TWO RESULTS CONCERNING MULTICOLORING

by

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The m-chromatic number $\chi_m(G)$ of a graph $G = (V,E)$ is the least integer k such that there exists a mapping $f:V \to \{S \subseteq \{1,2,\ldots k\}: |S| = m\}$ having the property that $f(u) \cap f(v) = \phi$ whenever $\{u,v\} \in E$. This is a generalization of the standard notion of chromatic number and arises in connection with mobile telephone frequency assignments. Answering a question

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The m-chromatic number $\chi_m(G)$ of a graph $G = (V,E)$ is the least integer k such that there exists a mapping $f:V \rightarrow \{S \subseteq \{1,2,\ldots,k\}:|S| = m\}$ having the property that $f(u) \cap f(v) = \phi$ whenever $\{u,v\} \in E$. This is a generalization of the standard notion of chromatic number and arises in connection with mobile telephone frequency assignments. Answering a question of Lovász, our first result shows that for any $m \geq 1$ and any $\varepsilon > 0$, there exists a graph G for which $\chi_{m+1}(G)/\chi_m(G) > 2-\varepsilon$. This shows that the known bound of 2 for all m and G is essentially best possible. Our second result shows that the least integer m_0 for which $\chi_{m_0}(G)/m_0 = \lim_{m\to\infty} \chi_m(G)/m$ can be asymptotically as large as $e^{\sqrt{(n\log n)/2}}$ for some n vertex graphs, though it can never exceed $e^{(n\log n)/2}$.

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I. INTRODUCTION

The following generalization of the standard notion of graph coloring has been of recent interest [1,3,4,6,7,8]. A <u>multicoloring</u> of a graph $G = (V,E)$ is a function f defined on V whose values are sets (of "colors") satisfying $f(u) \cap f(v) = \phi$ whenever $\{u,v\} \in E$. For positive integers k,m, a <u>(k,m)-coloring</u> of $G = (V,E)$ is a multicoloring f of G such that $|f(v)| = m$ for each $v \in V$ and $|\bigcup_{v \in V} f(v)| = k$. The <u>m-chromatic number</u> $\chi_m(G)$ is the least integer k such that there exists a (k,m)-coloring of G. (This last definition differs from that of [6,7] by a factor of m.) Notice that for $m = 1$ these definitions correspond to the usual graph coloring notions. The purpose of this note is to resolve two questions about multicoloring conveyed to us by P. Erdös [2].

The first question deals with the relationship between $\chi_m(G)$ and $\chi_{m+1}(G)$. It is not difficult to see that

$$\chi_{m+1}(G) \le \chi_m(G) + \chi_1(G) \le 2 \cdot \chi_m(G)$$

with equality possible in the right-hand inequality only for $m = 1$. Lovász asked [2] whether, for each value of m, there exist graphs G such that $\chi_{m+1}(G) > (2-\epsilon)\chi_m(G)$. We shall answer this question in the affirmative.

The graphs we shall use are defined as follows: for positive integers $n \geq 2m$, the graph G_m^n has vertex set consisting of all m-element subsets of $\{1,2,\ldots,n\}$ and has an edge between two such vertices exactly when their intersection is empty. It is easy to see that $\chi_m\left(G_m^n\right) \leq n$ merely by considering the multicoloring provided by the definition of G_m^n and, in fact, it is proved in [7,8] that $\chi_m\left(G_m^n\right) = n$. Thus, to answer the question of Lovász, it suffices to prove the following theorem:

<u>Theorem 1</u>. For each $m \geq 2$, there exists a constant c such that for all sufficiently large n

$$\chi_{m+1}\left(G_m^n\right) \geq 2n - c.$$

In order to prove Theorem 1, we require the following lemma, which is an immediate consequence of a special case of Theorem 3 in [5].

<u>Lemma 1</u>. For fixed $m \geq 2$ and n sufficiently large, there exists a constant a_0 such that the number of m-element subsets of $\{1,2,\ldots,n\}$ which can be chosen so that no two are disjoint but there is no element common to all is at most $a_0 n^{m-2}$.

<u>Proof of Theorem 1</u>. Fix m. We merely need to show that, for all sufficiently large n,

$$\chi_{m+1}(G^n_m) - \chi_{m+1}\left(G^{n-1}_m\right) \geq 2$$

and the result will follow by induction. So suppose we have a (k,m+1)-coloring of G^n_m such that $k = \chi_{m+1}\left(G^n_m\right)$, where n is any integer sufficiently large that the conclusion of Lemma 1 holds and such that $\binom{n-1}{m-1} > ma_0n^{m-2}$, where a_0 is the constant of Lemma 1. We first claim that there must be at least n+1 colors which each appear on more than a_0n^{m-2} vertices.

Suppose there are n or fewer colors which each appear on more than a_0n^{m-2} vertices. By Lemma 1, each such color can appear on at most $\binom{n-1}{m-1} > a_0n^{m-2}$ vertices since they must all share a common element. Thus, since each of the $\binom{n}{m}$ vertices receives exactly m+1 colors, we must have

$$(m+1) \binom{n}{m} \leq n \binom{n-1}{m-1} + (k-n)a_0n^{m-2}$$

$$\leq m \binom{n}{m} + (k-n)a_0n^{m-2}$$

or, rewriting,

$$\binom{n}{m} \leq (k-n)a_0n^{m-2}$$

Since

$$k = \chi_{m+1}\left(G^n_m\right) \leq \chi_m\left(G^n_m\right) + \chi\left(G^n_m\right)$$

$$\leq 2\chi_m\left(G^n_m\right) = 2n$$

it follows that we must have

$$\binom{n}{m} \le a_0 n^{m-1} .$$

However this is a contradiction, since n was chosen sufficiently large that $\binom{n}{m} = \frac{n}{m}\binom{n-1}{m-1} > \frac{n}{m} m a_0 n^{m-2} = a_0 n^{m-1}$, and the claim follows.

Thus there are at least n+1 colors which each appear on more than $a_0 n^{m-2}$ vertices. The set of vertices on which any color i appears must form a collection of pairwise-intersecting m-element subsets of $\{1,2,\ldots,n\}$, by definition of G_m^n. Thus, by Lemma 1, whenever color i appears on more than $a_0 n^{m-2}$ vertices, all those vertices must contain some common element e_i. Since there are more than n such colors, we must have $e_i = e_j$ for some i and j. If we delete from G_m^n all the vertices containing $e_i = e_j$, we obtain a copy of G_m^{n-1} and a (k-2,m+1)-coloring of it, since colors i and j have disappeared. Therefore

$$\chi_{m+1}\left(G_m^{n-1}\right) \le k-2 = \chi_{m+1}\left(G_m^n\right) - 2$$

and the theorem is proved. ∎

The second question involves what we call the <u>ultimate multichromatic number</u> $\chi^*(G)$ defined by

$$\chi^*(G) = \inf_m \chi_m(G)/m.$$

It is proved in [1,7] that the value of $\chi^*(G)$ is always achieved for some finite m. One easy way to see this is to formulate the problem of determining $\chi^*(G)$ as a linear programming problem (as done in [4]): Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of G and let $S_1, S_2, \ldots, S_\ell$ be an ordering of the independent sets of G. Define x_{ij} to be 1 whenever $v_i \in S_j$ and 0 otherwise. Then the value of $\chi^*(G)$ is given by

$$\chi^*(G) = \min \sum_{j=1}^{\ell} r_j$$

subject to: $\quad r_j \geq 0, \quad 1 \leq j \leq \ell;$

$$\sum_{j=1}^{\ell} x_{ij} r_j = 1, \quad 1 \leq i \leq n.$$

One can show easily, using Hadamard's Theorem, that no basis matrix for this problem can have determinant exceeding $n^{n/2}$ and this is an upper bound on the value of m required.

This upper bound however seems ridiculously large. Erdös asked [2] (as did the authors, independently) whether $\chi^*(G)$ could always be achieved for an m not exceeding the number of vertices of G. We answer this in the negative, constructing graphs for which extremely large values of m are necessary to achieve $\chi^*(G)$.

Let C_p denote the graph which is a cycle on p vertices. The <u>join</u> G_1+G_2 of two graphs G_1 and G_2, having disjoint vertex sets, consists of all edges and vertices in the two given graphs along with all edges joining a vertex from G_1 to a vertex from G_2. We use the following two lemmas in our construction:

<u>Lemma 2</u>. [4,7] For all integers $p \geq 1$,

$$\chi^*(C_{2p+1}) = 2 + (1/p).$$

<u>Lemma 3</u>. [7] For all graphs G_1 and G_2,

$$\chi^*(G_1+G_2) = \chi^*(G_1) + \chi^*(G_2).$$

Let p_i denote the i^{th} prime and define the graph $G(i)$ to be $C_{2p_1+1} + C_{2p_2+1} + \cdots + C_{2p_i+1}$. The number of vertices n of G(i) is given by

$$n = i + 2 \sum_{j=1}^{i} p_j .$$

Applying Lemmas 2 and 3, we obtain

$$\chi^*(G(i)) = 2i + \sum_{j=1}^{i} (1/p_j) .$$

Since $\chi_m(G(i))$ must always be an integer, it follows that the least value of m for which $\chi^*(G(i)) = \chi_m(G(i))/m$ can be no less that $\prod_{j=1}^{i} p_j$ (and in fact that value of m will work). Using the Prime Number Theorem and expressing this lower bound in terms of n, we obtain the asymptotic lower bound of

$$e^{\sqrt{(n\log n)/2}} .$$

Thus, though this is still quite far from the upper bound of

$$n^{n/2} = e^{(n\log n)/2}$$

we see that extremely large values of m <u>can</u> be required in order to achieve $\chi^*(G)$.

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