A NEW PROOF OF GLOBAL CONVERGENCE FOR THE TRIDIAGONAL QL ALGORITHM

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Abstract

By exploiting the relation of the QL algorithm to inverse iteration we obtain a proof of global convergence which is more conceptual and less computational than previous analyses. The proof uses a new, but simple, error estimate for the first step of inverse iteration.

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Contents

1. Introduction

The QR Algorithm has become the preferred method for finding *all* the eigenvalues of a given matrix, symmetric or nonsymmetric. One of the high points in the field of matrix computations is Wilkinson's discovery [Wilkinson, 1968] that the algorithm, when used with the proper shift strategy, converges for *all* symmetric, tridiagonal matrices. This result permits us to write clean efficient programs for computing these eigenvalues; there is no need for routine checking for rare unacceptable cases. The excellent asymptotic convergence rate for the method was already known.

Each iteration in the algorithm is effected by making a sequence of specially chosen plane rotations. Wilkinson's proof is based on a careful scrutiny of the last three of these rotations and a rather complicated computation is involved. A careful, detailed exposition of the proof can be found in [Lawson and Hanson, 1974, Appendix B].

The result is so nice that one is tempted to seek a proof which does not require explicit formulae for the elements of the next matrix in the QR sequence. The one presented here abandons the plane rotations in favor of the relation of the QR algorithm to inverse iteration, see for instance [Parlett and Poole, 1973]. The discussion is in terms of the QL algorithm which is a convenient variation of the original QR algorithm. Section 3 gives more details.

We try to adhere to the standard notational conventions: lower case roman letters for column vectors, lower case greek letters for scalars (all real here), and upper case roman letters for matrices (reserving symmetric letters for symmetric matrices). We write Z^T for the transpose of Z, I for $(e_1, e_2, \ldots, e_n)$, and $A-\lambda$ for $A-\lambda I$. All matrices are $n \times n$ unless the contrary is stated, $\|x\| = \sqrt{x^T x}$ and we write tridiagonal matrices A as shown below:

$$A = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & & \cdot & \cdot \\ \beta_1 & \alpha_2 & \beta_2 & & \cdot & \cdot \\ 0 & \beta_2 & \alpha_3 & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \beta_{n-1} \\ \cdot & \cdot & \cdot & & \beta_{n-1} & \alpha_n \end{bmatrix},$$

The QL transformation, with shift σ, transforms symmetric tridiagonal A into symmetric tridiagonal $\hat{A} = Q^T AQ$ where $Q = (q_1, q_2, \ldots, q_n)$ is orthogonal and depends on σ.

For the busy reader who is familiar with the subject we present a brief outline of the argument now. One special piece of notation is needed: MP_k denotes the set of all monic polynomials (leading coefficient 1) of degree k. We observe that

$$|\hat{\beta}_1 \hat{\beta}_2| = \min_{\phi \in MP_2} \|\phi(A) q_1\|, \quad \text{Lanczos},$$

$$\leq \|(A-\alpha_1)(A-\sigma) q_1\|, \quad \text{the artful choice},$$

$$= \|(A-\alpha_1) e_1 \tau\|, \quad \text{the connection with inverse iteration, Lemma 2},$$

$$= |\beta_1| \tau, \quad \text{since A is tridiagonal},$$

$$< |\alpha_1 - \sigma| \bullet |Bp|, \quad \text{if } \sigma \text{ is Wilkinson's shift, Lemma 4},$$

$$\leq |\beta_1 \beta_2|, \quad \text{by a characteristic property of Wilkinson's shift.}$$

Only the strict inequality is really new and a sharper form of it is used in Sections 5 and 6 to show that $(\beta_1^{(k+1)} \beta_2^{(k+1)})^2 <$ $< (2/5)(\beta_1^{(k-1)} \beta_2^{(k-1)})^2$ for all k and also that $\hat{\beta}_1^2 \leq |\beta_1 \beta_2|/\sqrt{2}$. This establishes global convergence, i.e. $\beta_1^{(k)} \to 0$, in a clean way.

2. Orthogonal Reduction to Tridiagonal Form

Any symmetric matrix M may be reduced to tridiagonal form $\hat{A}$ by an orthognoal similarity transformation. In symbols

$$(0) \qquad \hat{A} = G^T M G, \qquad I = G^T G = G G^T .$$

In fact, when the off diagonal elements $\hat{\beta}_j$ are not zero then $\hat{A}$ is completely determined by g_1 (or by g_n). Our interest is in expressions for products of the $\hat{\beta}_j$, $j = 1,2,\ldots$. From the pioneering work [Lanczos, 1950] we can deduce that

$$|\hat{\beta}_1 \cdots \hat{\beta}_j| = \min \|\phi(M) g_1\|$$

over all monic polynomials ϕ of degree j with equality only when $\phi(\lambda)$ is the leading principal $j \times j$ minor of X-A.

However we prefer to use some alternative formulas which yield rather more information and are also quite well known.

A useful way of understanding the relationships hidden in (0) is to equate columns on each side of the equation

$$(1) \qquad G\hat{A} = MG$$

and deduce that the columns $\{g_1, g_2, \ldots, g_j\}$ form an orthonormal basis for the so-called Krylov subspace K_j of $\mathbb{R}^n$ which is spanned by

$$g_1, Mg_1, M^2 g_1, \ldots, M^{j-1} g_1 .$$

Let P_j denote the orthogonal projection of $\mathbb{R}^n$ onto K_j and let $\bar{P}_j$ be its complement. For example, $P_1 = g_1 g_1^T$, $P_2 = g_1 g_1^T + g_2 g_2^T$.

LEMMA 1. *Let* $G\hat{A} = MG$ *with* $G = (g_1,\dots,g_n)$ *orthogonal, then*

$$g_2\hat{\beta}_1 = \bar{P}_1 M g_1 ,$$
$$g_3\hat{\beta}_2\hat{\beta}_1 = \bar{P}_2 M \bar{P}_1 M g_1 .$$

Proof. By equating the (1,1), (1,2), and (2,2) elements on each side of (1) we find

$$(2) \qquad \hat{\alpha}_1 = g_1^T M g_1 , \qquad \hat{\beta}_1 = g_1^T M g_2 = g_2^T M g_1 , \qquad \hat{\alpha}_2 = g_2^T M g_2 .$$

Now equate first columns on each side of the equation $G\hat{A} = MG$ and rearrange:

$$(3) \qquad \begin{aligned} g_2\hat{\beta}_1 &= Mg_1 - g_1\hat{\alpha}_1 , \\ &= Mg_1 - g_1(g_1^T M g_1) , \text{ using (2) }, \\ &= (I - g_1 g_1^T) M g_1 , \\ &= \bar{P}_1 M g_1 . \end{aligned}$$

Next equate the second columns on each side and rearrange:

$$(4) \qquad \begin{aligned} g_3\hat{\beta}_2 &= Mg_2 - g_2\hat{\alpha}_2 - g_1\hat{\beta}_1 , \\ &= Mg_2 - g_2(g_2^T M g_2) - g_1(g_1^T M g_2) , \text{ using (2) }, \\ &= (I - g_2 g_2^T - g_1 g_1^T) M g_2 , \\ &= \bar{P}_2 M g_2 . \end{aligned}$$

Multiply (4) by $\hat{\beta}_1$ and use (3) to obtain the formulas in the lemma. □

In the next section we will apply this lemma to the case when $M = A$ is also tridiagonal and Mg_1 lies in the plane of g_1 and e_1.

3. The QL Transform and Inverse Iteration

The QL transform of A is denoted by $\hat{A}$, has the same form as A, and is defined by

$$\hat{A} = Q^T A Q \tag{1}$$

where Q is the orthogonal matrix which satisfies

$$A - a = QL \tag{2}$$

and L is lower triangular with positive diagonal elements. The scalar σ is called the shift. Note that Q is the result of performing the Gram-Schmidt orthonormalizing process to the columns of A-a from right to left. The QL algorithm iterates the QL transform, choosing an appropriate shift at each step.

The QL transform is related to the earlier QR transform in a very simple way: if $\tilde{I} = (e_n, e_{n-1}, \ldots, e_1)$ and $\hat{A}$ is the QL transform of A then $\tilde{I}\hat{A}\tilde{I}$ is the QR transform of $\tilde{I}A\tilde{I}$. The QL algorithm has some minor advantages from the programmer's point of view and has become the preferred method. Consequently we will present our results in its terminology.

In practice the matrix Q which turns A into $\hat{A}$ is never formed explicitly. Even in theory the columns of Q are determined in the order $q_n, q_{n-1}, \ldots, q_2, q_1$. Nevertheless $\hat{A}$ is completely determined by q_1 and q_1 connects the QL transformation with simpler processes like inverse iteration.

We are now going to formulate a result which is quite well known.

LEMMA 2. *Let* $Q^TAQ = \hat{A}$ *be the QL transform of unreduced tridiagonal* A *with real shift* σ. *Then* $q_1 \equiv Qe_1$ *satisfies*

$$(A-\sigma)q_1 = e_1\tau .$$

If σ *is an eigenvalue* of A *then* $\tau = 0$; *otherwise* τ *is the scale factor which ensures that* $\|q_1\| = 1$; so $\tau = 1/\|(A-\sigma)^{-1}e_1\|$.

Proof. Transpose equation (2) above, post multiply by Q, and use $Q^TQ = I$ to find

$$(3) \qquad (A-\sigma)Q = L^T .$$

Equating column 1 on each side shows that

$$(4) \qquad (A-\sigma)q_1 = e_1\ell_{11} , \quad \ell_{11} \geq 0 .$$

If σ is not an eigenvalue then

$$(5) \qquad 1 = \|q_1\| = \|(A-\sigma)^{-1}e_1\| \cdot \ell_{11}$$

and we have written τ for ℓ_{11}. If σ is an eigenvalue then

$$0 = \det(A-\sigma) = \det Q \cdot \det L .$$

The Gram-Schmidt process begins with $q_n \equiv (A-\sigma)e_n/\ell_{nn}$. Because A is unreduced $\ell_{nn} = \|(A-\sigma)e_n\| \neq 0$. Moreover, for the same reason, the last (n-1) columns of $A-\sigma$ are linearly independent. Consequently

$$\ell_{jj} > 0 , \quad j = n,n-1,\ldots,3,2 ,$$

for all σ. It follows that on the last step of the Gram-Schmidt process a null vector is obtained. Hence $\ell_{11} \equiv \tau = 0$ and q_1 may be any unit vector orthogonal to all the other q's. This gives only a choice of sign for q_1 and in either case $(A-\sigma)q_1 = 0$. □

Equation (4) shows that the first column of Q is the normalized result of one step of inverse iteration with shift σ.

We now use Lemma 2 to get expressions for the off diagonal elements which are produced in the course of the QL algorithm.

LEMMA 3. *Let* $\hat{A} \equiv Q^T AQ$ *be the QL transform of* A *with real shift* σ. *Then*

$$|\hat{\beta}_1| = \tau|\sin\theta_1| ,$$
$$|\hat{\beta}_1\hat{\beta}_2| = \tau|\beta_1 \sin\theta_2| ,$$

where θ_i *is the angle between* e_i *and the Krylov space* K_i, $i = 1,2$.

<u>Proof.</u> Recall that in Lemma 1 $\bar{P}_1 = I - q_1q_1^T$, $\bar{P}_2 = I - q_1q_1^T - q_2q_2^T$. We have

$$\begin{aligned} q_2\hat{\beta}_1 &= \bar{P}_1Aq_1 &&, \text{ Lemma 1} \\ &= \bar{P}_1(q_1\sigma + e_1\tau) &&, \text{ Lemma 2} \\ &= \bar{P}_1e_1\tau \\ &= \tau(e_1 - q_1 \cos\theta_1) . \end{aligned}$$

Further

$$\begin{aligned} q_3\hat{\beta}_1\hat{\beta}_2 &= \bar{P}_2A\bar{P}_1Aq_1 , && \text{Lemma 1} \\ &= \tau\bar{P}_2A(e_1 - q_1\cos\theta_1) , && \text{two lines up,} \\ &= \tau\bar{P}_2(e_1\alpha_1 + e_2\beta_1 - q_1\cos\theta_1) , && \text{A is tridiagonal,} \\ &= \tau\beta_1P_2e_2 , && \bar{P}_2 \text{ annihilates } K_2 = \text{span}(q_1,Aq_1) \\ & && = \text{span}(q_1,e_1) . \end{aligned}$$

On taking norms the results follow. □

Using the same technique it can be shown that

$$\hat{\beta}_1\hat{\beta}_2 \cdots \hat{\beta}_j = \beta_1\beta_2\cdots\beta_{j-1}\,\tau \sin\theta_j \ .$$

Lemma 3 holds for any shift strategy but the global convergence follows from a simple bound on τ which holds when Wilkinson's shift is used.

4. Wilkinson's Shift

Given A then Wilkinson's shift ω is that eigenvalue of $\begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_1 & \alpha_2 \end{bmatrix}$ which is closer to $a_{1'}$ In case of a tie either eigenvalue may be used. So we have

$$(\alpha_1 - \omega)(\alpha_2 - \omega) - \beta_1^2 = 0$$

and

$$|\alpha_1 - \omega| \le |\alpha_1 - \omega'| \ .$$

Let us write $\delta = (\alpha_2 - \alpha_1)/2$ and observe that

$$\omega,\ \omega' = (\alpha_1 + \alpha_2)/2 \pm \sqrt{\delta^2 + \beta_1^2} \ .$$

This shows that

$$|\alpha_1 - \omega| \le |\alpha_2 - \omega|$$

with equality if, and only if, $\delta = 0$. By noting that $|\beta_1|$ is the geometric mean of $|\alpha_1 - \omega|$ and $|\alpha_2 - \omega|$ we have

$$\frac{|\alpha_1 - \omega|}{\beta_1} = \frac{\beta_1}{|\alpha_2 - \omega|} = \sqrt{\left|\frac{\alpha_1 - \omega}{\alpha_2 - \omega}\right|} \le 1$$

with e-quality if, and only if, $8 = 0$.

5. A Residual Estimate for Inverse Iteration

Since A is symmetric and tridiagonal we know that when β_1 is small compared with $|\alpha_1-\alpha_2|$ then e_1 is a good approximation to an eigenvector and Wilkinson's shift ω is an even better eigenvalue approximation than α_1. A well known way to obtain an improved normalized eigenvector is to solve for q_1 the equation

$$(1) \qquad (A-\omega)q_1 = e_1\tau$$

where τ is the positive scale factor which ensures that $\|q_1\| = 1$.

Our concern here is at the opposite extreme. If β_1 is not necessarily small and e_1 is a poor approximation to an eigenvector of A how bad can (ω,q_1) be as an approximate eigenpair? A good measure for this approximation is,

$$\tau/\|A\| ,$$

which is the norm of the "residual" vector $(A-\omega)q_1$ relative to $\|A\|$.

We now show that (ω,q_1) cannot be arbitrarily bad; in fact $\tau < |\beta_2|$. For convenience we write

$$\bar{\alpha}_i = \alpha_i - \omega$$

and define $p = (\pi_1,\pi_2,\pi_3,\ldots)^T$ by

$$(2) \qquad (A-\omega)p = e_1 .$$

LEMMA 4. When *Wilkinson's shift* ω *is used* in (1) then

$$\tau^2 < \left[\frac{1}{\bar{\alpha}_1^2+\beta_1^2} + \frac{\beta_1^2}{\bar{\alpha}_1^2\beta_2^2}\right]^{-1} < \frac{\bar{\alpha}_1^2\beta_2^2}{\beta_1^2}$$

Proof. If ω is an eigenvalue of A then the QL transform will make q_1 an eigenvector, so $\tau = 0$. From now on assume that A-w is invertible. From (1) and (2) we have

$$q_1 = p/\|p\|$$

and

(3) $$\tau^2 = 1/\|p\|^2 \leq 1/(\pi_1^2+\pi_2^2+\pi_3^2) .$$

The first two equations in (2) are

(4) $$\bar{\alpha}_1\pi_1 + \beta_1\pi_2 = 1 ,$$

(5) $$\beta_1\pi_1 + \alpha_2\pi_2 + \beta_2\pi_3 = 0 .$$

Recall the definition of ω and form $(\beta_1/\bar{\alpha}_1) \times (4) - (5)$ to find

(6) $$0 + 0 - \beta_2\pi_3 = \beta_1/\bar{\alpha}_1 .$$

In fact (6), together with the fact, from (4), that π_1 and π_2 cannot vanish simultaneously, is sufficient to prove that $\tau < |\beta_2|$. However, we can easily bound $\pi_1^2 + \pi_2^2$ away from 0. By elementary geometry the distance of the origin from the line (4) in the π_1, π_2 plane is $1/\sqrt{\bar{\alpha}_1^2+\beta_1^2}$. Hence

(7) $$\pi_1^2 + \pi_2^2 \geq 1/(\bar{\alpha}_1^2+\beta_1^2)$$

and the result follows readily from using (6) and (7) in (3). □

The surprisingly simple expression (6) for π_3 ensures, by itself, that $\beta_1\beta_2$ is monotone decreasing. The extra information contained in (7) shows that the decrease is linear right from the start.

Lemma 4 gives more information than we need. To simplify later discussion we use the harmonic mean, defined for positive ξ, η by

$$H(\xi,\eta) \equiv 2/(\xi^{-1}+\eta^{-1}) .$$

On majorizing $|\bar{\alpha}_1|$ by β_1 Lemma 4 simplifies to the useful

COROLLARY. $\tau^2 < H(\beta_1^2, \frac{1}{2}\beta_2^2)$

6. Global Convergence of the QL Algorithm

The QL algorithm produces a sequence of unreduced symmetric tridiagonal matrices $A^{(k)}$, $k = 1,2,\ldots$ and the glorious fact is that, always, $\beta_1^{(k)} \to 0$ rapidly as $k \to \infty$, revealing $\alpha_1^{(k)}$ as an increasingly good approximation to an eigenvalue of $A^{(1)}$. When $\beta_1^{(k)}$ is accepted as negligible the algorithm continues to transform all but the first row and column of $A^{(k)}$ and thus all the eigenvalues may be found in turn.

The convergence of $|\beta_1^{(k)}|$ need not be monotonic but the key fact is that $\{|\beta_1^{(k)}\beta_2^{(k)}|, k=1,2,\ldots\}$ is monotone decreasing and its limit is 0. Using the corollary of Lemma 4 in Lemma 3 and noting that $H(\xi,\eta) \le \sqrt{\xi\eta}$ we obtain

LEMMA 5. *When Wilkinson's shift is used in the QL algorithm,*

(a) $\hat{\beta}_1^2 < \tau^2 < \min\{2\beta_1^2, \beta_2^2, |\beta_1\beta_2|/\sqrt{2}\}$,

(b) $(\hat{\beta}_1\hat{\beta}_2)^2 < \beta_1^2\tau^2 < (\beta_1\beta_2)^2 H(\beta_1^2/\beta_2^2, \frac{1}{2}) < (\beta_1\beta_2)^2$.

This establishes the monotonic decline of $|\beta_1^{(k)}\beta_2^{(k)}|$ but to see that the limit is zero it suffices to consider two successive steps in the algorithm and so the superscript k can be dropped.

Lemma 5(b) shows that the reduction in $\beta_1\beta_2$ is substantial unless $|\beta_2/\beta_1|$ is small. However Lemma 5(a) shows that such an unfortunate ratio cannot persist. The next result makes this precise. We recall that the harmonic mean, $H(\xi,\eta)$, of positive numbers ξ and η is symmetric, homogeneous (of degree 1) and monotonic increasing in each of its arguments separately,

THEOREM 1. *Let* A, $\hat{A}$, $\mathring{A}$ *be three successive terms in the QL sequence using Wilkinson's shift. Then*

$$(\mathring{\beta}_1\mathring{\beta}_2)^2 < (\beta_1\beta_2)^2 2/(3+\sqrt{5}) < (2/5)(\beta_1\beta_2)^2 .$$

<u>Proof.</u>

$$\begin{aligned}
\mathring{\beta}_1^2\mathring{\beta}_2^2 &< \hat{\beta}_1^2\hat{\tau}^2 , && \text{Lemma 5b for } \hat{A}, \\
&< \hat{\beta}_1^2 H(\hat{\beta}_1^2, \tfrac{1}{2}\hat{\beta}_2^2) , && \text{Lemma 4 Corollary for ii}, \\
&= H(\hat{\beta}_1^4, \tfrac{1}{2}\hat{\beta}_1^2\hat{\beta}_2^2) , && \text{homogeneity of } H, \\
&< H(\tau^4, \tfrac{1}{2}\beta_1^2\tau^2) , && \text{monotonicity and Lemma 5 for } A, \\
&= \tau^2 H(\tfrac{1}{2}\beta_1^2, \tau^2) , && \text{homogeneity and symmetry of } H, \\
&< H(\beta_1^2, \tfrac{1}{2}\beta_2^2)\cdot H(\tfrac{1}{2}\beta_1^2, H(\beta_1^2, \tfrac{1}{2}\beta_2^2)) , && \text{Lemma 4 Corollary for } A, \\
&= \beta_1^2\beta_2^2 H(\tfrac{1}{2}, \rho^{-1}) H(\tfrac{1}{2}, H(1, \tfrac{1}{2}\rho)) , && \text{homogeneity of } H,\ \rho = \beta_2^2/\beta_1^2 , \\
&< \beta_1^2\beta_2^2 2/(3+\sqrt{5}) , && \text{maximizing over all } \rho > 0.
\end{aligned}$$

We note that

$$\begin{aligned}
H(\tfrac{1}{2}, \rho^{-1})\cdot H(\tfrac{1}{2}, H(1, \tfrac{1}{2}\rho)) &= \left(\frac{2}{2+\rho}\right)\cdot\left(\frac{2}{2+(\frac{1}{2}+\frac{1}{\rho})}\right) , \\
&= 1/[\tfrac{3}{2} + (\tfrac{5}{4}\rho + \rho^{-1})/2] , \\
&\leq 1/[\tfrac{3}{2} + \sqrt{\tfrac{5}{4}}] .
\end{aligned}$$

□

COROLLARY 1. *For the QL algorithm with Wilkinson's shift*

$$\beta_1^{(k)}\beta_2^{(k)} \to 0 \text{ as } k \to \infty .$$

Proof. $|\beta_1^{(2k+1)}\beta_2^{(2k+1)}| < |\beta_1^{(2k)}\beta_2^{(2k)}| < (2/5)^k|\beta_1^{(1)}\beta_2^{(1)}|$. □

The asymptotic convergence rate is much better than this. What is remarkable is that convergence is linear, with a good ratio, right from the start.

> COROLLARY 2. *For the QL algorithm with Wilkinson's shift*
>
> $$\beta_1^{(k)} \to 0 \quad \textit{as} \quad k \to \infty .$$

Proof. By Lemma 5(a), $\hat{\beta}_1^2 < |\beta_1\beta_2|/\sqrt{2}$. Convergence follows from Corollary 1. □

7. Local Convergence

We suppress the fact that all the elements α_1, β_1, etc. depend on k, the iteration count. We know that as $k \to \infty$ both $\bar{\alpha}_1 \to 0$, $\beta_1 \to 0$.

In the usual case $\beta_2 \to 0$ as well. In this case $\bar{\alpha}_2 \to \delta \neq 0$ because the eigenvalues of an unreduced tridiagonal matrix are distinct (although sometimes very close). The question we take up now is the asymptotic convergence rate. From Lemma 3 $|\hat{\beta}_1| = \tau|\sin\theta_1|$, but the estimate for τ in Lemma 4 does not reflect the asymptotic regime. In fact, as $k \to \infty$,

$$\tau = 1/\|p\| = O(1/|\pi_1|)$$

where $(A-\omega)p = e_1$.

Solving these equations as before yields

$$\pi_3 = \frac{-\alpha_2}{\beta_1\beta_2},$$

$$\pi_2 = \frac{\bar{\alpha}_2\bar{\alpha}_3}{\beta_1\beta_2^2} - \frac{\beta_3\pi_4}{\beta_2},$$

$$\pi_1 = -\frac{\bar{\alpha}_2^2\bar{\alpha}_3}{\beta_1^2\beta_2^2} + \frac{\bar{\alpha}_2\beta_3\pi_4}{\beta_1\beta_2} + \frac{\alpha_2}{\beta_1^2}.$$

In the usual regime

$$|\sin\theta_1| = \left(\frac{\pi_2^2 + \cdots + \pi_n^2}{\pi_1^2+\pi_2^2+\cdots+\pi_n^2}\right)^{1/2} = O\left(\left|\frac{\pi_2}{\pi_1}\right|\right)$$

and using the first terms in the expressions for π_1 and π_2

$$\hat{\beta}_1 \sim \beta_1^3\beta_2^2/\alpha_2^3 = O(\beta_1^3\beta_2^2).$$

This is better than cubic convergence.

We have not been able to prove that the analysis given above always obtains. The possibility remains open that $\bar{\alpha}_2 \to 0$, $\beta_2 \to \eta \neq 0$. In this case π_1 still dominates the other elements of π but it is the third term in the above expression for π_1 which brings this about, Thus, in such a case

$$\tau \sim \beta_1^2/|\bar{\alpha}_2| = |\bar{\alpha}_1| ,$$

$$|\sin \theta_1| \sim \beta_1\bar{\alpha}_3/\beta_2^2 ,$$

and

$$|\hat{\beta}_1| = O(|\bar{\alpha}_1\beta_1|) = O(\beta_1^2) .$$

Thus quadratic convergence occurs even in this unstable, and very special, eventuality.

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