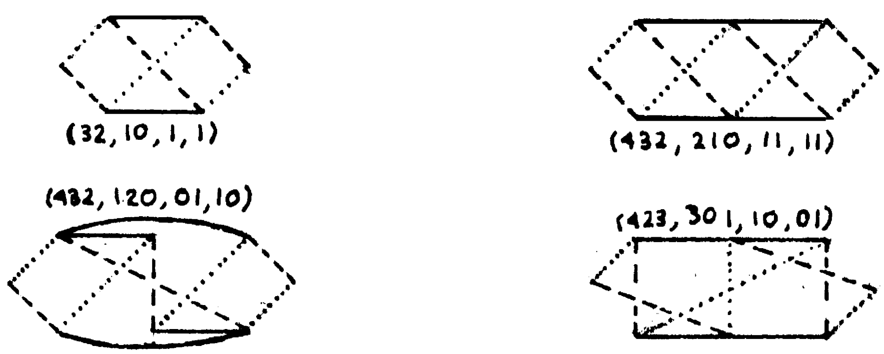

Figure 3. Small NOHO-graphs

G has 180° rotational symmetry when drawn as in Figure 3 if and only if (P',Q',S',T')=(P,Q,S,T). This occurs for all the graphs in Figure 3. If (P',Q',S',T')≠(P,Q,S,T), then G is counted twice when the quadruples are enumerated. In the next section we enumerate the symmetric solutions, so we will know the extent to which NOHO-graphs are overcounted here,

8. Symmetric NOHO-graphs

In this section we count the symmetric NOHO-graphs. We define a symmetric quadruple as a realizable quadruple for which $(P',Q',S',T')=(P,Q,S,T)$. A symmetric NOHO-graph is one where the vertex permutation interchanging x_k^i and x_k^j for all k leaves the graph unchanged, As noted earlier,

REMARK (27). G is a symmetric NOHO-graph if and only if $(P(G),Q(G),S(G),T(G))$ is a symmetric quadruple,

The following remark applies to all P(G), and is useful in determining the number of symmetric ones.

REMARK (28). In a realizable P, $P_i=j$ implies $i+j \geq m+3$.

Proof: By (18.i), the number of positions after i in P must be at least as big as one less than the number of elements less than P_i, so $m-i \geq j-3$. □

LEMMA (29). The number of symmetric realizable P is $2^{\lfloor (m-1)/2 \rfloor}$.
Proof: P symmetric requires $P_j=i$ if $P_i=j$, so P corresponds to a matching of the positions $(2,\ldots,m)$. Some positions maybe matched to themselves, if $P_i=i$. (In fact, this can only happen twice.) Note we always have $P_1=m+1$ and $P_{m+1}=1$. We construct P match by match from m down to $\lceil (m+3)/2 \rceil$, matching P_j on step m-j.

At each step there are two choices. By $(18.i), P_m \in \{2,3\}$

and at step j $P_{m-j} \in \{2,3,\ldots,m-j+3\}$. However, j of these have already been matched with higher positions on previous steps, This leaves two choices for P_{m-j}, one of which is $m-j+3$, since it was not available before, Upon reaching $P_{\lceil (m+3)/2 \rceil}$, the choices are $\lfloor (m+3)/2 \rfloor$ and one lower value. If m is odd, we choose between matching them to each other or to themselves. If m is even, set $P_{\lceil (m+3)/2 \rceil}$ equal to one of them and match the remaining one to itself, Now we have made $m-\lceil (m+3)/2 \rceil+1 = \lfloor (m-1)/2 \rfloor$ choices and completed the matching, Every P so coustructed satisfies (18), and these are all the symmetric P which do so. By (21),(25), they are all realizable. []

Examining the construction in the proof above, we can define a binary sequence B(P), indexed from $\lceil (m+3)/2 \rceil$ to m, where $B_j=0$ if $P_j=m-j+3$ and $B_j=1$ if $P_j<m-j+3$. Now we can count the graphs associated with each P.

LEMMA (30). Suppose P is realizable by a symmetric NOHO-graph. Then the number of symmetric NOHO-graphs realizing P is 2^q, where q is the number of ones in B(P).

Proof: We consider how many ways symmetric Q can be constructed so that (P,Q) satisfies (16). We claim that each way determines a unique symmetric quadruple, By (25) it determines a unique realizable quadruple. Using the algorithms in (25) we generate S and T. Reflecting and applying the algorithms again, we find S'=S and T'=T, since P and Q are symmetric. So by (27), the NOHO-graph realizing the quadruple is symmetric,

First suppose $B(P)=(0,\ldots,0)$. Then $P=(m+1,2,m,m-1,\ldots,4,3)$. There is one reversion after P_1, so (21) and (25) imply there is one realizable quadruple with this P. The corresponding Q is $(m,0,m-1,\ldots,4,3,1)$, which is symmetric as desired,

Now suppose $B(P)\neq(0,\ldots,0)$. By the way B(P) is constructed, $B_k=1$ implies. Pk begins a reversion in P, The uppermost 1 occurs when $P_k=2$, beginning the last reversion. That postpones picking m-k+3 until the next lower 1 in B, at which point it must begin a reversion, and so on.

Recalling (20), the elements of S are unrestricted if and only if they correspond to the last element of a reversion other than the last one, So, covering the index range $\lceil(m+1)/2\rceil$ to m, there are 2^q ways to write down this portion of a realizable (P,S). Using the algorithm in (25), we can write down what the corresponding segment of Q must be.

Determine the rest of Q by setting $Q_j=k$ if $Q_k=j$, where $k\geq(m+1)/2$. That this is well-defined is ensured by (28). Q is now symmetric and completely defined, We need only verify that (P,Q) satisfies (16).

For (16.iii), we have guaranteed $Q_k=0$ placed where $P_k=2$, since $B(P)\#(0,\ldots,0)$ and the last reversion begins in the "good" segment. By symmetry $P_2=k$ and k is the element missing from Q. (16.i,ii) hold for all elements of Q at (m+1)/2 or later. suppose $Q_i=P_j=k$ with $j<i<(m+1)/2$. Then by symmetry and (28), $P_k<Q_k$ with $k>(m+1)/2$, violating (16.i). Finally, suppose $P_j<Q_j$

with $j<(m+1)/2$. Applying symmetry and (28) again, we violate (16.ii) in the good segment.

To summarize, we have shown that there are 2^q symmetric Q that might be paired with P, and that all such pairs are realizable and determine symmetric quadruples. ▯

THEOREM (31). The number of symmetric NOHO-graphs is $3^{\lfloor p/2 \rfloor}$.

Proof: If B(P) has q ones, they may occur at any of the $\lfloor p/2 \rfloor$ steps in constructing P. So (29), (30), and the binomial theorem yield $\sum_{q=0}^{\lfloor p/2 \rfloor} \binom{\lfloor p/2 \rfloor}{q} 2^q = 3^{\lfloor p/2 \rfloor}$ as the number of symmetric solutions. ▯

Symmetric quadruples are one-to-one with symmetric NOHO-graphs. Other realizable quadruples are two-to-one with other NOHO-graphs. So we have from (26), (27), (31)

COROLLARY (32). The number of NOHO-graphs on $n \geq 6$ vertices, n even (other than Q* when n=8) is $(3^{p-1}+3^{\lfloor p/2 \rfloor})/2$.

9. Concatenation of NOHO-graphs

Before defining the concept of an irreducible NOHO-graph, we need to define a way of combining NOHO-graphs. Suppose we have two NOHO-graphs G_1 and G_2 on n_1 and n_2 vertices $\{x_j^i\}$ and $\{y_j^i\}$, with associated quadruples (P^1,Q^1,S^1,T^1) and (P^2,Q^2,S^2,T^2). We define the concatenation of G_1 and G_2, denoted G_1+G_2, as a new graph G_3 constructed as follows.

To obtain the edge set of G_3, unite those of Cl and G_2, deleting the edges incident to x_0^2 and y_0^1. The vertex set of G_3 is the union of the vertex sets of G_1 and G_2, with x_0^2 and y_0^1 deleted, Furthermore, identify $h(x_0^2)$ with y_1^1 and $h(y_0^1)$ with x_1^2. Now G_3 is a graph on $n_3 = n_1 + n_2 - 4$ vertices, with $2n_1 - 4 + 2n_2 - 4 - 4 = 2n_3 - 4$ edges.

For the ordering of edges, any edge that was a first edge or last edge in G_1 or G_2 remains a first edge or last edge, The order between two edges from the same Gi is preserved, In addition, every edge from $C^1(G_1)$ is set less than every edge from $C^1(G_2)$, and every edge from $C^2(G_2)$ is set less than every edge from $C^2(G_1)$.

Figure 4 gives an example of concatenation,

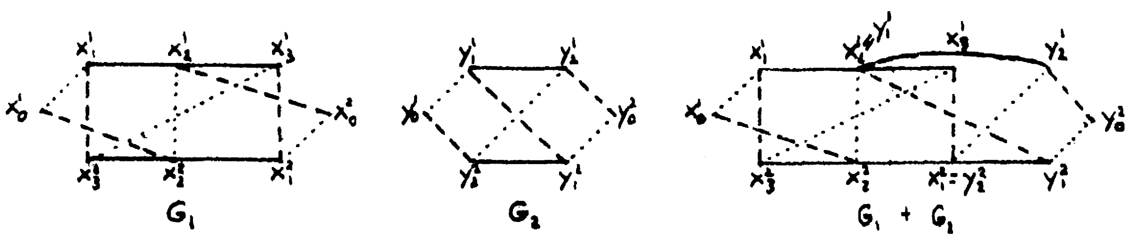

Figure 4. Concatenation

Note that concatenation is not a commutative operator, Also, if we label the vertices of the 4-cycle $\{x_0^1, x_1^1, x_0^2, x_2^2\}$, it becomes an identity element under concatenation. In fact, NOHO-graphs not in Q* form a non-commutative semi-group under concatenation, Associativity is clear from the construction. The next lemma verifies closure, LEMMA (33). If G_1 and G_2 are NOHO-graphs, then the concatenation G_1+G_2 is also a NOHO-graph.

Proof: We need only show that G_1+G_2 contains paths between all pairs of vertices and satisfies NOHO. We may consider the identified vertices as elements of either of the original graphs. Any path wholly within one of the component graphs is still present in G_1+G_2, unless it used one of the deleted vertices,\ The only paths which used them as non-endpoints are $(x_1^2, x_0^2, h(x_0^2))$ and $(y_1^1, y_0^1, h(y_0^1))$. In the concatenation these paths can be replaced as follows. Since we have identified x_1^2 with $h(y_0^1)$ and $h(x_0^2)$ with y_1^1, we can consider the endpoints as originating from the other summand graph, The transmission path between these vertices in that graph uses none of the deleted edges,

Obtaining an increasing path from a vertex of Gi to a vertex in G_j is quite simple, If v lies in G_1 and w in G_2, $v \to w$ can be formed by attaching $y_1^1 \to w$ from G_2 to the end of $v \to h(x_0^2)$ from G_1. Similarly, $w \to v$ can be formed by attaching $x_1^2 \to v$ from G_1 to the end of $w \to h(y_0^1)$ from G_2. These constructions work because every edge incident to y_1^1 in G_1+G_2 that comes from G_2 is greater than every such edge from G_1, and every edge incident to x_1^2 in G_1+G_2 from G_1 is greater than every such edge from G_2. The edges that could have violated that were the edges deleted from the union.

Finally, to prove NOHO we note that no increasing path which starts at a vertex from Gi can leave those vertices and later return, This would require traveling along $C^i(G_1+G_2)$, crossing to $C^j(G_1+G_2)$, and returning, The crossover could only use a first edge or last edge, which would prohibit including the earlier or later portion of the path. On the other hand, no path violating NOHO can lie entirely within the edges coming from one of the summands, since they are NOHO-graphs. ▯

To determine (P,Q,S,T) for $G_1+G_2=G_3$, we obtain $S(G_3)$ and $T(G_3)$ by concatenating in the usual sense $S(G_1)$ and $T(G_1)$ with $S(G_2)$ and $T(G_2)$. That is, with $m_i=n_i/2-1$, $S(G_3)$ contains $S(G_1)$ in positions 2 through m_1, and it contains $S(G_2)$ in positions m_1+1 through $m_3=m_1+m_2-1$. $S_{m_1}(G_3)$ describes what happens when $C^1(G_3)$ reaches y_2^1, which is the same as what happened when $C^1(G_1)$ reached $x^1_{m_1+1}$. The remainder of C^1 (Gi) is as before, The same argument applies to T, P and Q can be determined as in (25), or they can be determined directly by adjusting and combining $P(G_i)$ and $Q(G_i)$ as was done with S and T, This requires dropping an element, adding p_1 or p_2 to the elements in one portion, and concatenating.

If is natural to call a realizable quadruple or a NOHO-graph <u>irreducible</u> if it cannot be expressed as a concatenation of two smaller ones, In the next section we will count the number of realizable quadruples in subclasses involving irreducibility.

10. Irreducibility and NOHO-graphs

Before discussing irreducibility, we introduce some standard terminology about compositions of integers. A composition of an integer p is an ordered sequence of positive integers whose parts sum to p. Again, we are using p here because (n-4)/2 adds simply in concatenation. The i^{th} partial sum q_i of a composition is the sum of the first i parts, A refinement of a composition of p is a composition of p with as least as many parts whose partial sums contain the partial sums of the original composition, The least refinement of two compositions is the composition whose partial sums are the union of the partial sums of the original compositions. For example, the least refinement of (2,3,5) and (1,3,1,4,1) is (1,1,2,1,4,1).

This terminology will be useful for the following lemma, which states a very convenient fact about concatenation, Namely, NOHO-graphs are "uniquely factorable" into irreducible pieces, In algebraic terms, this means the irreducible solutions are the generators of the semigroup of NOHO-graphs under concatenation.

LEMMA (34). Any realizable quadruple can be uniquely expressed as a concatenation of irreducible quadruples,

Proof: Any such decomposition of a quadruple breaks up (S,T) into segments which each determine NOHO-graphs. For example, describing graphs as G(S,T), we have G(101010,111101) = G(1,1)

+ G(010,111) + G(10,01). We can describe the decomposition by a composition of the integer $p=(n-4)/2$.

We claim the least refinement of two compositions of p which correspond to decompositions of G also corresponds to a decomposition of G, If one composition is a refinement of the other, we are finished, If not then the least refinement has two consecutive partial sums $q_i=r$ and $q_{i+1}=s$, where r is a partial sum for exactly one of the compositions, and s is a partial sum only for the other. Performing the decomposition, we have indices j and k such that the segments (S_j,T_j) through (S_{s+1},T_{s+1}) and (S_{r+2},T_{r+2}) through (S_k,T_k) determine NOHO-graphs G_1 and G_2. (We have assumed $r<s$.)

Define another graph G_3, whose vertices and edges include the vertices and edges that lie both in G_1 and in G_2, plus two vertices y and z of degree two. (By "both in G_1 and in G_2" we mean when the vertices are labeled as the fit into G.) The neighbors of y and z are defined by $f_{G_3}(y)=x^1_{r+1}$, $h_{G_3}(y)=h_{G_2}(f_{G_2}(x^1_{r+1}))$, $f_{G_3}(z)=x^2_{m-s}$, and $h_{G_3}(z)=h_{G_1}(f_{G_1}(x^2_{m-s}))$.

G_3 is a NOHO-graph, and the proof of this rests on the fact that increasing paths which leave G_3 can never return to it, When such a path leaves G_3 it simultaneously leaves G_1 or G_2. By the same argument used to verify NOHO in (33), it cannot return, So, the increasing paths in G between vertices of G_3 must lie wholly within G_3. Information is transmitted for y and z also, since y takes the place of a vertex in G_2 Of degree 2 and z does the same in G_1. That NOHO is true follows because any increasing path in G_3 appears in G_1 or in G_2 (except $y \to z$ and $z \to y$), and they satisfy NOHO.

Let G_1' be obtained from G_1 by deleting vertices and edges belonging to G_3. Add a vertex w of degree two with $f_{G_1'}(w) = h_{G_3}(y)$ and $h_{G_1'}(w) = f_{G_3}(y)$. By the arguments in (33) and above, it is easy to see G_1' is a NOHO-graph and $G_1' + G_3 = G_1$. So G_1 was not irreducible,

Repeating-this argument over all decompositions of G, we see the only decomposition into irreducible segments is the least refinement of all the decompositions. ▯

Having proved unique decomposition, it becomes easy to count various classes of solutions by induction.

THEOREM (35). The number of realizable quadr ples formed by concatenating k irreducible quadruples is $\binom{p-1}{k-1}2^{p-k}$.

Proof: By induction on p. Examining Figure 3 yields the basis steps for p=1 and p=2. Assume the theorem is true for smaller values than p.

First consider k>1. To obtain such a quadruple we determine a composition of p and fill the quadruple with irreducible (S,T)-segments of those lengths. p is the eventual length of S and T from positions 2 through $m=n/2-1$. By induction, each segment of length r can be filled by 2^{r-1} irreducible pairs, Filling each segment in all possible ways, (33) says these are realizable, and (34) says there are no others. So, for each composition $r_1+\dots+r_k=p$ with-k parts, there are $2^{r_1-1}\cdots 2^{r_k-1}$ $=2^{p-k}$ quadruples of this type, There are $\binom{p-1}{k-1}$ Compositions of P with k parts, so the total number of solutions is $\binom{p-1}{k-1}2^{p-k}$.

This holds also for k=1, since that is precisely how many remain of the 3^{p-1} quadruples counted in (26). The binomial theorem says there are 2^{P-1} irreducible quadruples, ▯

THEOREM (36). The number of symmetric NOHO-graphs formed by concatenating k irreducible parts is $t_{p,k}$, where

$$t_{p,k} = \begin{cases} \binom{p/2-1}{k/2-1}2^{(p-k)/2} & ;\ p \text{ even, } k \text{ even} \\ 0 & ;\ p \text{ odd, } k \text{ even} \\ 2^{\lfloor p/2\rfloor - r}\sum_{q=1}^{\lfloor (p-1)/2\rfloor}\binom{q-1}{r-1} & ;\ k \text{ odd, } k>1,\ r=(k-1)/2 \\ 2^{\lfloor p/2\rfloor} & ;\ k=1 \end{cases}$$

Proof: We use a similar induction to the above. Figure 3 again provides the basis, though now p=1 and p=2 are both necessary. Assume the theorem is true for smaller values ofan p.

First consider k>1. If k is even, p must be even to allow symmetry. We determine a composition of the first p/2 places into k/2 parts, fill it with irreducible (S,T)-segments, and then obtain the rest by reflection (27). (33) and(34) again justify the conclusion that this counts everything, There are $\binom{p/2-1}{k/2-1}$ compositions and $2^{(p-k)/2}$ solutions for each one.

If k is odd and k>1, determine a composition of q with

$r=(k-1)/2$ parts, where $2q<p$. The middle segment of (S,T) will have length $p-2q$. In that segment we place a symmetric irreducible segment, of which by induction there are $2^{\lfloor (p-2q)/2 \rfloor}$. There are 2^{q-r} ways to fill the remainder, With the usual arguments about reflection, number of compostions, and the correctness of the count, we have $t_{p,k} = \sum_{q=1}^{\lfloor (p-1)/2 \rfloor} \binom{q-1}{r-1} 2^{q-r} 2^{\lfloor (p-2q)/2 \rfloor}$ $= 2^{\lfloor p/2 \rfloor - r} \sum_{q=1}^{\lfloor (p-1)/2 \rfloor} \binom{q-1}{r-1}$; $r=(k-1)/2$.

To compute $t_{p,1}$, we subtract the other $t_{p,k}$ from $3^{\lfloor p/2 \rfloor}$, the total number- of symmetric NOHO-graphs, derived in (31). Note that

$$\begin{aligned}
\sum_{r=1}^{\lfloor (p-1)/2 \rfloor} 2^{\lfloor p/2 \rfloor - r} \sum_{q=1}^{\lfloor (p-1)/2 \rfloor} \binom{q-1}{r-1} &= \sum_{q=1}^{\lfloor (p-1)/2 \rfloor} 2^{\lfloor p/2 \rfloor - q} \sum_{r=1}^{q} (q-1) 2^{q-r} \\
&= 2^{\lfloor p/2 \rfloor - 1} \sum_{q=1}^{\lfloor (p-1)/2 \rfloor} 2^{1-q} 3^{q-1} \\
&= 2^{\lfloor p/2 \rfloor - 1} [(3/2)^{\lfloor (p-1)/2 \rfloor} - 1] / [(3/2) - 1] \\
&= \begin{cases} 2 \cdot 3^{p/2-1} - 2^{p/2} & ; p \text{ even} \\ 3^{(p-1)/2} - 2^{(p-1)/2} & ; p \text{ odd} \end{cases}
\end{aligned}$$

-When p is even, we must also consider k even. If $s=k/2$, we have $\binom{p/2-1}{s-1} 2^{p/2-s} = 3^{p/2-1}$ as the number of these solutions, So,

$$\begin{aligned}
t_{p,1} &= 3^{\lfloor p/2 \rfloor} - \begin{cases} 3^{p/2-1} + 2 \cdot 3^{p/2-1} - 2^{p/2} & ; p \text{ even} \\ 3^{(p-1)/2} - 2^{(p-1)/2} & ; p \text{ odd} \end{cases} \\
&= 2^{\lfloor p/2 \rfloor}
\end{aligned}$$

□

11. Planarity and Hamiltonicity

In this section we note two properties of NOHO-graphs that are commonly of interest. Constructions are given for both, First, a quick lemma,

LEMMA (37). In a NOHO-graph, consider a path R that begins at x_0^1 and alternates along first edges and last edges, Then

i) The path alternates between C^1 and C^2, reaching C^1 always on first edges and C^2 on last edges,

ii) R eventually reaches x_0^2.

iii) From x_0^1 to x_0^2, R is a simple path.

iv) Among the x_i^1 and x_j^2 that appear along R until x_0^2, the indices i increase and the indices j decrease,

Proof: (i) is obvious, We verify the remainder in reverse order. For (iv), it suffices to consider pairs of consecutive appearances, If $x_{i'}^1 = f(h(x_i^1))$ so that $f(x_{i'}^1) = h(x_i^1)$, then (16.i) says $i' > i$. If $x_{j'}^2 = h(f(X_j^2))$ SO that $x_{j'}^2 = h(x_k^1)$ and $x_j^2 = f(x_k^1)$, then (16.i) says $j' < j$. (i) implies the consecutive appearances are as described. (iv) immediately implies (iii). Since the path connot contiue in the same direction forever, (iv) also implies (ii), ▯

THEOREM (38). In a NOHO-graph (other than Q*), uniting the first edges and last edges yields a Hamiltonian circuit,

Proof: Consider the alternating paths guaranteed by (37) that emerge from x_0^1 and proceed to x_0^2. One begins with a first edge, one with a last edge, Call them R_1 and R_2, respectively. We claim Rl and R_2 intersect only in $\{x_0^1, x_0^2\}$.

If not, let v be the first vertex where they meet after x_0^1. If it is before x_0^2, it lies in C^1 or in C^2, By (37.i), both paths reach it via the same type of edge, i.e. first or last, But F(G) and L(G) are matchings by (2), so there is only one such edge incident to v. This means the paths had to meet at the previous vertex.

So, uniting R_1 and R_2 yields a simple circuit, It is easy to see it must be Hamiltonian, If v lies outside it, we can begin paths there that proceed alternately along first edges and last edges, By the argument of (37), one such path R_v proceeds to x_0^2. The next-to-last vertex on it is in R_1 or R_2, since R_1 and R_2 reach x_0^2 separately and $d(x_0^2)=2$. It also lies in C^1 or C^2. As in the preceding paragraph, all of R_v including v lies in that same R_1 or R_2. ▯

THEOREM (39). Every NOHO-graph (excluding Q*) is planar.

Proof: We construct a planar representation. Place the vertices on the boundary of the shadow of a sausage. Put x_0^1 at the left end, x_0^2 at the right end, x_1^1 along the top edge from left to right, and x_1^2 to x_m^2 along the bottom edge from right to left,

Let R_1 and R_2 be as in the previous proof, Draw in R_1 as a path of chords, By (37.iii,iv), there are no crossings, Blur the boundary of the sausage so that the top and bottom boundaries become doubled, still meeting at the endpoints, Let the vertices of R_1 remain on the inside boundary, and move the vertices of R_2 to the outside boundary, R_2 can be drawn as a path of non-crossing chords in the outside infinite face, again by (37).

We still must show that the edges in the caterpillars can be added without crossings, The interior of the boubled boundary has not yet been entered by any edge, No edge of the caterpillars joins two vertices on the same R_i, i.e. on the same side of the doubled boundary, If so, (14.iii) and (37.iv) require a triangle. So, we can draw the caterpillar edges as chords across the interior of the boundary,

We claim there are no crossings. Since the vertices have been placed in order, (x^i_j, x^i_k) cannot cross (x^i_r, x^i_s) with $\max\{j,k\}<\min\{r,s\}$. If a crossing exists, we may assume $j<r<k$, $r<s$. By (14.iii), (x^i_j, x^i_r) is an edge. Similarly, if $k<s$ then (x^i_r, x^i_k) is an edge, while if $k>s$ then (x^i_j, x^i_s) is an edge. Either way, we have created a triangle in a tree, using x^i_j, x^1_r, and one of $\{x^i_k, x^i_s\}$. ▯

Figure 5 shows a representation drawn with this method.

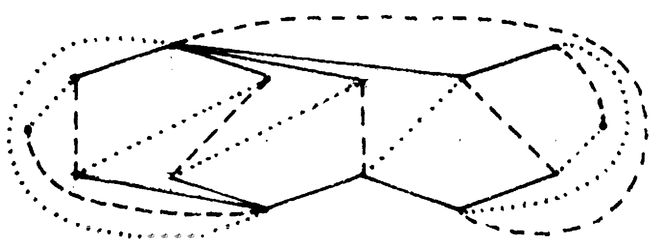

Figure 5. A planar representation

12. Related Gossip Questions

Golumbic[6] and Harary and Schwenk[8] have shown that any connected graph with n vertices, 2n-4 edges, and a 4-cycle admits an edge-ordering which solves the usual gossip problem. By 2n-4 edges, we mean 2n-4 calls will be made using "allowed" edges, Of course, most of these violate NOHO. The question remains open, however, whether every optimal solution of the gossip problem contains a 4-cycle. An affirmative answer would characterize these solutions, Examining (12) and (16.iii), we *note*

REMARK (40). Every optimal solution of the gossip problem satisfying NOHO has a 4-cycle.

It may be possible to prove the conjecture by applying this remark,

Graph theorists have also considered solutions of the gossip problem in which no transmission of information is duplicated, so there is a unique increasing path from each vertex to every other, Usually this includes the condition NOHO. Paradoxically, forbidding wastage requires more work, if indeed the problem can be solved at all. In other words, the information cannot be transmitted in 2n-4 calls unless n=4 or n=8, which follows from

REMARK (41). Every NOHO graph other than C_4 and those in Q* duplicates some transmission,

Proof: C_4 and graphs in Q*, as remarked in (12), duplicate no transmission. Consider any other NOHO-graph, and suppose $s_m=0$, We claim there are two paths from x_2^2 to x_m^1. By (16.iii), $f(x_2^2)=h(x_0^2)$. By (14.iii), $(x_m^1, h(x_0^2))$ is an edge. So $(x_2^2, h(x_0^2), x_m^1)$ forms one such path. By (14.i) there is an increasing pathe in C^2 from x_2^2 to every other vertex of C^2, including $h(x_m^1)$, which completes the path, On the other hand, if $S_m=1$, then $x_m^1=h(x_0^2)$. Applying (16.iii) again, there exist increasing paths $(x_2^2, x_m^1, h(x_1^2), x_1^2)$ and (x_2^2, x_1^2). □

Finally, we describe a generalization of the problem considered here, Consider an n by n "transmission matrix" on vertices $\{v_1, \ldots, v_n\}$ with entries from $\{1,0,-1\}$. If $a_{ij}=1$, we require an increasing path from v_i to v_j. If $a_{ij}=-1$ we forbid such a path, If $a_{ij}=0$ we don't care, We ask whether a calling scheme satifying the matrix exists, what is the least number of calls in such a scheme, what schemes achieve the minimum, and so on. The original gossip problem results when diagonal entries are 0 and off-diagoanal entries are 1, Changing the diagonal entries to -1 yields the subject here, The problem with ones above the diagonal and zeros on or below it is clearly optimized by a chain of n-1 edges, For a matrix in block diagonal form, we require the sum of the calls required by the smaller problems. Here's another example:

REMARK (42). Consider a transmission matrix with $a_{ii}=0$, $a_{ij}=0$ for $i>r\geq j$, and all other $a_{ij}=1$. The smallest graph solving this gossip problem has 2n-7 edges, This remains true if aii=-1, n even, r even,

Proof: Take an ordinary (2r-4)-edge solution H_1 on $\{v_1,\ldots,v_r\}$ and an ordinary (2n-2r-4)-edge solution H_2 on $\{v_{r+1},\ldots,v_n\}$. Order the edges so all those of H_2 occur after all those of H_1, Add an edge joining a vertex of the last edge in H_1 to the first edge in H_2, and let it occur between them. This uses 2n-7 calls and satisfies the matrix.

To show optimality, take any solution and delay all edges not wholly within $\{v_1,\ldots,v_r\}$, in order, until after every edge within that set, The resulting scheme still satisfies the matrix, But now it must consist of an ordinary scheme on r vertices, followed by at least one connecting edge and a solution on n-r vertices, So, there are at least 2n-7 calls,

If aii=-1, simply use NOHO-graphs in the H_1, H_2 construction. This requires n and r even. ▯

There are innumerable variations.

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