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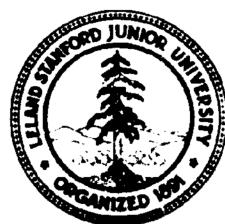
TWO RESULTS CONCERNING MULTICOLORING

by

V. Chvatal, M. R. Garey and D. S. Johnson

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ABSTRACT

The m -chromatic number $x_m(G)$ of a graph $G = (V, E)$ is the least integer k such that there exists a mapping $f: V \rightarrow \{S \subseteq \{1, 2, \dots, k\}: |S| = m\}$ having the property that $f(u) \cap f(v) = \emptyset$ whenever $\{u, v\} \in E$. This is a generalization of the standard notion of chromatic number and arises in connection with mobile telephone frequency assignments.

Answering a question of Lovász, our first result shows that for any $m \geq 1$ and any $\epsilon > 0$, there exists a graph G for which $x_{m+1}(G)/x_m(G) > 2-\epsilon$. This shows that the known bound of 2 for all m and G is essentially best possible. Our second result shows that the least integer m_0 for which $x_{m_0}(G)/m_0 =$

$\lim_{m \rightarrow \infty} x_m(G)/m$ can be asymptotically as large as $e^{(n \log n)/2}$ for some n vertex graphs, though it can never exceed $e^{(n \log n)/2}$.

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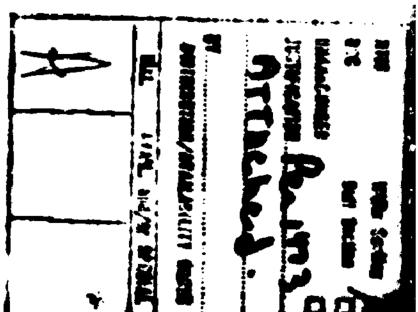
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I. INTRODUCTION

The following generalization of the standard notion of graph coloring has been of recent interest [1,3,4,6,7,8]. A multicoloring of a graph $G = (V, E)$ is a function f defined on V whose values are sets (of "colors") satisfying $f(u) \cap f(v) = \emptyset$ whenever $\{u, v\} \in E$. For positive integers k, m , a (k, m) -coloring of $G = (V, E)$ is a multicoloring f of G such that $|f(v)| = m$ for each $v \in V$ and $|\cup_{v \in V} f(v)| = k$. The m -chromatic number $x_m(G)$ is the least integer k such that there exists a (k, m) -coloring of G . (This last definition differs from that of [6,7] by a factor of m .) Notice that for $m = 1$ these definitions correspond to the usual graph coloring notions. The purpose of this note is to resolve two questions about multicoloring conveyed to us by P. Erdős [2].

The first question deals with the relationship between $x_m(G)$ and $x_{m+1}(G)$. It is not difficult to see that

$$x_{m+1}(G) \leq x_m(G) + x_1(G) \leq 2 \cdot x_m(G)$$



with equality possible in the right-hand inequality only for $m = 1$. Lovász asked [2] whether, for each value of m , there exist graphs G such that $\chi_{m+1}(G) > (2-\epsilon)\chi_m(G)$. We shall answer this question in the affirmative.

The graphs we shall use are defined as follows: for positive integers $n \geq 2m$, the graph G_m^n has vertex set consisting of all m -element subsets of $\{1, 2, \dots, n\}$ and has an edge between two such vertices exactly when their intersection is empty. It is easy to see that $\chi_m(G_m^n) \leq n$ merely by considering the multicoloring provided by the definition of G_m^n and, in fact, it is proved in [7,8] that $\chi_m(G_m^n) = n$. Thus, to answer the question of Lovász, it suffices to prove the following theorem:

Theorem 1. For each $m \geq 2$, there exists a constant c such that for all sufficiently large n

$$\chi_{m+1}(G_m^n) \geq 2n - c.$$

In order to prove Theorem 1, we require the following lemma, which is an immediate consequence of a special case of Theorem 3 in [5].

Lemma 1. For fixed $m \geq 2$ and n sufficiently large, there exists a constant a_0 such that the number of m -element subsets of $\{1, 2, \dots, n\}$ which can be chosen so that no two are disjoint but there is no element common to all is at most $a_0 n^{m-2}$.

Proof of Theorem 1. Fix m . We merely need to show that, for all sufficiently large n ,

$$x_{m+1}(G_m^n) - x_{m+1}(G_m^{n-1}) \geq 2$$

and the result will follow by induction. So suppose we have a $(k, m+1)$ -coloring of G_m^n such that $k = x_{m+1}(G_m^n)$, where n is any integer sufficiently large that the conclusion of Lemma 1 holds and such that $\binom{n-1}{m-1} > m a_0 n^{m-2}$, where a_0 is the constant of Lemma 1. We first claim that there must be at least $n+1$ colors which each appear on more than $a_0 n^{m-2}$ vertices.

Suppose there are n or fewer colors which each appear on more than $a_0 n^{m-2}$ vertices. By Lemma 1, each such color can appear on at most $\binom{n-1}{m-1} > a_0 n^{m-2}$ vertices since they must all share a common element. Thus, since each of the $\binom{n}{m}$ vertices receives exactly $m+1$ colors, we must have

$$\begin{aligned} (m+1) \binom{n}{m} &\leq n \binom{n-1}{m-1} + (k-n)a_0 n^{m-2} \\ &\leq m \binom{n}{m} + (k-n)a_0 n^{m-2} \end{aligned}$$

or, rewriting,

$$\binom{n}{m} \leq (k-n)a_0 n^{m-2}$$

Since

$$\begin{aligned} k = x_{m+1}(G_m^n) &\leq x_m(G_m^n) + x(G_m^n) \\ &\leq 2x_m(G_m^n) = 2n \end{aligned}$$

it follows that we must have

$$\binom{n}{m} \leq a_0^{n^{m-1}}$$

However this is a contradiction, since n was chosen sufficiently large that $\binom{n}{m} = \frac{n}{m} \binom{n-1}{m-1} > \frac{n}{m} m a_0^{n^{m-2}} = a_0^{n^{m-1}}$, and the claim follows.

Thus there are at least $n+1$ colors which each appear on more than $a_0^{n^{m-2}}$ vertices. The set of vertices on which any color i appears must form a collection of pairwise-intersecting m -element subsets of $\{1, 2, \dots, n\}$, by definition of G_m^n . Thus, by Lemma 1, whenever color i appears on more than $a_0^{n^{m-2}}$ vertices, all those vertices must contain some common element e_i . Since there are more than n such colors, we must have $e_i = e_j$ for some i and j . If we delete from G_m^n all the vertices containing $e_i = e_j$, we obtain a copy of G_m^{n-1} and a $(k-2, m+1)$ -coloring of it, since colors i and j have disappeared. Therefore

$$x_{m+1}(G_m^{n-1}) \leq k-2 = x_{m+1}(G_m^n) - 2$$

and the theorem is proved. \blacksquare

The second question involves what we call the ultimate multichromatic number $\chi^*(G)$ defined by

$$\chi^*(G) = \inf_m x_m(G)/m.$$

It is proved in [1,7] that the value of $\chi^*(G)$ is always achieved for some finite m . One easy way to see this is to formulate the problem of determining $\chi^*(G)$ as a linear programming problem (as done in [4]): Let v_1, v_2, \dots, v_n be an ordering of the vertices of G and let S_1, S_2, \dots, S_k be an ordering of the independent sets of G . Define x_{ij} to be 1 whenever $v_i \in S_j$ and 0 otherwise. Then the value of $\chi^*(G)$ is given by

$$\chi^*(G) = \min \sum_{j=1}^k r_j$$

subject to: $r_j \geq 0, 1 \leq j \leq k;$

$$\sum_{j=1}^k x_{ij} r_j = 1, 1 \leq i \leq n.$$

One can show easily, using Hadamard's Theorem, that no basis matrix for this problem can have determinant exceeding $n^{n/2}$ and this is an upper bound on the value of m required.

This upper bound however seems ridiculously large. Erdős asked [2] (as did the authors, independently) whether $\chi^*(G)$ could always be achieved for an m not exceeding the number of vertices of G . We answer this in the negative, constructing graphs for which extremely large values of m are necessary to achieve $\chi^*(G)$.

Let C_p denote the graph which is a cycle on p vertices. The join G_1+G_2 of two graphs G_1 and G_2 , having disjoint vertex sets, consists of all edges and vertices in the two given graphs along with all edges joining a vertex from G_1 to a vertex from G_2 . We use the following two lemmas in our construction:

Lemma 2. [4,7] For all integers $p \geq 1$,

$$\chi^*(C_{2p+1}) = 2 + (1/p).$$

Lemma 3. [7] For all graphs G_1 and G_2 ,

$$\chi^*(G_1+G_2) = \chi^*(G_1) + \chi^*(G_2).$$

Let p_i denote the i^{th} prime and define the graph $G(i)$ to be $C_{2p_1+1} + C_{2p_2+1} + \dots + C_{2p_i+1}$. The number of vertices n of $G(i)$ is given by

$$n = i + 2 \sum_{j=1}^i p_j.$$

Applying Lemmas 2 and 3, we obtain

$$\chi^*(G(i)) = 2i + \sum_{j=1}^i (1/p_j).$$

Since $\chi_m(G(i))$ must always be an integer, it follows that the least value of m for which $\chi^*(G(i)) = \chi_m(G(i))/m$ can be no less than $\prod_{j=1}^i p_j$ (and in fact that value of m will work). Using the Prime Number Theorem and expressing this lower bound in

terms of n , we obtain the asymptotic lower bound of

$$e^{\sqrt{(n \log n)/2}}.$$

Thus, though this is still quite far from the upper bound of

$$n^{n/2} = e^{(n \log n)/2}$$

we see that extremely large values of m can be required in order to achieve $\chi^*(G)$.

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