THE STATIONARY P-TREE FOREST

by

Arne Jonassen

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At present at Computer Science Department, Stanford University, Stanford, California 94305, U.S.A. The work with this report has also been supported by The Norwegian Research Council for Science and the Humanities.

Abstract

This paper contains a theoretical analysis of the conditions of a priority queue strategy after an infinite number of alternating insert/remove steps. Expected insertion time, expected length, etc. are found.

Kev words

Analysis of algorithms, priority queues, random deletions, binary trees.

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1. Summary.

In [2] Ole-Johan Dahl and the author studied an algorithm for priority queue maintenance, first used in the work with the language SIMULA in the beginning of the 1960's. The strategy uses special binary trees called p-trees, and algorithms to maintain those structures.

The main part of [2], as well as of the more detailed treatment in [1] and [3], was devoted to a mathematical analysis of the efficiency of the structure after n successive insertions. Each new key was supposed to be independent of the other keys and to have equal probability of falling in any of the intervals defined by those keys already in the queue.

This paper is concerned with the efficiency of the algorithm after a large number of alternating remove-best/insert-random steps, starting with the situation after n successive insertions.

The famous ergodic theorem of Markov chain theory ensures us that there exists a stationary state, called the stationary p-tree forest, which the process approaches. We will find approximate values for properties of the stationary p-tree forest, as an application of general methods which will be developed for the analysis of such algorithms.

.Let F denote the normal p-tree forest and S the stationary p-tree forest. The following table compares some of the aspects of these two random structures:

	F	S
Expected left path length	$2H_n - 1$	$\frac{1}{3} H_n^2 + \frac{5}{3} H_n + O(1)$
Expected insertion time	$\frac{1}{3} H_n^2 + \frac{10}{9} H_n + O(1)$	$\frac{1}{3} H_n^2 + \frac{5}{3} H_n + O(1)$
Expected recursion depth	$\frac{2}{3} H_{n+1} + \frac{1}{9}$	$\frac{2}{3} H_n - \frac{1}{6} + O\left(\frac{H_n}{n}\right)$

The stationary p-tree forest S is more "skinny" than the normal p-tree forest F . Near the root, S is approximately equal to F ; for example the expected right path length tends to the same limit, and the probabilities of the value of the node next to the root are nearly the sane. However at the end of the left path S is quite different from F . The expected length of the left path of the last right subtree of the left path is shown to approach 1 , while the corresponding value of F approaches $\frac{3}{2}$. Similarly, the probability for the node next to the left leaf to be a, is shown to have the approximate values:

$$\left.\begin{array}{ll} \text{in S:} & \frac{2}{3a\cdot(a+1)} + \frac{1}{3n(n-1)} \\ \text{in F:} & \frac{1}{a(a+1)} + \frac{1}{n(n-1)} \end{array}\right\} \quad 4 \le a \le n-2$$

if $a = 3$

$$\text{S:} \quad \frac{1}{9} + \frac{2}{3}\,\frac{H_n}{n} \qquad \text{F:} \quad \frac{1}{6}$$

and if $a = 2$

$$\text{S:} \quad \frac{2}{3} - \frac{2}{3}\,\frac{H_n}{n} \qquad \text{F:} \quad \frac{1}{2} \quad .$$

In Chapter 2, more general aspects of the queuing phenomenon are presented. It should be pointed out that the text primarily deals with the particular problem of finding measures of the efficiency of the stationary p-tree forest, despite the fact that some of the methods have obvious generalizations.

In Chapter 3 is found a detailed definition of the stationary p-tree forest and its prerequisites. We also discuss a function, the characteristic left path polynomial attached to the forest, which will be essentially useful later in the paper. By arguments in Chapter 3 the function is defined for S .

In Chapter 4 one will find a deductive proof of the probabilities for the value of the node next to the left leaf. The derivation involves techniques from discrete mathematics, especially involving binomial coefficients.

In Chapter 5 we collect the information to derive the measures for S .

2. Models

2.1 The Queuing Phenomenon.

In the general case of the queuing phenomenon we have a Source (S) consisting of a number of independent devices, generating units to be served at some Service Processor (SP). SP for some reason (for example, its capacity) will not serve the units at arrival, and therefore it depends on some type of Queue Controller (QC) which arranges the units in some kind of priority sequence according to key values assigned to each unit. QC usually makes use of some predefined strategy working with special-types of storage structures in the queue itself (e.g. linear lists, binary trees, index tables). At request, the QC releases the unit having the best key value, for service by the SC (Best-In-First-Out (BIFO) strategy).

The process of placing a new unit in the queue is called an Insertion (I) and-the process of taking the best unit out of the queue is called a Remove (R) .

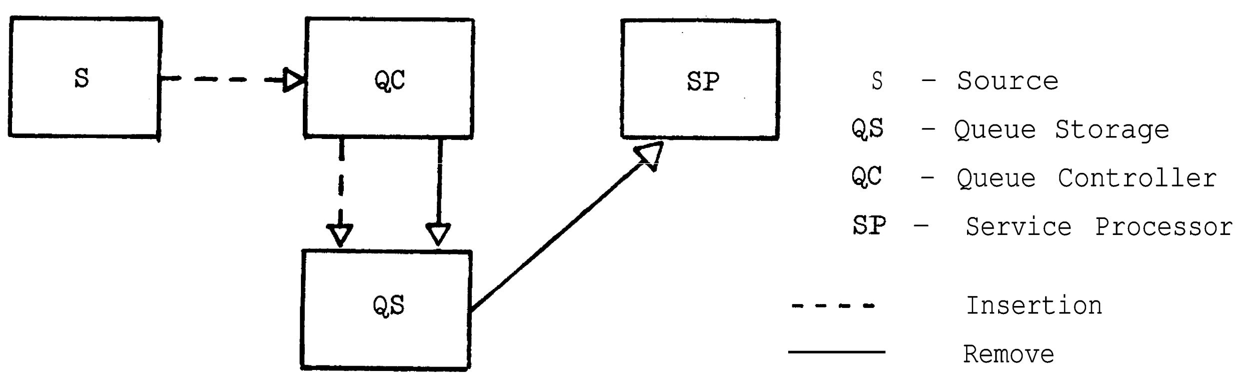


Figure 1.

We shall deal only with the queuing process and will assume that the units consist of the key value only.

A very simple way of assigning key values is to define some kind of time function according to the arrival at the QC . The best strategy is then probably to use a simple linear list in the QS . However, in the general case keys emanate from the source with values according to some distribution function; they may be adjusted by the QC prior to insertion and even be changed during their stay in the QS . We will use the term key pattern for the complex of rules according to which keys are assigned.

The queuing process may be regarded as a discrete time sequence of events. At each time t (t = 1,2,3,...) either an insertion or a removal takes place. In general we may have a case where the event to take place is subject to selection according to some distribution function. We will use the term I/R-pattern for the complex of rules according to which the insert/remove sequence takes place.

Maintaining a priority queue requires selection of a strategy for the structural ordering of the keys and algorithms for insertion and removal of keys. Linear lists, AVL-trees, and "heaps" are examples of such strategies. Each strategy provides algorithms for insertion and removal, as well as a mechanism for representation of the data, and we shall call it the queue strategy.

The purpose of this paper is to study a specific combination of the three elements in the queuing phenomenon, as described in the next sections. Some of our methods and resultshave obvious generalizations; however, we shall not attempt such generalizations in this paper, but concentrate on obtaining results for our special case.

2.2 Models for Key and I/R-patterns.

We will assume that our source generates keys as an infinite sequence of real numbers

$$X_1, X_2, \ldots, X_s, \ldots$$

being independent random variables chosen according to the exponential distribution with mean λ $(0 < \lambda)$, having the density distribution function:

$$(2.2.1)\quad f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } 0<x \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we will adopt the following assumption

(2.2.2) | Upon entry to the queue controller, each new key is increased by the value of the key last removed from the queue.

To demonstrate the effect of (2.2.2) we give an example.

Example 2.2.1.

Let the first five keys from the source be

0.8 , 1.9 , 1.1 , 0.1 , 2.0

and suppose the I/R-pattern is

IIIRIRRI

Time	I/R	Key from source	Key to the queue	The keys in the queue	Last key removed
1	I	0.8	0.8	0.8	0.0
2	I	1.9	1.9	0.8 , 1.9	0.0
3	I	1.1	1.1	0.8 , 1.9 , 1.1	0.0
4	R			1.9 , 1.1	0.8
5	I	0.1	0.9	1.9 , 1.1 , 0.9	0.8
6	R			1.9 , 1.1	0.9
7	R			1.9	1.1
8	I	2.0	3.1	1.9 , 3.1	

□

Restricting ourselves to a source generating keys which'are independent exponential random variables is not uncommon. Biasing the keys as described in (2.2.2) needs some motivation. If no adjustment were made we would run into cases where we would have smaller keys in the queue than some of those removed on earlier stages. Not biasing keys also means that large keys will have a tendency to be trapped in the queue, because smaller keys keep coming in with non-vanishing probability. The example below, quoted from [2], gives a practical example of bias occurrence:

Example 2.2.2.

Let the source contain n $(n \geq 1)$ independent exponentially distributed event patterns, with common parameter $\lambda > 0$.

The n devices each deliver an event time X_j $(j = 1,2,\ldots,n)$ to an initial queue. From that time on the best key, say X_k, is executed and the device k delivers a new key

$$X_k' = X_k + E$$

where E is exponentially distributed. Since X_k is the smallest of such keys in the queue at the present time, we have a situation conforming with (2.2.1) and (2.2.2). □

The key pattern described above is denoted K_0 .

Suppose Δ is some fixed (i.e., not subject to probabilistic. changes) I/R pattern, and let $A(t)$ $(1 < t)$ denote the t-th event (I or R) . (In Chapter 3 we will concentrate on a few such Δ , at present it is left unspecified.)

If we at any time t are left with an empty queue (i.e., if the number of I's having occurred is equal to the number of R's having occurred up to and including time t), we clearly are in a trivial situation equivalent to the original state; previous counts have no effect on the subsequent ones. Thus we may neglect this situation.

We will allow Δ to be infinite, but will assume that it is bounded in the sense that the queue never will contain a number of keys larger than some predetermined number M .

The latter two assumptions may be formulated as follows.

(2.2.3) Let $N_\Delta(t)$ be the difference between the number of I's and the number of R's having occurred in Δ up to and including time t . Then

$$0 < N_\Delta(t) < M$$

for all times $t = 1,2,\ldots$, where M is some predetermined number.

K_0 and Δ together uniquely define the queue at all times $t = 1,2,\ldots$, when the initial stage $(t = 0)$ is defined by the empty queue. The content of the queue will be denoted as follows.

n_t	$(= N_\Delta(t))$
$X_1^{(t)},\ldots,X_{n_t}^{(t)}$	the keys in the queue at time t , in sequence according to their arrival in the queue
δ_t	the value of the key last removed from the queue.

The notations apply to the situation after execution at time t (A(t)) .

Initially

$$n_0 = \delta_0 = 0 .$$

Our combination of K_0 and Δ have the nice property of leaving invariant the simultaneous density distribution function for the differences between the keys and the value of the last removed key, as stated in the following proposition.

Proposition 2.2.1. Using the notations above, let $1 \leq t$ and define the stochastic variables:

$$W_j^{(t)} = x_j^{(t)} - \delta_t \qquad 1 \leq j \leq n_t = n .$$

Then the W's have the following simultaneous density distribution function:

$$f(w_1, w_2, \ldots, w_n) = \begin{cases} \lambda^n e^{-\lambda(w_1 + w_2 + \cdots + w_n)} & \text{if } 0 \leq w_1, w_2, \ldots, w_n \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.4}$$

Proof. The proof follows from standard results and methods of probability theory.

As A(1) = I , and the first X from the source is exponentially distributed, we have $n_1 = 1$, $\delta_1 = 0.0$ and the correct distribution function. So the proposition is true for $t = 1$.

Assume the proposition to be true for some t , $1 \leq t$.

If $\Delta(t+1) = I$, let the new key from the source be

$$x = W + \delta_t$$

where the density function of W is given by (2.2.1). At time $t+1$ we will have:

$$n_{t+1} = n_t + 1$$

$$\delta_{t+1} = \delta_t$$

and the queue sequence:

$$(X_1^{(t+1)}, X_2^{(t+1)}, \ldots, X_{n_{t+1}}^{(t+1)}) = (X_1^{(t)}, X_2^{(t)}, \ldots, X_{n_t}^{(t)}, X) .$$

The W's at time t+1 are therefore defined by:

$$W_j^{(t+1)} = W_j^{(t)} \qquad j = 1,2,\ldots,n_{t+1}-1$$

$$W_{n_{t+1}}^{(t+1)} = w .$$

As W is independent of $W_1^{(t+1)}, \ldots, W_{n_t}^{(t+1)}$ we obviously have the required simultaneous density distribution function at time t+1 .

<u>If $\Delta(t+1) = R$</u>, let

$$V^{(t)} = \min(W_1^{(t)}, W_2^{(t)}, \ldots, W_{n_t}^{(t)})$$

and

$Y_1^{(t)}, Y_2^{(t)}, \ldots, Y_{n_t-1}^{(t)}$ be the remaining $W^{(t)}$'s, conserving the sequence.

By symmetry, the simultaneous density distribution function for $V^{(t)}, Y_1^{(t)}, \ldots, Y_{n_t-1}^{(t)}$ is:

$$f(v, y_1, y_2, \ldots, y_{n_t-1}) = \begin{cases} n_t \lambda^{n_t} e^{-\lambda(v + y_1 + \ldots + y_{n_t-1})} & \text{if } 0 \le v \le y_1, \ldots, y_{n_t-1} \\ 0 & \text{otherwise.} \end{cases}$$

Removing the-smallest of the $X^{(t)}$'s is equivalent to removing the smallest of the $W^{(t)}$'s, leaving us with the following situation:

$$n_{t+1} = n_t - 1 \; ; \quad \delta_{t+1} = V^{(t)} + \delta_t \; ;$$

and

$$W_j^{(t+1)} = Y_j^{(t)} - V^{(t)} \qquad j = 1,2,\ldots,n_{t+1} \; .$$

The simultaneous density distribution function for the $W^{(t+1)}$'s is hence:

$$f(w_1,w_2,\ldots,w_{n_{t+1}}) = \int_0^\infty n_t \lambda^{n_t} e^{-\lambda(n_t v + (w_1 + \ldots + w_{n_{t+1}}))} dv$$

when $0 \le w_1,w_2,\ldots,w_{n_{t+1}}$ (0 otherwise) because

$$V^{(t)} + Y_1^{(t)} + \ldots + Y_{n_t-1}^{(t)} = V^{(t)} + (W_1^{(t+1)} + V^{(t)} + \ldots + W_{n_{t+1}}^{(t+1)} + V^{(t)})$$

$$= n_t V^{(t)} + (W_1^{(t+1)} + \ldots + W_{n_{t+1}}^{(t+1)}) \; .$$

Simple integration yields the desired density function.

Proposition 2.2.1 has now been proved by induction. □

Another useful property of our (K_0,Δ) complex is the fact that a key to be inserted has equal probability of falling into any of the intervals defined by the keys already in the queue, as is readily seen from the symmetry properties of the density distribution function of Proposition 2.2.1:

Proposition 2.2.2. Using the notations above, assume

$$A(t+1) = I \; .$$

Let $X = W_t + \delta_t$ be the key to be inserted, W being distributed according to (2.2.1).

Let $Z_1^{(t)}, Z_2^{(t)}, \ldots, Z_{n_t}^{(t)}$ be the ordering variables of $X_1^{(t)}, X_2^{(t)}, \ldots, X_{n_t}^{(t)}$.

Then for $j = 1,2,\ldots,n_t-1$:

$$\text{Prob}(X < Z_1^{(t)}) = \text{Prob}(Z_j^{(t)} \le X < Z_{j+1}^{(t)}) = \text{Prob}(Z_{n_t}^{(t)} < X) = \frac{1}{n_t + 1} \; .$$

The results in Propositions 2.2.1 and 2.2.2 enable us to replace the continuous key pattern K_0 by a discrete key pattern D_0, described below. The replacement is easily seen to carry no loss of generality, for-queue strategies that depend only on the relative order of keys.

The key pattern D_0 involves renumbering of the key values in the queue at each step. However this will not alter the internal arrangement of the key

equivalent to those of K_0.

(2.2.5)

Key pattern D_0.

-- At the end of each time t the queue contains a permutation of the integers $1,2,\ldots,n_t$.

-- If $A(t+1) = I$, the source generates an X from the set

$$\mathcal{T}_{n_t} = \left\{ \frac{1}{2}, \frac{3}{2}, \ldots, n_t + \frac{1}{2} \right\}$$

with discrete probability distribution

$$\text{Prob}(X = x) = \frac{1}{n_t + 1} \qquad \forall x \in \mathcal{T}_{n_t} .$$

Having inserted x in the queue the keys are renumbered according to their size.

-- If $A(t) = R$, the key 1 is removed and the remaining key values are decreased by 1.

Note that in D_0 (as in K_0) all permutations (all relative orderings) are equally likely to occur, and that inserted X's (both in D_0 and K_0) have the same probability of falling in any of the n_t+1 intervals defined by the queue keys.

2.3 The Queue Strategy: p-trees.

The queue strategy ρ studied in this paper is the use of p-trees with algorithms for insert and remove, as described in [1] and [2]. In these papers, as well as in [3] and [6], one will find theoretical and practical results concerning ρ . We will assume familiarity with ρ .

using ρ , the queue structures are postfix ordered binary trees, being elements of a subset of the set of all binary trees. We will denote by

$\mathcal{B}^{(n)}$ the set of all binary trees with n nodes $(n \geq 1)$

$\mathcal{F}^{(n)}$ the set of all p-trees.

(We recall that a tree $T \in \mathcal{B}^{(n)}$ is a p-tree if and only if each node having a right successor also has a left successor.)

We will agree to define $\mathcal{B}^{(0)}$ and $\mathcal{F}^{(0)}$ to consist of one tree, viz. the empty tree ω .

When using p-trees we will adopt some conventional notations.

Let $T \in \mathcal{F}^{(n)}$ $(2 \leq n)$.

-- The length of the left path will be denoted τ .

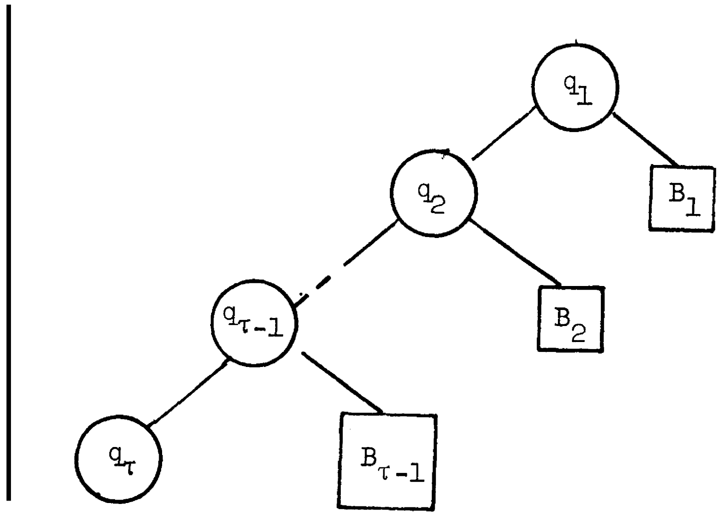
-- The values of the left path nodes in postfix order, from top to bottom, will be denoted by

(2.3.1) $$n = q_1 > q_2 > \cdots > q_\tau = 1$$

-- The right subtrees of the τ-1 first left path nodes (left leaf excluded) will be denoted by

$$B_1, B_2, \ldots, B_{\tau-1}$$

agreeing that node values are adjusted to range from 1 upwards (if nonempty),



(2.3.2) A p-tree forest F is defined as the pair of items:

$$F = (\mathcal{F}^{(n)}, \Phi)$$

where Φ is some probability model containing for each tree T in $\mathcal{F}^{(n)}$ a probability P_T to occur.

Using the key pattern D_0, some I/R-pattern Δ and ρ, will at each time t leave us with a p-tree forest, denoted by

$$F_\Delta^{(t)} = (\mathcal{F}^{(n)}, \Phi_A^{(t)})$$

where

$$\Phi_\Delta^{(t)} = \{P_\Delta^{(t)}(T) \mid T \in \mathcal{F}^{(n)}\} .$$

In [1] and [2] are presented theoretical results of the so-called "**normal** p-tree forest", being the pair

$$F_0^{(n)} = (\mathcal{F}^{(n)}; \Phi^{(n)}_{\Delta_0^{(n)}})$$

where $\Delta_0^{(n)} = II...I$ is the I/R-pattern consisting of n successive insertions.

One of the important properties of the normal p-tree forest, due to the recursiveness of the insertion algorithm is that:

(2.3.3) The set of all right **subtrees** of a fixed left path node position is composed of a set of copies of normal p-tree forests.

This property is called the basic p-tree property (BPP). Formally, the BPP may be described as follows:

(2.3.4)

Let $F = (\mathcal{F}^{(n)}, \Phi)$ be any p-tree forest $(n > 2)$ with

$$\Phi = \{P_T \mid T \in \mathcal{F}^{(n)}\} .$$

Let $1 \leq j \leq (n-1)$ be any fixed left path node position, and define

$$\alpha_U^{(j)} = \sum_{\substack{T \in \mathcal{F}^{(n)} \\ B_j \cdot - U}} P_T$$

for all trees U :

$$U \in \bigcup_{s=0}^{n} \mathcal{F}^{(s)} .$$

Let

$$A_k^{(j)} = \sum_{U \in \mathcal{F}^{(k)}} \alpha_U^{(j)} \qquad (0 \leq k \leq n-j-1)$$

and

$$\varphi_k^{(j)} = \left\{ q_U^{(j)} = \frac{\alpha_U^{(j)}}{A_k^{(j)}} \;\middle|\; U \in \mathcal{F}^{(k)} \right\} .$$

Then F is said to have the BPP if the forests

$$Q_k^{(j)} = (\mathcal{F}^{(k)}, \varphi_k^{(j)})$$

are normal p-tree forests for all j, k : $0 \leq k \leq n-j-1$; $1 \leq j \leq n-1$.

A proof of the fact that $F_0^{(n)}$ have the BPP is found in [1].

2.4 The Characteristic Left Path Polynomial.

Adopt the notations in the previous section and let $T \in \mathcal{F}^{(n)}$, $(n \geq 2)$. The polynomial

$$(2.4.1) \qquad h_T(z,w) = \sum_{j=1}^{\tau-1} z^{q_j} w^{q_{j+1}}$$

is called T's left path polynomial (LPP).

Let $F = (\mathcal{F}^{(n)}, \Phi)$ be any p-tree forest $(n > 2)$. The polynomial

$$(2.4.2) \qquad H_F(z,w) = \sum_{T \in \mathcal{F}^{(n)}} P_T \; h_T(z,w)$$

is called F's characteristic left path polynomial (CLPP).

Being a polynomial in z and w having terms of the type $z^a w^b$ with $1 < b < a \leq n$ we see that we may write

$$(2.4.3) \qquad H_F(z,w) = \sum_{1 \leq b < a \leq n} \beta_{a,b}^{(F)} \, z^a \; w^b$$

where

(2.4.4) $\beta_{a,b}^{(F)}$ is the probability of a tree in F to have a and b as values of adjacent nodes on the left path

For convenience we will adopt the conventions

$$(2.4.5) \qquad \begin{cases} h_T(z,w) = H_F(z,w) = 0 & \text{if } n = 1 \\ h_T(z,w) = H_F(z,w) = -z & \text{if } n = 0 \,. \end{cases}$$

From the CLPP of a p-tree forest F, we may deduce the expected left path length- L_F. Because-each tree T has exactly one more node on its left path than the number of terms in its LPP we find

$$\tau = 1 + h_T(1,1)$$

leading to

(2.4.6) $$L_F = 1 + H_F(1,1) \qquad (0 \le n) .$$

Assume that $F = (\mathcal{F}^{(n)}, \Phi)$ $(n > 2)$ has the BPP, defined in the previous section. We may then use F 's CLPP to establish the expected number of key comparisons (S_F) necessary to insert a random $x \in \mathcal{J}_n$, being subject to the equiprobability distribution as in D_0 in the trees.

We split S_F into two parts:

(2.4.7) $$S_F = SL_F + SR_F$$

where SL_F is the expected number of key comparisons involving left path nodes, and SR_F is the expected number of key comparisons involving nodes in the right subtrees.

SL_F.

Let T be any tree, and use notations as in (2.3.1):

if $x < 1$ we use τ comparisons;

if $q_{j+1} < x < q_j$ for some $j = 1,2,\ldots,\tau-1$ we use $j+1$ comparisons;

if $n < x$ we use 1 comparison;

leading to the expected number of left path comparisons in T :

$$SL_T = \left(\tau + \sum_{j=1}^{\tau-1}(q_j - q_{j+1})(j+1) + 1\right)\Big/(n+1)$$

$$= 1 + \frac{1}{n+1} \sum_{j=1}^{\tau-1} q_j \quad .$$

Now

$$\sum_{j=1}^{\tau-1} q_j = \left[\frac{\partial}{\partial z} \sum_{j=1}^{\tau-1} z^{q_j} w^{q_{j+1}} \right]_{z=w=1}$$

so that

$$SL_T = 1 + \frac{1}{n+1} \left[\frac{\partial h_T(z,w)}{\partial z} \right]_{Z=w=1}$$

and obviously

$$\text{(2.4.8)} \qquad SL_F = 1 + \frac{1}{n+1} \left[\frac{\partial H_F(z,w)}{\partial z} \right]_{Z=w=1} .$$

$\underline{SR_F}$.

Let the number of key comparisons necessary to insert x in the right **subtree** of **left** path node j of the tree T be

$$s_T(x,j)$$

provided $q_{j+1} < x < q_j$. This process is clearly equivalent **with** inserting $x-q_{j+1}$ in B_j (where node values have been adjusted), denoted **by** $s_{B_j}(x)$, because of the BPP.

We then find

$$SR_F = \sum_{T \in \mathcal{F}(n)} \sum_{j=1}^{t-1} \sum_{\substack{x \in \mathcal{T}_n \\ q_{j+1} < x < q_j}} P_T \frac{1}{n+1} s_T(x,j)$$

$$= \sum_{j=1}^{n-1} \sum_{k=0}^{n-j-1} \sum_{U \in \mathcal{F}(k)} \sum_{x \in \mathcal{T}_k} \frac{1}{n+1} s_U(x) \sum_{\substack{T \in \mathcal{F}(n) \\ B_j = U}} P_T$$

Using the notations of (2.3.4), knowing that F has the BPP, we find

$$\sum_{\substack{T \in \mathcal{F}^{(n)} \\ B_j = u}} P_T = \alpha_U^{(j)} = q_U^{(j)} \; A_k^{(j)}$$

and hence

$$SR_F = \sum_{j=1}^{n-1} \sum_{k=0}^{n-j-1} \frac{k+1}{n+1} A_k^{(j)} \; V_{j,k}$$

where

$$V_{j,k} = \sum_{U \in \mathcal{F}^{(k)}} \; \sum_{x \in \mathcal{T}_k} q_U^{(j)} \; s_U(x) \; \frac{1}{k+1} \cdot S_{F_0^{(k)}}$$

because $(\mathcal{F}^{(k)}, \varphi_k^{(j)})$ is a normal p-tree forest.

$A_k^{(j)}$ is the probability of finding a tree in F with right sub-tree j of size k . Each time a term $z^{q_j} w^{q_{j+1}}$ with $q_j - q_{j+1} - 1 = k$ occurs in the LPP's of the trees in F we get a contribution P_T to the corresponding term in F's CLPP. Summing over all possible j 's will correspond in the CLPP to summing the coefficients of all possible $z^a w^b$ with a-b-1 = k . Hence

$$\sum_{j=1}^{n-k-1} A_k^{(j)} = \sum_{b=1}^{n-k-1} \beta_{b+k+1,b}^{(F)}$$

and

$$\text{(2.4.9)} \qquad SR_F = \sum_{k=0}^{n-2} \frac{k+1}{n+1} \; S_{F_0^{(k)}} \sum_{b=1}^{n-k-1} \beta_{b+k+1,b}^{(F)} \quad .$$

Bringing (2.4.8) and (2.4.9) together we find

$$\text{(2.4.10)} \qquad S_F = 1 + \frac{1}{n+1} \left[\frac{\partial H_F(z,w)}{\partial z} \right]_{Z=w=1} + \sum_{k=0}^{n-2} \frac{k+1}{n+1} \; S_{F_0^{(k)}} \; \rho_k \qquad (n \geq 2)$$

where

(2.4.11) $$\rho_k = \sum_{b=1}^{n-k-1} \beta^{(F)}_{b+k+1,b} \quad .$$

The ρ_k's may be found as follows:

(2.4.12) $$\sum_{k=0}^{n-2} \rho_k z^k = \frac{1}{z} H_F\left(z , \frac{1}{z} \right) \quad .$$

The quantities $S_{F_0}(n)$ are known from [1] and [2]:

(2.4.13) $$\begin{cases} S_{F_0}(n) = \frac{1}{3} H^2_{n+1} + \frac{10}{9} H_{n+1} - \frac{1}{3} H^{(2)}_{n+1} - \frac{28}{27} & (n \geq 2) \\ S_{F_0}(0) = 0 \quad ; \quad S_{F_0}(1) = 1 \quad . \end{cases}$$

Similar to the methods used above for L_F and S_F we may establish formulae for the quantities

- R_F the expected length of the right path in F
- RL_F the expected length of the left path of the last right subtree
- C_F the expected recursion depth

all quantities being examined in [2]. The appropriate formulae turn out to be

(2.4.14) $$R_F = 1 + \sum_{k=0}^{n-2} R_{F_0}(k) \; \beta^{(F)}_{n,n-k-1} \qquad (n \geq 2)$$

(2.4.15) $$RL_F = \sum_{k=0}^{n-2} L_{F_0}(k) \; \beta^{(F)}_{k+2,1} \qquad (n \geq 2)$$

(2.4.16) $$c_{F} = 1 + \sum_{k=0}^{n-2} \frac{(k+1)}{(n+1)} \; c_{F_0^{(k)}} \; \rho_k \qquad (n \geq 2)$$

(with ρ_k defined in (2.4.11)).

We shall demonstrate the effects of formulae (2.4.6), (2.4.10), (2.4.14), (2.4.15) and (2.4.16) when applied to the normal p-tree forests. We assume $n \geq 2$ and $F_0^{(n)}$.

$F_0^{(n)}$ has the BPP and the CLPP:

(2.4.17) $$H_{F_0^{(n)}}(z,w) = \sum_{a=2}^{n} \sum_{b=1}^{a-1} \left(\frac{1}{(n+1-b)(n-b)} + \frac{1}{(a-1)a} \right) z^a w^b \quad .$$

The latter formula was established in [2] on basis of considerations on the correspondence between the set of all permuations of the numbers $1,2,\ldots,n$ and $F_0^{(n)}$.

From (2.4.17) we deduce

(2.4.18) $$H_{F_0^{(n)}}(1,1) = \sum_{a=2}^{n} \sum_{b=1}^{a-1} \left(\frac{1}{(n+1-b)(n-b)} + \frac{1}{(a-1)a} \right)$$

$$= 2(H_n - 1)$$

and (according to (2.4.12)):

(2.4.19) $$\left| \begin{aligned} \sum_{k=0}^{n-2} \rho_k z^k &= \sum_{a=2}^{n} \sum_{b=1}^{a-1} \left(\frac{1}{(n+1-b)(n-b)} + \frac{1}{(a-1)a} \right) z^{a-b-1} \\ &= \sum_{k=0}^{n-2} \frac{2(n-k-1)}{(k+1)n} z^k \end{aligned} \right.$$

and finally

$$
(2.4.20)\qquad \left[\frac{\partial H_{F_0^{(n)}}(z,w)}{\partial z}\right]_{z=w=1} = \sum_{a=2}^{n}\ \sum_{b=1}^{a-1}\left(\frac{1}{(n+1-b)(n-b)} + \frac{1}{(a-1)a}\right)a
$$

$$
= (n+1)H_n - \frac{(n+3)}{2}.
$$

Inserting (2.4.18) - (2.4.20) in (2.4.6), (2.4.X)) and (2.4.14)- (2.4.16) we find ($n \geq 2$) :

$$
(2.4.21)\qquad
\begin{aligned}
L_{F_0}(n) &= 2H_n - 1 \\
S_{F_0}(n) &= H_n + \frac{(n-1)}{2(n+1)} + \frac{2}{n(n+1)} \sum_{k=0}^{n-2} (n-k-1) S_{F_0}(k) \\
R_{F_0}(n) &= 1 + \sum_{k=0}^{n-2} R_{F_0}(k) \cdot \left(\frac{1}{n(n-1)} + \frac{1}{(k+1)(k+2)} \right) \\
RL_{F_0}(n) &= \sum_{k=0}^{n-2} L_{F_0}(k) \left(\frac{1}{n(n-1)} + \frac{1}{(k+1)(k+2)} \right) = \frac{3}{2}\ \frac{4}{ii} - + \frac{1}{n(n-1)} \\
C_{F_0}(n) &= 1 + \frac{2}{(n(n+1))} \sum_{k=0}^{n-2} (n-k-1) C_{F_0}(k) \quad .
\end{aligned}
$$

These formulae confirm those of [2]. Detailed treatment of (2.4.21), **may** be found in [1] and [3].

The main advantage obtained by use of the CLPP relative to [1] is the establishment of the term $SL_{F_0}(n)$ from (2.4.8).

3. The Stationary p-tree Forest.

3.1 General Considerations.

Consider the key pattern D_0 (see (2.2.5)), the queue strategy ρ and any p-tree forest $F = (\mathcal{F}^{(n)}, \Phi)$ where $n > 1$ and

$$\Phi = (q(T) \mid T \in \mathcal{F}^{(n)}\} \quad (3.1.1)$$

Let Δ be the I/R pattern consisting of an infinite number of alternating insertions and removals:

$$\Delta(2s-1) = I \ , \ \Delta(2s) = R \qquad s = 1, 2, \ldots \quad (3.1.2)$$

Suppose we start at time 0 with F and apply Δ . At each time $t = 2s$, $s = 1,2,\ldots$ we are left with a p-tree forest, denoted by

(3.1.3)
$$F^{(s)} = (\mathcal{F}^{(n)}, \Phi^{(s)})$$
$$\Phi^{(s)} = [illegible] \mid [illegible] \in \mathcal{F}^{(n)}\} \quad .$$

We also define $F^{(0)} = F$.

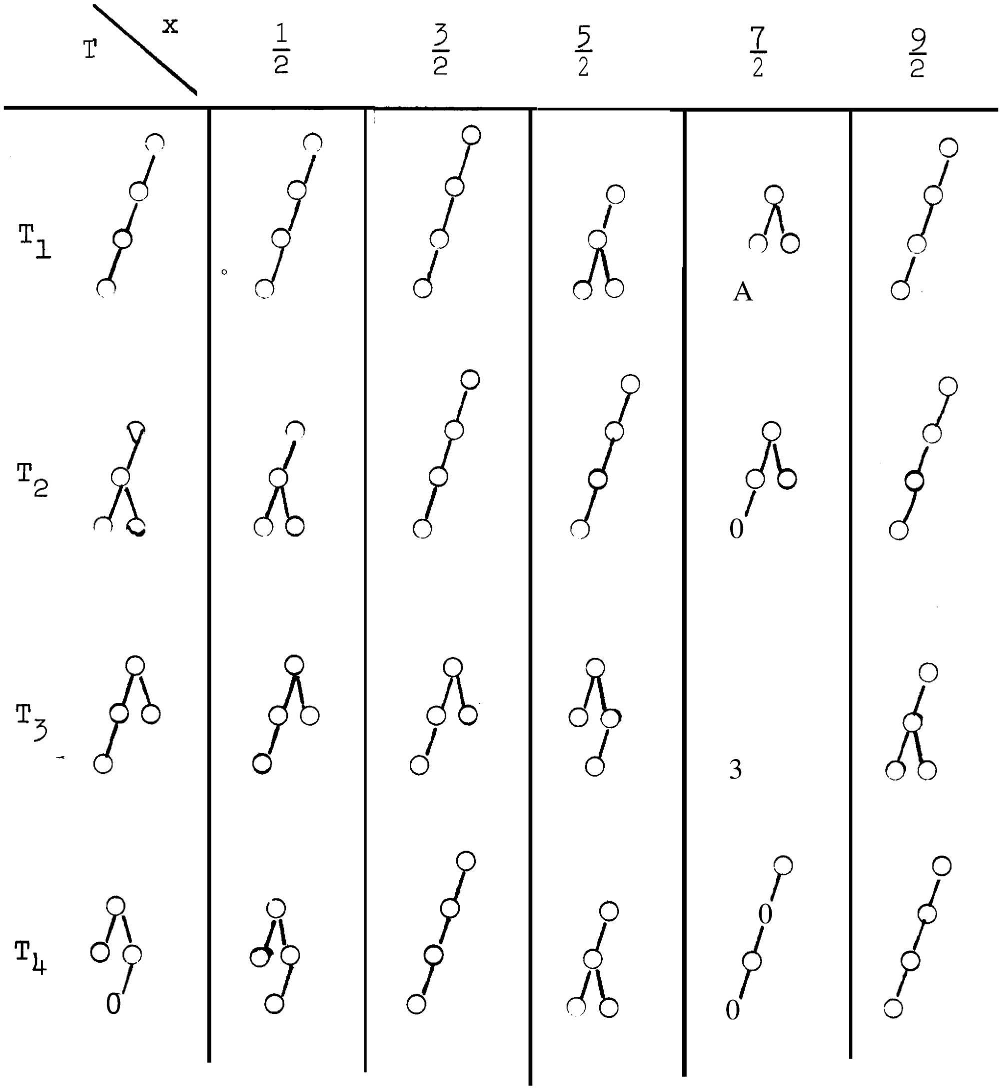
The sequence $F^{(0)}, F^{(1)}, \ldots, F^{(s)}, \ldots$ may be regarded as an infinite Markov chain, where the possible stages are the trees of $\mathcal{F}^{(n)}$ and where the transition matrix

$$\mathcal{M} = (m_{i,j})$$

is an $N \times N$ matrix (N being the number of elements in $\mathcal{F}^{(n)}$) whose elements $m_{i,j}$ are the probabilities of mapping tree i from $\mathcal{F}^{(1)}$ to tree j in $\mathcal{F}^{(n)}$ in one complete insert-remove operation. (We fix some numbering of the trees in $\mathcal{F}^{(1)}$.)

To demonstrate this transition, let us consider the case when n = 4 .

<u>Example 3.1.1.</u> The left column in the table below contains the possible trees in $\mathcal{F}^{(4)}$, the horizontal line contains the possible x's and the table entries are the resulting trees when inserting x and removing the left leaves.



The transition matrix is therefore

$$\mathcal{M} = \begin{pmatrix} \frac{3}{5}, & \frac{1}{5}, & \frac{1}{5}, & 0 \\ \frac{3}{5}, & \frac{1}{5}, & \frac{1}{5}, & 0 \\ 0, & \frac{1}{5}, & \frac{2}{5}, & \frac{2}{5} \\ \frac{3}{5}, & \frac{1}{5}, & 0, & \frac{1}{5} \end{pmatrix}$$

and one complete insert-remove step may be described as

$$\bar{P}^{(s+1)} = \bar{P}^{(s)} \cdot \mathcal{M}$$

where $\bar{P}^{(s)}$ is the row vector

$$(\varphi^{(s)}(T_1), \varphi^{(s)}(T_2), \varphi^{(s)}(T_3), \varphi^{(s)}(T_4)) .$$

In the general case, if we agree to denote

$$P^{(S)} = (\varphi^{(s)}(T_1), \varphi^{(s)}(T_2), \ldots \varphi^{(s)}(T_N))$$

for some predetermined enumeration of the N trees in $\mathcal{F}^{(n)}$, we have

$$(3.14) \qquad \bar{P}^{(s+1)} = \bar{P}^{(s)} \mathcal{M} \qquad (s \geq 0)$$

and

$$(3.15) \qquad \bar{P}^{(s)} = \bar{P}^{(0)} \mathcal{M}^s \qquad (s \geq 0) .$$

$\mathcal{M}$ is a sparse matrix, the number of positive elements in each row being at most $n+1$, while N is very large (consult [2]). However it is easy to see that

(3.16) $\mathcal{M}^n$ is a positive matrix.

This is deduced from the fact that D_0 gives a positive probability of reaching any tree T_1 in n steps, regardless of what the original tree T_0 was.

To see this, we refer to [1] where it is shown that any p-tree may be created by selecting an appropriate permutation of the numbers $1,2,\ldots,n$ and then performing n successive insertions using ρ. (Conversely, picking any permutation, performing n successive insertions using ρ, of course gives us a p-tree.) Let therefore $(a_1,a_2,\ldots,a_n)$ be a permutation of of the numbers $1,2,\ldots,n$ corresponding to the tree T_1. Select the x 's to be inserted: $x_1,x_2,\ldots,x_n$ into T_0 in such a manner that at any stage the inserted nodes are larger than those from T_0, maintaining the order according to the permutation $(a_1,a_2,\ldots,a_n)$. The tree T_1 will then gradually be built in the upper part of the tree while the original nodes will be removed one by one.

Since $\mathcal{M}^n$ is a positive matrix, $\mathcal{M}$ is a regular matrix in the terminology of Markov chain theory (see for example [4]). The famous ergodic theorem of Markov chain theory then gives the following statements:

(3.1.6)

'There exists a uniquely defined p-tree forest

$$S = \{\mathcal{F}^{(n)}, \Psi\}$$

with

$$\Psi = (\Psi(T) \mid T \in \mathcal{F}^{(n)})$$

such that $F^{(s)} \xrightarrow[s \to \infty]{} S$ in the sense that

$$\lim_{s \to \infty} \sum_{T \in \mathcal{F}^{(n)}} |\varphi^{(s)}(T) - \Psi(T)| = 0 .$$

The probability vector $\bar{P}$ of Ψ is defined by

$$\bar{P} = \bar{P}\mathcal{M} \quad , \quad \sum \bar{P} = 1 .$$

S is independent of F .

<u>Example 3.1.2.</u> To find S for $n = 4$ we have to solve the equations:

$$P_1 = \frac{}{3} P_1 + \frac{}{3} P_2 + \frac{3}{5} P_4$$

$$P_2 = \frac{1}{5} P_1 + \frac{}{5} P_2 + \frac{}{5} P_3 + \frac{}{5} 1 1 \frac{1}{} P_4$$

$$P_3 = \frac{\;}{5} P_1 + \frac{\;}{15} P_2 + \frac{2}{\;} P\ 3$$

$$P_4 = \frac{2}{5} P_3 + \frac{1}{5} P\ 1\ _{5\ \ 4}$$

$$1 = P_1 + P_2 + P_3 + P_4 .$$

The first four of these equations have a determinant equal to 0 , as the column sums in $\mathfrak{M}$ are all 1 . We find, for example,

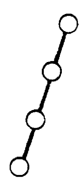
$$P_1 + P_2 + P_3 + P_4 = 1$$

$$-\frac{2}{5} P_1 + \frac{3}{5} P_2 \qquad + \frac{3}{5} P_4 = 0$$

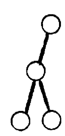
$$\frac{1}{5} P_1 + \frac{4}{5} P_2 + \frac{1}{5} P_3 + \frac{1}{5} P_4 = 0$$

$$\frac{1}{5} P_1 + \frac{1}{5} P_2 - \frac{3}{5} P_3 \qquad = 0$$

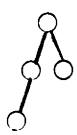
giving us S for n = 4 :



$Y = \frac{7}{15}$



$Y = \frac{1}{5}$



$\Psi = \frac{2}{9}$

$\Psi = \frac{1}{9}$.

S depends <u>only</u> on $\mathfrak{M}$, defined by Δ , and it is therefore characterized by Δ , D_0 , ρ and the number n . We will call S the <u>stationary p-tree forest</u> (of degree n).

Since starting with $F = F_0$, the normal p-tree forest, will maintain the BPP (basic p-tree property (see (2.3.3) and (2.3.4)) for all $F^{(s)}$, it is easy to see that

$$S = \lim_{s \to \infty} F^{(s)}$$

must have the BPP:

(3.1.9) The stationary p-tree forests have the basic p-tree property.

We also see that the characteristic left path polynomials (CLPP) of the $F^{(s)}$'s must approach the CLPP of the stationary forest, in the sense that each coefficient in the CLPP of the $F^{(s)}$'s approaches the corresponding coefficient in the CLPP of S. We shall later on establish a transition matrix, corresponding to $\mathcal{M}$ for the coefficients of the CLPP's.

3.2 Defining $S_1^{(n)}$.

Prom now on we will assume that $n > 2$. We shall be interested only in the process of alternating insert/remove so we define:

(3.2.1)
$$\Delta_1(2t-1) = I \qquad t = 1,2,\dots$$
$$\Delta_1(2t) = R \qquad t = 1,2,\dots .$$

using D_0, ρ and the initial forest $F_0^{(n)}$ (the normal p-tree forest)

(3.2.2) The p-tree forest at time 2t (t = 0,1,2,...) will be denoted

$$F_1^{(t)} = \{\mathcal{F}^{(n)}, \Phi^{(t)}\}$$

and at time 2t+1 (t = 0,1,2,...)

$$G_1^{(t)} = \{\mathcal{F}^{(n+1)}, \Psi^{(t)}\} \quad .$$

According to the results of the previous sections

(3.2.3) $F_1^{(t)}$ tends to a limit, the stationary p-tree forest, having the BPP, and n elements, denoted by

$$S_1^{(n)} \quad .$$

It is not hard to see that the sequence $G_1^{(1)}, G_1^{(2)}, \ldots$ also approaches a limit, viz. that obtained by inserting a node in $S_1^{(n)}$, denoted here by $T_1^{(n+1)}$.

Eventually we will be interested in the average left path lengthy the number of key comparisons to insert x in $S_1^{(n)}$, etc. Using the methods of Section 2.4 we will need the CLPP's of the forests. We will denote

CLPP of $F_1^{(t)}$ by:

$$H_1^{(t)}(z,w) = \sum_{1<b<a\leq n} \alpha_{a,b}^{(t)} z^a w^b \tag{3.2.4}$$

CLPP of $G_1^{(t)}$ by:

$$I_1^{(t)}(z,w) = \sum_{1<b<a\leq n+1} \beta_{a,b}^{(t)} z^a w^b \tag{3.2.5}$$

CLPP of $S_1^{\prime(n)}$ by:

(3.2.6) $$A_1^{(n)}(z,w) = \sum_{1 \leq b < a \leq n} \eta_{a,b}^{(n)} z^a w^b$$

and finally the CLPP of $T_1^{\prime(n)}$ by:

(3.2.7) $$B_1^{(n+1)}(z,w) = \sum_{1 \leq b < a \leq n+1} \mu_{a,b}^{(n+1)} z^a w^b \quad .$$

The interpretation of the α's, β's, η's and μ's (see 2.4.4), together with the statement (3.1.6), makes it easy to see that

(3.2.8) $$\lim_{t \to \infty} H_1^{(t)}(z,w) = A_1^{(n)}(z,w)$$

and

(3.2.9) $$\lim_{t \to \infty} I_1^{(t)}(z,w) = B_1^{(n+1)}(z,w) \quad .$$

In the next section we will establish relations between these CLPP.

3.3 Relations Between the CLPP's.

Suppose we have any p-tree forest X with the BPP and the CLPP:

$$H(z,w) = \sum_{1 \leq b < a \leq n} r_{a,b} z^a w^b$$

(X having n nodes).

We will establish the CLPP's of three p-tree forests X_1^*, X_2^* and X_3^* as functions of H(z,w) :

-- X_1^* is the result of one single insertion in X . X_1^* has n+1 nodes.

-- X_2^* is the result of one single removal in X . X_2^* has n-1 nodes.

-- X_3^* is the result of the combination of an insertion and a removal in succession.

In terms of the notations in the previous section, we then will have:

-- if X is $F^{(t)}$ $(t > 0)$, then X_1^* is $G^{(t)}$ and X_3^* is $F^{(t+1)}$

-- if X is $G^{(t)}$ then X_2^* is $F^{(t+1)}$

-- if X is $S^{(n)}$, then X_1^* is $T^{(n+1)}$ and X_3^* is $S^{(n)}$

-- if X is $T^{(n)}$, then X_2^* is $S^{(n-1)}$.

Having established the equations for the **CLLP's** for X_1^* , X_2^* and X_3^* below we may therefore concentrate on one single relation, viz. the one arising from the relation

(3.3.1) if X is $S^{(n)}$ then X_3^* is $S^{(n)}$;

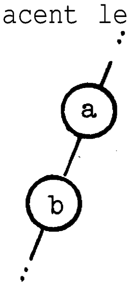
as we indeed will in Chapter 4.

Below a and b are integers satisfying $1 \leq b < a \leq n$, and T is some tree in $\mathcal{F}^{(n)}$.

<u>Case 1. X_1^*</u>

The CLPP of X_1^* will be denoted $H_1^*(z,w)$. Let T have a and b as values of two adjacent left path nodes:

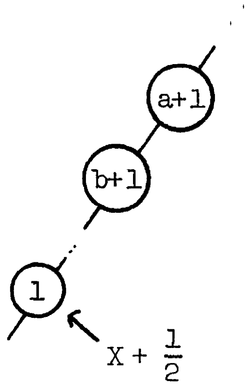
T:



The tree T* resulting from T is then:

if $x < 1$:

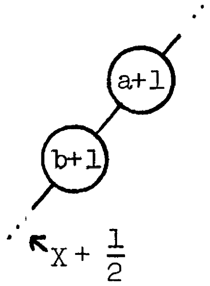
T^*



new left leaf

if $1 < x < b$

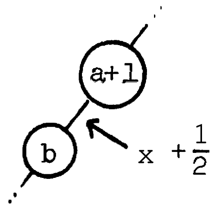
T^*



no new nodes on left path

if $b < x < a$

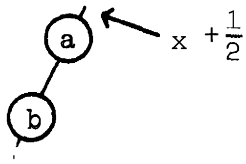
T^*



no new nodes on left path

if $a < x < n$

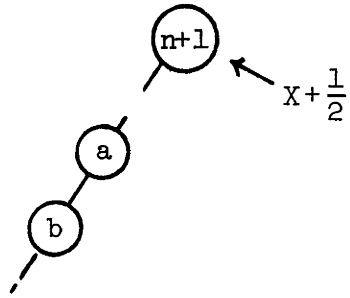
T^*



no new nodes on left path

if $n<x$

T^*



new root.

Summing over the entire forest we see:

$$
(3.3.2) \quad \left| \begin{array}{l} H_1^*(z,w) = \frac{1}{n+1} z^2 w^1 + \frac{1}{n+1} z^{n+1} w^n \\ \\ \sum_{1 \le b < a \le n} r_{a,b} \left(\frac{b}{n+1} z^{a+1} w^{b+1} + \frac{a-b}{n-1} z^{a+1} w^b + \frac{n+1-a}{n+1} z^a w^b \right) \end{array} \right.
$$

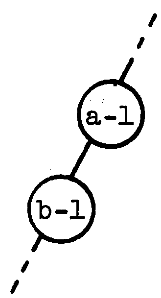
Case 2. X_2^*

The CLPP of X_2^* will be denoted $H_2^*(z,w)$ and we recall the CLPP's of the normal p-tree forests: $H_0^{(k)}(z,w)$ from (2.4.17).

As in the previous case we assume T to have a and b as adjacent **left** path node values:

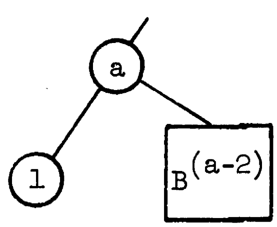
If $1 < b$ we get

T^*

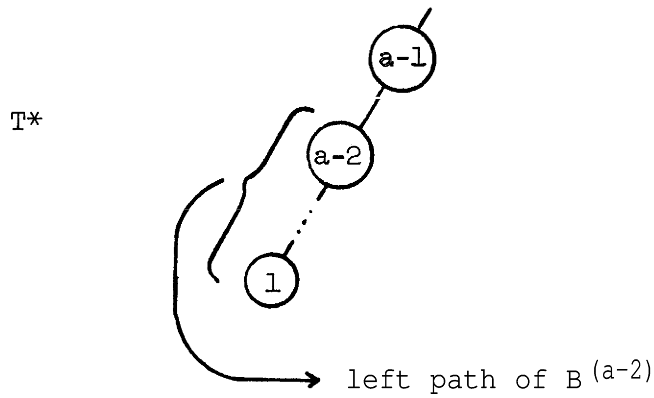


If $b = 1$ we have

T



where $B^{(a-2)}$ is the last right subtree of the left path having a-2 nodes $(a \geq 2)$. As $B^{(a-2)}$ is appended to the left path, we will have the new left path of T^* :



Summing over the total forest X , recalling that X has the BPP, we find

$$(3.3.3) \quad \begin{aligned} H_2^*(z,w) &= \sum_{2 \leq b < a \leq n} r_{a,b}\, z^{a-1} w^{b-1} \\ &+ \sum_{2 \leq a \leq n} r_{a\,1} \left(z^{a-1} w^{a-2} + H^{(a-2)}(z,w) \right) . \end{aligned}$$

(The latter formula being justified by the conventions made earlier:

$$H^{(0)}(z,w) = -z$$

$$H^{(1)}(z,w) = 0 \quad .)$$

Case 3. X_3^*

In this case we could use "geometric" considerations as in the two previous cases. However we may establish H_3^* by means of (3.3.2) and (3.3.3) .

Introducing the convention

$$r_{a,b} = 0 \qquad \text{if } (a,b) \notin \{(c,d) \mid 1 \leq d < c < n\}$$

we have from (3.3.2)

$$H_1^*(z,w) = \sum_{1 \leq b < a \leq n+1} \frac{z^a w^b}{(n+1)} ((b-1)r_{a-1,b-1} + (a-1-b)r_{a-1,b} + (n+1-a)r_{a,b})$$

$$+ \frac{1}{n+1} z^2 w^1 + \frac{1}{n+1} z^{n+1} w^n \quad .$$

Inserting this in (3.3.3) we obtain

$$(3.3.4) \quad \left| \begin{array}{l} H_3^*(z,w) = \displaystyle\sum_{1<b<a<n} z^a w^b \left(\frac{1}{n+1} (b\, r_{a,b} + (a-b-1)r_{a,b+1} + (n-a)r_{a+1,b+1}) \right) \\ \\ + \displaystyle\sum_{2 \leq a \leq n} \frac{1}{n+1} (z^a w^{a-1} + H_0^{(a-1)}(z,w))((a-1)r_{a,1} + (n-a)r_{a+1,1}) \\ \\ + \dfrac{1}{n+1} z^n w^{n-1} \quad . \end{array} \right.$$

3.4 The CLPP of $S_1^{(n)}$.

From (3.3.1) and (3.3.4) we obtain the following polynomial identity for the CLPP of the stationary p-tree forest $S_1^{(n)}$, using the notation (3.2.6):

$$
(3.4.1)\quad
\begin{aligned}
&\sum_{a=2}^{n}\ \sum_{b=1}^{a-1} \eta_{a,b}^{(n)} z^a w^b \ = \ A_1^{(n)}(z,w) \\
&= \frac{1}{n+1} \cdot \sum_{a=2}^{n}\ \sum_{b=1}^{a-1} z^a w^b \left(b\,\eta_{a,b}^{(n)} + (a-b-1)\ \eta_{a,b+1}^{(n)} + (n-a)\ \eta_{a+1,b+1}^{(n)} \right) \\
&= \frac{1}{n+1} \cdot \sum_{a=2}^{n} \left((a-1)\eta_{a,1}^{(n)} + (n-a)\eta_{a+1,1} \right)\left(z^a w^{a-1} + H_0^{(a-1)}(z,w) \right) \\
&+ \frac{1}{n+1} z^n w^{n-1}
\end{aligned}
$$

Here we recall

$$
(3.4.2)\quad
\begin{aligned}
&H_0^{(k)} = \sum_{a=2}^{k}\ \sum_{b=1}^{a-1} \left(\frac{1}{(k+1-b)(k-b)} + \frac{1}{(a-1)a} \right) z^a w^b \qquad (k \geq 2) \\
&H_0^{(0)} = 0 \qquad (k = 1)
\end{aligned}
$$

and the convention:

$$
(3.4.3)\quad \eta_{a,b}^{(n)} = 0 \qquad \text{if } (a,b) \notin \{(r,s) \mid 1 \leq s < r \leq n\} \ .
$$

(3.4.1) is in fact a set of $M = \frac{n\cdot(n-1)}{2}$ simultaneous linear equations in the M variables

$$
\eta_{a,b}^{(n)} \qquad (1 \leq b < a \leq n) \ .
$$

The uniqueness of the solutions follows from the existence of $S^{(1)}$, but we could also prove it directly from (3.4.1).

The solution of 3.4.1 for the first few n's proves to be:

n = 2

a \ b	1
2	1

n = j

a \ b	1	2
2	$\frac{3}{4}$	
3	$\frac{1}{4}$	$\frac{3}{4}$

n = 4

a \ b	.12		3
2	$\frac{31}{45}$		
3	$\frac{1}{5}$	$\frac{7}{15}$	
4	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{3}$

n = 5

a \ b	1	2	3	4
2	$\frac{2584}{N}$			
3	$\frac{698}{N}$	$\frac{1555}{N}$		
4	$\frac{363}{N}$	$\frac{624}{N}$	$\frac{1443}{N}$	
5	$\frac{243}{N}$	$\frac{405}{N}$	$\frac{810}{N}$	$\frac{2430}{N}$

N = 3888

There seems to be no simple solution to (3.4.1), except for:

$$\eta_{n,1}^{(n)} = \frac{1}{(n-1)^2}$$

For example, we may show the general formulae:

$$\eta^{(n)}_{n-1,1} = \frac{1}{(n-1)(n-2)} \left[1 + \frac{\frac{1}{3}}{\binom{n}{1}-1} + \frac{\frac{4}{3}}{(\binom{n}{1}-1)(\binom{n}{2}-1)} \right]$$

$$\eta^{(n)}_{n-2,1} = \frac{1}{(n-1)^3} \left[n - \frac{4}{3} + \frac{\frac{2}{3}}{((p-1)} + \frac{\frac{8}{3}\,n-1}{(\binom{n}{1}-1)(\binom{n}{2}-1)} + \frac{2n^2 - 2n + 3}{(\binom{n}{1}-1)(\binom{n}{2}-1)(\binom{n}{3}-1)} \right]$$

We will therefore settle for approximate solutions. Fortunately we need not have solutions for all $\eta^{(n)}_{a,b}$ to establish the quantities described in Section 2.4, it will turn out in Section 5 that in order to establish L , S , R , RL and C for $S^{(n)}_1$ we need only the values of the corresponding quantities for $F^{(k)}_0$ $(k \geq 0)$ and the $\eta^{(n)}_{a,1}$'s and the $\eta^{(n)}_{n,b}$'s.

In the next chapter we shall deduce from (3.4.1) an equation for the $\eta^{(n)}_{a,1}$'s and find an approximate solution for them.

4. Approximate Probabilities for the Next Last Node Value on Left Paths of $S_1^{'(n)}$.

4.1 Summary.

In this chapter we will prove the following formulae to be true.

Proposition 4.1.1. We have

$$(4.1.1)\quad \eta_{t,1}^{(n)} = \frac{2}{3t(t-1)} + \frac{1}{3n(n-1)} + \eta_{t,1}^{(n)} \cdot O\left(\frac{1}{n}\right) \qquad (4 \leq t \leq n)$$

$$(4.1.2)\quad \eta_{3,1}^{(n)} = \frac{1}{9} + \frac{2}{3}\,\frac{H_n}{n} - \left(\frac{8}{27} + \frac{4}{3} H_n^{(2)}\right)\frac{1}{n} + O\left(\frac{H_n}{n^2}\right)$$

$$(4.1.3)\quad \eta_{2,1}^{(n)} = \frac{2}{3} - \frac{2}{3}\,\frac{H_n}{n} + \frac{20}{9n} + O\left(\frac{H_n}{n^2}\right) .$$

H_n and $H_n^{(2)}$ are the harmonic numbers:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

$$H_n^{(2)} = \sum_{k=1}^{n} \frac{1}{k^2} .$$

The $O(f(n))$ notations should be interpreted as follows:

$$g(n) = O(f(n))$$

iff there exists a constant that

$$|g(n)| < M|f(n)| \qquad \text{for all } n = 1,2,\ldots .$$

During this and the following chapter we will make extensive use of standard formulae from combinatorics and discrete mathematics, referring for example to [5].

Please notice the difference between the $\eta^{(n)}_{t,1}$'s above and the corresponding probabilities in the normal p-tree forest:

$$\frac{1}{t(t-1)} + \frac{1}{n(n-1)} \quad .$$

4.2 Linear Equations Involving Only $\eta^{(n)}_{a,1}$'s.

The goal of this section is to prove

$$\sum_{a=2}^{n} z^{a-2} \eta^{(n)}_{a,1} \binom{n-2}{a-2} n\cdot(n-1) \equiv (z+1)^{n-2}$$

$$(4.2.1) \quad + \sum_{a=2}^{n} (\eta^{(n)}_{a,1}(a-1) + \eta^{(n)}_{a+1,1}(n-a))(n-a+1)(z+1)^{a-2}$$

$$+ \sum_{a=2}^{n} (\eta^{(n)}_{a,1}(a-1) + \eta^{(n)}_{a+1,1}(n-a))(X^{(a-1)}(z) + Y^{(a-1)}(z)) \quad ,$$

where $\eta^{(n)}_{n+1,1} = 0$ and

$$(4.2.2) \quad X^{(k)}(z) = \sum_{r=2}^{k} \sum_{s=1}^{r-1} \frac{(z+1)^{s-1} z^{r-s-1}}{(k+1-s)(k-s)} (n-r+1)\binom{n-s}{n-r+1}$$

$$(4.2.3) \quad Y^{(k)}(z) = \sum_{r=2}^{k} \sum_{s=1}^{r-1} \frac{(z+1)^{s-1} z^{r-s-1}}{(r-1)r} (n-r+1)\binom{n-s}{n-r+1}$$

for $1 \leq k-$ (k = 1 leaves empty sums, being 0).

In (3.4.1) we multiply both sides with (n+1) and move the double sum to the left, obtaining the equation:

$$(4.2.4)\quad \begin{aligned} &\sum_{a=2}^{n}\sum_{b=1}^{a-1} z^a w^b\left((n+1-b)\eta_{a,b}^{(n)} - (a-b-1)\eta_{a,b+1}^{(n)} - (n-a)\eta_{a+1,b+1}^{(n)}\right) \\ &= z^n w^{n-1} + \sum_{a=2}^{n}\left((a-1)\eta_{a,1}^{(n)} + (n-a)\eta_{a+1,1}^{(n)}\right)\left(z^a w^{a-1} + H_0^{(a-1)}(z,w)\right). \end{aligned}$$

We introduce the new quantity:

$$(4.2.5)\quad \sigma_{a,b}^{(n)} = (n-a+2)(n-a+1)\binom{n-b+1}{n-a+2}\eta_{a,b}^{(n)} \qquad \text{for all } a, b .$$

The left hand side coefficients then transform to (provided $1 \le b < a-2 \le n-3$) ,

$$(n+1-b)\,\eta_{a,b}^{(n)} - (a-b-1)\,\eta_{a,b+1}^{(n)} - (n-a)\,\eta_{a+1,b+1}$$

$$= \frac{\sigma_{a,b}^{(n)}(n-b+1)}{(n-a+1)(n-a+2)\binom{n-b+1}{n-a+2}} - \frac{\sigma_{a\ b+1}(a-b-1)}{(n-a+1)(n-a+2)\binom{n-b}{n-a+2}} \quad \frac{\sigma_{a+1,b+1}(n-a)}{(n-a)\ (n-a+1)\ \binom{n-b}{n-a+1}}$$

$$= \frac{1}{(n-a+1)\binom{n-b}{n-a+1}}\left(\sigma_{a,b} - \sigma_{a,b+1} - \sigma_{a+1,b+1}\right) .$$

This transformation is easily checked to be valid for the cases $a = n$, and $b = a-2$ or $b = a-1$ with $2 \le a \le n$ also.

We use this result in (4.2.4) and multiply each term $z^a w^b$ with

$$(n-a+1)\binom{n-b}{n-a+1}$$

obtaining

$$
(4.2.6)\quad \left|\begin{aligned}
&\sum_{a=2}^{n}\sum_{b=1}^{a-1} z^a w^b(\sigma^{(n)}_{a,b} - \sigma^{(n)}_{a,b+1} - \sigma^{(n)}_{a+1,b+1}) \\
&= z^n w^{n-1} + \sum_{a=2}^{n}((a-1)\eta^{(n)}_{a,1} + (n-a)\eta^{(n)}_{a+1,1})((n-a+1)z^a w^{a-1} + K^{(a-1)}(z,w))
\end{aligned}\right.
$$

where

$$
(4.2.7)\quad K^{(k)}(z,w) = \sum_{r=2}^{k}\sum_{s=1}^{r-1} \frac{1}{(k+1-b)\cdots} + \frac{1}{(a-1)\cdot a}\,(n+1-r)\binom{n-s}{n-r+1} z^r w^s .
$$

Now,

$$
\sum_{a=2}^{n}\sum_{b=1}^{a-1} z^a w^b(\sigma^{(n)}_{a,b} - \sigma^{(n)}_{a,b+1} - \sigma^{(n)}_{a+1,b+1})
$$

$$
= \sum_{a=2}^{n}\sum_{b=1}^{a-1} z^a w^b\,\sigma^{(n)}_{a,b} - \sum_{a=3}^{n}\sum_{b=2}^{a-1} z^a w^{b-1}\,\sigma^{(n)}_{a,b} - \sum_{a=3}^{n}\sum_{b=2}^{a-1} z^{a-1} w^{b-1}\,\sigma^{(n)}_{a,b}
$$

$$
= \left(1 - \frac{1}{w} - \frac{1}{zw}\right)\sum_{1\le b<a\le n} z^a w^b\,\sigma^{(n)}_{a,b} + \left(1 + \frac{1}{z}\right)\sum_{a=2}^{n} z^a\,\sigma^{(n)}_{a,1}
$$

so that by putting $w = \frac{z+1}{z}$, followed by division of $z\cdot(z+1)$ we obtain from (4.2.6)

$$
\sum_{a=2}^{n} z^{a-2}\,\sigma_{a,1} = (z+1)^{n-2}
$$

$$
+ \sum_{a=?}^{n}((a-1)\eta^{(n)}_{a,1} + (n-a)\eta^{(n)}_{a+1,1})\left((1-a+1)(z+1)^{a-2} + \frac{K^{(a-1)}\left(z, \frac{z+1}{z}\right)}{z(z+1)}\right)
$$

We have from (4.2.7)

$$\frac{K^{(k)}\left(z, \frac{z+1}{z}\right)}{z(z+1)} = \sum_{r=2}^{k} \sum_{s=1}^{r-1} \left(\frac{1}{(k+1-b)\ \ (k-b)} + \frac{1}{(a-1)a} \right)(n-r+1) \binom{n-s}{n-r-1} (z+1)^{s-1} z^{r-s-1}$$

$$= X^{(k)}(z,w) + Y^{(k)}(z,w)$$

so we arrive at (4.2.1), having realized from (4.2.5):

$$\sigma_{a,1}^{(n)} = (n-a+2)(n-a+1)\binom{n}{n-a+2}\eta_{a,1}^{(n)} = \binom{n-2}{a-2} n(n-1)\eta_{a,1}^{(n)} \quad .$$

4.3 <u>Properties of $X^{(k)}(z)$ and $Y^{(k)}(z)$</u> .

The complexity of (4.2.1) is primarily due to the sums involving the functions $X^{(k)}(z)$ and $Y^{(k)}(z)$ as defined in (4.2.2) and (4.2.3). In this section we shall concentrate on simplifying these polynomials.

We will make use of the following differential operator:

$$(4.3.1) \quad \mathcal{A}_j = \left\{ j - (z+1) \quad \frac{d}{dz} \right\} \quad ; \quad 0 \leq j$$

so that $\mathcal{A}_j$ applied to a function $f(z)$ is

$$\mathcal{A}_j f(z) = j\ f(z) - (z+1)\ \frac{df(z)}{dz} \ .$$

In particular we will make use of

$$(4.3.2) \quad \mathcal{A}_j (z+1)^i = (j-i)(z+1)^i \qquad \text{(all } i \text{)}.$$

This section contains the proof of the following three statements, all valid for $1 \leq k \leq n-1$:

$$(4.3.3)\quad a_{n-2}\; X^{(k)}(z) = \sum_{s=2}^{k} z^{s-2}\left(\binom{n}{s} - (n-k)\binom{k-1}{s-1} - \binom{k}{s}\right)$$

$$(4.3.4)\quad a_{n}\; Y^{(k)}(z) = \sum_{s=2}^{k} z^{s-2}\binom{n}{s}$$

$$(4.3.5)\quad a_{n-2}\left(X^{(k)}(z) + Y^{(k)}(z)\right) = a_{n-1}\left(2\,Y^{(k)}(z) - \frac{(z+1)^{k-1} - 1}{z}\right) .$$

From (4.2.2) we obtain:

$$X^{(k)}(z) = \sum_{s=1}^{k-1} \frac{(z+1)^{s-1}(n-s)}{(k-s)(k-s+1)} \sum_{r=s+1}^{k} z^{r-s-1}\binom{n-s-1}{r-s-1} .$$

The inner sum of this expression may be written as a polynomial in (z+1) :

$$\sum_{r=s+1}^{k} z^{r-s-1}\binom{n-s-1}{r-s-1} = \sum_{r=s+1}^{k}\sum_{t=0}^{r-s-1} \binom{n-s-1}{r-s-1}\binom{r-s-1}{t}(z+1)^{t}(-1)^{r-s-t-1}$$

$$= \sum_{t=0}^{k-s-1} (z+1)^{t} \sum_{r=t+s+1}^{k} \binom{n-s-1}{t}\binom{n-s-t-1}{r-s-t-1}(-1)^{r-s-t-1}$$

$$= \sum_{t=0}^{k-s-1} (z+1)^{t}\binom{n-s-1}{t}\binom{n-s-t-2}{k-s-t-1}(-1)^{k-s-t-1}$$

so that

$$X^{(k)}(z) = \sum_{s=1}^{k-1}\sum_{t=0}^{k-s-1} \frac{(z+1)^{s+t-1}(n-s)}{(k-s)(k-s+1)}\binom{n-s-1}{t}\binom{n-s-t-2}{k-s-t-1}(-1)^{k-s-t-1} .$$

Now :

$$\binom{n-s-1}{t}\binom{n-s-t-2}{k-s-t-1} = \binom{n-s-1}{n-s-t-1}\binom{n-s-t-1}{k-s-t}\frac{k-s-t}{n-s-t-1}$$

$$= \binom{n-s-1}{n-s-t-1}\binom{n-s-t-1}{n-k-1}\frac{k-s-t}{n-s-t-1}$$

$$= \binom{n-s-1}{n-k-1}\binom{k-s}{k-s-t}\frac{k-s-t}{n-s-t-1}$$

$$= \binom{n-s}{n-k-1}\frac{k-s+1}{n-s}\cdot\binom{k-s-1}{k-s-t-1}\frac{k-s}{n-s-t-1}$$

so that

$$x^{(k)}(z) = \sum_{s=1}^{k-1}\sum_{t=0}^{k-s-1}\frac{(z+1)^{s+t-1}\,-1^{k-s-t-1}}{(n-s-t-1)}\binom{n-s}{n-k-1}\binom{k-s-1}{k-s-t-1} .$$

Applying a_{n-2} to $(z+1)^{s+t-1}$ we obtain

$$a_{n\ 2}(z+1)^{s+t-1} = (n-s-t-1)(z+1)^{s+t-1}$$

so that

$$a_{n-2}\,X^{(k)}(z) = \sum_{s=1}^{k-1}\binom{n-s}{n-k-1}(z+1)^{s-1}\sum_{t=0}^{k-s-1}(z+1)^{t}(-1)^{k-s-1-t}\binom{k-s-1}{t}$$

$$= \sum_{s=1}^{k-1}\binom{n-s}{n-k-1}(z+1)^{s-1}z^{k-s-1}$$

$$= \sum_{s=1}^{k-1}\sum_{t=0}^{s-1} z^{(k-s-1)+(s-t-1)}\binom{n-s}{n-k-1}\binom{s-1}{t}$$

$$= \sum_{t=o}^{k-s} z^{k-t-2}\sum_{s=t+1}^{k-1}\binom{n-s}{n-k-1}\binom{s-1}{t}$$

$$\sum_{t=0}^{k-2} z^{k-t-2}\left[\left(\sum_{s=t+1}^{k+1} \binom{n-s}{n-k-1}\binom{s-1}{t}\right) - \binom{n-k}{n-k-1}\binom{k-1}{t} - \binom{n-k}{n-k}\binom{k}{t}\right]$$

$$\sum_{t=0}^{k-2} z^{k-t-2}\left[\binom{n}{n-k+t} - (n-k)\binom{k-1}{k-t-1} - \binom{k}{k-t}\right] .$$

Changing the summation index to k-t we obtain (4.3.3).

From (4.2.3) we see:

$$Y^{(k)}(z) = \sum_{r=2}^{k} \frac{(n-r+1)}{r\cdot(r-1)} \sum_{s=1}^{r-1} z^{r-s-1}(z+1)^{s-1}\binom{n-s}{n-r+1} .$$

The inner sum may be transformed as follows:

$$\sum_{s=1}^{r-1} z^{r-s-1}(z+1)^{s-1}\binom{n-s}{n-r+1}$$

$$= \sum_{s=1}^{r-1} \sum_{t=0}^{s-1} \binom{s-1}{t} z^{s-t-1} z^{r-s-1}\binom{n-s}{n-r+1}$$

$$= \sum_{t=0}^{r-2} z^{r-t-2} \sum_{s=t+1}^{r-1} \binom{s-1}{t}\binom{n-s}{n-r+1}$$

$$= \sum_{t=0}^{r-2} z^{r-t-2}\binom{n}{n\ \ r+t+2}$$

$$= \sum_{t=0}^{r-2} z^{t}\binom{n}{t} .$$

Using this transformation, and applying $\mathcal{A}_n$ we obtain:

$$\mathcal{A}_n Y^{(k)}(z) = \sum_{r=2}^{k} \sum_{s=0}^{r-2} \frac{n-r+1}{r\cdot(r-1)} \binom{n}{s}(n z^{s} - (z+1)\, s\, z^{s-1})$$

$$= \sum_{r=2}^{k} \frac{n-r+1}{r\cdot(r-1)} \left[\sum_{s=0}^{r-2} n\cdot\binom{n-1}{s} z^{s} - \sum_{s=0}^{r-2} n\cdot\binom{n-1}{s-1} z^{s-1} \right|$$

$$= \sum_{r=2}^{k} \frac{n-r+1}{r\cdot(r-1)} \cdot n \cdot \binom{n-1}{r-2} z^{r-2}$$

$$= \sum_{r=2}^{k} \binom{n}{r} z^{r-2}$$

and (4.3.4) is proven.

From (4.3.3) and (4.3.4) we find:

$$\mathcal{a}_{n-2}(X^{(k)}(z) + Y^{(k)}(z)) = \sum_{s=2}^{k} z^{s-2}\binom{n}{s} - (n-k)\sum_{s=2}^{k} z^{s-2}\binom{k-1}{s-1} \sum_{s=2}^{k} z^{s-2}\binom{k}{s}$$

$$+ (\mathcal{a}_n Y^{(k)}(z)) \div 2\, Y^{(k)}(z)$$

$$= 2(\mathcal{a}_n Y^{(k)}(z)) - 2\, Y^{(k)}(z) - (n-k) \frac{(z+1)^{k-1} - 1}{Z} \frac{(z+1)^{k} - kz - 1}{z^2}.$$

Now :

$$(\mathcal{a}_n Y^{(k)}(z)) - Y^{(k)}(z) = a_{n-1} Y^{(k)}(z)$$

and

$$\mathcal{a}_{n-1} \frac{(z+1)^{k-1} - 1}{Z} = (n-1) \frac{(z+1)^{k-1} - 1}{z} - (z+1)\left[\frac{(k-1)(z+1)^{k-2}}{Z} \quad \frac{(z+1)^{k-1} - 1}{z^2}\right]$$

$$= (n-k) \frac{(z+1)^{k-1} - 1}{z} - \frac{(z+1)^{k} - kz - 1}{z^2}$$

proving (4.3.5).

4.4 Revision of Equation (4.2.1).

In this section we will obtain a simplified version of (4.2.1), using the results of Section 4.3. We will use a new notation for the **unknown** quantities $\eta^{(n)}_{a,l}$:

$$(4.4.1) \qquad \rho_a = \eta^{(n)}_{n-a,1} .$$

The main result of this section is:

$$(4.4.2) \qquad \left| \begin{aligned} &\sum_{r=0}^{n-2} (z^r(r+2) - z^{r+1} \quad r \quad) \sum_{s=0}^{r} \binom{r}{s} \binom{n-1}{r+1} n \rho_s (-1)^{r-s} \\ &= 2\left(\sum_{r=0}^{n-1} z^r \right) - (n - (n-1)\rho_{n-2}) z^{n-1} \\ &\quad + \sum_{r=0}^{n-2} (z^r(r+2) - z^{r+1}(r+1))(\rho_r(n-r-1) + \rho_{r-1}\, r) . \end{aligned} \right.$$

As before we use the convention

$$\rho_a = 0 \quad \text{if } a < 0 \text{ or } n < a .$$

We will split (4.2.1) into three sums

$$(4.4.3) \quad S_1 = \sum_{a=2}^{n} z^{a-2} \eta^{(n)}_{a,1} \binom{n-2}{a-2} n(n-1)$$

$$(4.4.4) \quad S_2 = (z+1)^{n-2}$$

$$(4.4.5) \quad S_3 = \sum_{a=2}^{n} (\eta^{(n)}_{a,1}(a-1) + \eta^{(n)}_{a+1,1}(n-a))((n-a+1)(z+1)^{a-2} + X^{(a-1)}_{a-1}(z) + Y^{(a-1)}(z))$$

s; that

$$(4.4.6) \qquad S_1 = S_2 + S_3 .$$

We also introduce the notation

$$(4.4.7) \qquad K_a = \sum_{s=a+2}^{n} \binom{n-a-2}{s-a-2} \eta^{(n)}_{s,1} (-1)^{s-a-2} .$$

In (4.3.1) we defined the differential operator $\mathcal{a}_j$. The corresponding integration operator will be denoted $\mathcal{J}_j$. We have

(4.4.8) $$\mathcal{A}_j \ \mathcal{T}_j \ f(z) = f(z)$$

(4.4.9) $$\mathcal{T}_j \ 0 = C(z+1)^j \qquad \text{(C constant)}$$

(4.4.10) $$\mathcal{T}_j(z+1)^i = \frac{1}{(j-i)}(z+1)^i + C(z+1)^j \qquad \text{(C constant and } i \neq j) .$$

We will apply the operator

(4.4.11) $$\mathfrak{w} = z^2 \ \mathcal{A}_n \ \mathcal{D}_{n-1} \ \mathcal{A}_{n-2}$$

to (4.2.1), and then rearrange the polynomials using (z+1) as variable.

For S_2 we find

(4.4.12) $$\begin{aligned} \mathfrak{w}\, S_2 &= C\ z^2(z+1)^{n-1} \\ &= C((z+1)^{n+1} - 2(z+1)^n + (z+1)^{n-1}) \end{aligned}$$

where the constant C is assumed to represent the integration constant for the entire equation.

For S_1 we find

$$S_1 = \sum_{a=2}^{n} \sum_{b=0}^{a-2} (z+1)^b \binom{a-2}{b} (-1)^{a-2-b} \binom{n-2}{a-2} \eta_{a,1}^{(n)} \ n(n-1)$$

$$= \sum_{b=0}^{n-2} (z+1)^b \sum_{a=b+2}^{n} \binom{n-2}{b} n(n-1) \binom{n-2-b}{a-2-b} (-1)^{a-2-b} \ \eta_{a,1}^{(n)}$$

$$= \sum_{a=0}^{n-2} (z+1)^a \binom{n-2}{a} n \cdot (n-1) K_a$$

and

$$\mathfrak{w}\, S_1 = \sum_{a=0}^{n-2} z^2 \left(\frac{(n-a-2)(n-a)}{(n-a-1)} \right) (z+1)^a \binom{n-2}{a} n(n-1) K_a$$

leading to

$$(4.4.13) \quad \mathfrak{w}\; S_1 = \sum_{a=0}^{n-2} \quad (z+1)^{a+2} - 2(z+1)^{a+1} + (z+1)^a)(n-a-2)(n-a)\binom{n-1}{a} n\, K_a \ .$$

For S_3 we will have to involve ourselves in more complicated calculations, (see (4.3.4) and (4.3.5)).

$$\mathfrak{w}((n-a+1)(z+1)^{a-2} + X_{a-1}(z) + Y_{a-1}(z))$$

$$= z^2 \left[(n-a+1) \frac{(n-a+2)(n-a)}{(n-a+1)} (z+1)^{a-2} \right] + z^2 \left(a_n \left(2\; Y_{a-1}(z) - \frac{(z+1)^{a-2} - 1}{z} \right) \right)$$

$$= (n-a+2)(n-a)(z+1)^{a-2}\; z^2 + 2 \sum_{s=2}^{a-1} z^s \binom{n}{s}$$

$$- \left(n \frac{(z+1)^{a-2} - 1}{z} - (z+1) \left(\frac{(a-2)(z+1)^{a-3}}{z} - \frac{(z+1)^{a-2} - 1}{z^2} \right) \right) z^2$$

$$= 2 \sum_{s=0}^{a-1} z^s \binom{n}{s} + (n-a)(n-a+2)(z+1)^{a-2}\; z^2 - (n-a+2)(z+1)^{a-2}\, z - (z+1)^{a-1}$$

$$+ n\, z + (z+1) - 2n\, z - 2 \ .$$

Hence we may write

$$\mathfrak{w}\; S_3 = u+v+w$$

where

$$(4.4.14) \qquad U = \sum_{a=2}^{n} x_a\, u_a$$

$$(4.4.15) \qquad v = \sum_{a=2}^{n} x_a\; v_a$$

$$(4.4.16)\quad W = \sum_{a=2}^{n} x_a w_a$$

where

$$x_a = (\eta^{(n)}_{a,1}(a-1) + \eta^{(n)}_{a+1,1}(n-a))$$

$$u_a = 2\sum_{s=o}^{a-1} z^s \binom{n}{s}$$

$$V_a = (n-a)(n-a+2)(z+1)^{a-2} z^2 - (n-a+2)(z+1)^{a-2} z - (z+1)^{a-1}$$

$$w_a = n\,z + (z+1) - 2\,n\,z - 2$$

Now

$$U_a = 2\sum_{s=o}^{a-1} z^s\binom{n}{s} = 2\sum_{s=o}^{a-1}\sum_{t=0}^{s}(z+1)^t(-1)^{s-t}\binom{n}{s}\binom{s}{t}$$

$$= 2\sum_{t=o}^{a-1}(z+1)^t\binom{n}{t}\sum_{s=t}^{a-1}\binom{n-t}{s-t}(-1)^{s-t}$$

$$= 2\sum_{t=0}^{a-1}(z+1)^t\binom{n}{t}\binom{n-t-1}{a-t-1}(-1)^{a-t-1}$$

$$= 2\sum_{t=o}^{a-1}(z+1)^t\binom{n-1}{n-1-t}\frac{n}{n-t}\binom{n-t-1}{n-a}(-1)^{a-t-1}$$

$$= 2\,n\binom{n-1}{a-1}\sum_{t=0}^{a-1}\binom{a-1}{t}(-1)^{a-t-1}(z+1)^t\,\frac{1}{(n-t)}$$

so that

$$u = \sum_{a=2}^{n} x_a \; u_a$$

$$= \sum_{a=2}^{n} \eta_{a,1}^{(n)}(a-1)2n\binom{n-1}{a-1} \sum_{t=0}^{a-1} \binom{a-1}{t}(-1)^{a-t-1}(z+1)^t \frac{1}{(n-t)}$$

$$+ \sum_{a=2}^{n} \eta_{a+1,1}^{(n)}(n-a)2n\binom{n-1}{a-1} \sum_{t=0}^{a-1} \binom{a-1}{t}(-1)^{a-t-1}(z+1)^t \frac{1}{(n-t)}$$

$$= 2n(n-1)\Bigg[\sum_{a=2}^{n} \sum_{t=0}^{a-1} \eta_{a,1}^{(n)}\binom{n-2}{a-2}\binom{a-1}{t}(-1)^{a-t-1}(z+1)^t \frac{1}{(n-t)}$$

$$+ \sum_{a=3}^{n} \sum_{t=0}^{a-2} \eta_{a,1}^{(n)}\binom{n-2}{a-2}\binom{a-2}{t}(-1)^{a-t-2}(z+1)^t \frac{1}{(n-t)} \Bigg]$$

$$= 2n(n-1)\Bigg[\sum_{a=2}^{n} \sum_{t=0}^{a-1} \eta_{a,1}^{(n)}\binom{n-2}{a-2}(-1)^{a-t-1}\binom{a-2}{t-1}(z+1)^t \frac{1}{n-t}$$

$$- \eta_{2,1}^{(n)}\binom{n-2}{2-2}\binom{2-2}{0}(-1)^{2-0-2}(z+1)^0 \Bigg] \frac{1}{n-0}$$

$$= 2n(n-1)\Bigg[\sum_{a=2}^{n} \sum_{t=0}^{a-2} \eta_{a,1}^{(n)}(-1)^{a-t-2} \frac{1}{n-t-1}\binom{n-2}{a-2}\binom{a-2}{t}(z+1)^{t+1} - 2(n-1)\eta_{2,1}$$

$$= 2n(n-1)\Bigg[\sum_{t=0}^{n-2} (z+1)^{t+1}\binom{n-2}{t} \frac{1}{n-t-1} \sum_{a=t+2}^{n} \eta_{a,1}^{(n)}(-1)^{a-t-2}\binom{n-2-t}{a-2-t} \Bigg] - 2(n-1)\eta_{2,1}^{(n)} .$$

And hence

$$u = 2n \sum_{a=0}^{n-2} (z+1)^{a+1}\binom{n-1}{a}K_a - 2(n-1)\eta_{2,1}^{(n)} . \tag{4.4.17}$$

For the sum W we find

$$w = \sum_{a=2}^{n} x_a \; w_a$$

$$= \sum_{a=2}^{n} x_a(-nz+z-1)$$

$$= (-nz+z-1)\left[\sum_{a=2}^{n} \eta_{a,1}^{(n)}(a-1) + \sum_{a=3}^{n} \eta_{a,1}^{(n)}(n-a+1)\right]$$

$$= (-nz+z-1)\left[\sum_{a=2}^{n} \eta_{a,1}^{(n)}\ n - \eta_{2,1}^{(n)}(n-1)\right]$$

Knowing that $\sum_{a=2}^{n} \eta_{a,1}^{(n)} = 1$ we obtain

$$(4.4.18) \quad w = (n-(n-1)\eta_{2,1})((n-2)-(n-1)(z+1)) .$$

Applying w to (4.4.6), inserting (4.4.12), (4.4.13), (4.4.17) and (4.4.18) we obtain equality between two polynomials where the maximum exponent of $(z+1)$ is $(n+1)$, occurring only in wS_2 (4.4.12). Hence the integration constant $C = 0$ and we have transformed (4.2.1) to the equivalent identity:

$$(4.4.19) \quad \begin{aligned} &\sum_{a=0}^{n-2}((z+1)^{a+2} - 2(z+1)^{a+1} + (z+1)^{a})(n-a-2)(n-a)\binom{n-1}{a} nK_a \\ &= \sum_{a=0}^{n-2}(z+1)^{a+1}\binom{n-1}{a} 2nK_a - 2(n-1)\eta_{2,1}^{(n)} + \sum_{a=0}^{n-2} x_a v_a \\ &+ (n-(n-1)\eta_{2,1}^{(n)})((n-2)-(n-1)(z+1)) \end{aligned}$$

where

$$x_a = (\eta_{a,1}^{(n)}(a-1) + \eta_{a+1,1}^{(n)}(n-a))$$

and

$$\begin{aligned} v_a &= (n-a)(n-a+2)(z+1)^{a-2} z^2 - (n-a+2)(z+1)^{a-2} z - (z+1)^{a-1} \\ &= (n-a)(n-a+2)(z+1)^{a} - 2(n-a+1)^2(z+1)^{a-1} + (n-a)(n-a+2)(z+1)^{a-2} \\ &\quad - (n-a+1)(z+1)^{a-1} + (n-a+2)(z+1)^{a-2} . \end{aligned}$$

In (4.4.19) the first sum on the right hand side is moved to the left hand side, and we use $\mathcal{J}_n$ throughout the identity to simplify the terms:

$$\mathcal{J}_n[((z+1)^{a+2} - 2(z+1)^{a+1} + (z+1)^a)(n-a-2)(n-a) - 2(z+1)^{a+1}]$$

$$= \mathcal{J}_n[(z+1)^{a+2}(n-a-2)(n-a) - 2(z+1)^{a+1}(n-a-1)^2 + (z+1)^a(n-a-2)(n-a)]$$

$$= [(z+1)^{a+2}(n-a) - 2(z+1)^{a+1}(n-a-1) + (z+1)^a(n-a-2)] + C(z+1)^n$$

$$= z[(z+1)^{a+1}(n-a) - (z+1)^a(n-a-2)] + C(z+1)^n$$

for some constant C .

Furthermore

$$\mathcal{J}_n v_a = (n-a+2)(z+1)^a - 2(n-a+1)(z+1)^{a-1} + (n-a)(z+1)^{a-2} - (z+1)^{a-1} + (z+1)^{a-2}$$

$$= z((n-a+2)(z+1)^{a-1} - (n-a+1)(z+1)^{a-2})$$

*(neglecting the integration constant).

Application of $\mathcal{J}_n$ to (4.4.19) hence yields

$$(4.4.20)\quad \begin{cases} \sum_{a=0}^{n-2} z[(z+1)^{a+1}(n-a) - (z+1)^a(n-a-2)]\, n\binom{n-1}{a} K_a \\ = -\frac{1}{n}2(n-1)\eta^{(n)}_{2,1} + (n - (n-1)\eta^{(n)}_{2,1})\left(\frac{n-2}{n} - (z+1)\right) \\ + \sum_{a=2}^{n} (\eta^{(n)}_{a,1}(a-1) + \eta^{(n)}_{a+1,1}(n-a))((n-a+2)(z+1)^{a-1} - (n-a+1)(z+1)^{(a-2)})z \\ + C(z+1)^n \end{cases}$$

The coefficients of z^n are seen to be

$$(n-(n-2))\ n\binom{n-1}{n-2}K_{n-2} = (\eta^{(n)}_{n,1}(n-1))\cdot(n-n+2) + C$$

Now, following (4.4.7)

$$K_{n-2} = \binom{n-(n-2)-2}{n-(n-2)-2} \eta_{n,1}^{(n)} (-1)^{n-(n-2)-2} = \eta_{n,1}^{(n)}$$

so

$$C = \eta_{n,1}[2n(n-1) - 2(n-1)] = 2(n-1)^2 \eta_{n,1}^{(n)} .$$

Going back to (4.2.1) easily gives us $\eta_{n,1}^{(n)}$:

$$\eta_{n,1}^{(n)} \binom{n-2}{n-2} n(n-1) = 1 + \eta_{n,1}^{(n)} (n-1)\,(n-n+1)$$

$$\eta_{n,1}^{(n)} = \frac{1}{(n-1)^2}$$

and hence $C = 2$ in (4.4.20).

We insert this last result in (4.4.20), divide by z , and then change our variable from $(z+1)$ to z , obtaining

$$\text{(4.4.21)} \quad \begin{aligned} &\sum_{a=0}^{n-1} [z^{a+1}(n-a) - z^a(n-a-2)]\, n \binom{n-1}{a} K_a \\ &\quad = \sum_{a=2}^{n} (\eta_{a,1}^{(n)}(a-1) + \eta_{a+1,1}^{(n)}(n-a))((n-a+2)z^{a-1} - (n-a+1)z^{a-2}) \\ &\quad\quad + 2\,\frac{z^n - 1}{z-1} - (n - (n-1)\eta_{2,1}) . \end{aligned}$$

Recalling the definition of the ρ_a 's in (4.4.1) we find from (4.4.2)

$$\sum_{a=0}^{n-2} (z^{a+1}(n-a) - z^a(n-a-2))\, n \binom{n-1}{a} K_a$$

$$= \sum_{a=0}^{n-2} (z^{a+1}(n-a) - z^a(n-a-2))\, n \binom{n-1}{a} \sum_{s=a+2}^{n} \binom{n-a-2}{s-a-2} \rho_{n-s} (-1)^{s-a-2}$$

$$= \sum_{a=0}^{n-2} (z^{a+1}(n-a) - z^a(n-a-2))\, n \binom{n-1}{n-1-a} \sum_{s=0}^{n-a-2} \binom{n-a-2}{s} \rho_s (-1)^{n-s-a-2}$$

$$= \sum_{a=0}^{n-2} (z^{n-a-1}(a+2) - z^{n-a-2}\, a)\, n \binom{n-1}{a+1} \sum_{s=0}^{a} \binom{a}{s} \rho_s (-1)^{a-s}$$

and

$$\sum_{a=2}^{n} (\eta^{(n)}_{a,1}(a-1) + \eta^{(n)}_{a+1,1}(n-a))((n-a+2)z^{a-1} - (n-a+1)z^{a-2})$$

$$= \sum_{a=0}^{n-2} ((a+2)z^{n-a-1} - (a+1)z^{n-a-2}) (\eta^{(n)}_{n-a,1}(n-a-1) + \eta^{(n)}_{n-a+1,1}\, a)$$

$$= \sum_{a=0}^{n-2} ((a+2)z^{n-a-1} - (a+1)z^{n-a-2})(\rho_a (n-a-1) + \rho_{a+1}\, a)$$

Inserting the two last results in (4.4.21), dividing by z^{n-1} and finally changing the variable to $1/z$ we obtain (4.4.2).

4.5 Series Expansion of the ρ_a's.

The polynomial equation (4.4.2) contains n equations and the (n-1) variables $(\rho_0, \ldots, \rho_{n-2})$. However, by putting z = 1 we will see that the equations are dependent. Furthermore, it is not hard to see that the equation obtained from the coefficients of z^{n-1} may be ruled out, leaving an independent set of linear equations.

In this section we shall obtain series expansions for the ρ_a's, making it possible for us to obtain approximate solutions.

The following facts are trivial.

$$(4.5.1)\quad \begin{cases} 0 \le \rho_a \le 1 & 0 < a \le n-2 \\ \rho_a = 0 & a < 0 \quad \text{or} \quad n < a \\ \sum_{a=0}^{n-2} \rho_a = 1 & \\ \rho_0 = \dfrac{1}{(n-1)^2} & \end{cases}$$

We shall prove the following proposition:

Proposition 4.5.1. Define for $1 \le t \le n-2$

$$(4.5.2)\quad \alpha_t^{(0)} = \frac{2n}{3(n-1)} \frac{t+2}{\binom{n}{t+1}}$$

$$(4.5.3)\quad \alpha_t^{(r+1)} = \frac{1}{t(t+1)\binom{n}{t+1}} \sum_{k=1}^{t} \sum_{j=1}^{k} \binom{k}{j} k(k+1)\alpha_j^{(r)} \qquad (0 \le r)$$

$$(4.5.4)\quad \delta_t^{(r)} = \sum_{j=1}^{t} j\,\alpha_j^{(r)} \frac{\binom{t}{j}}{(j+2)(n-j-1)} \qquad (0 \le r)$$

Then

$$(4.5.5)\quad \rho_t = \frac{1}{(n-1)^2} + \sum_{r=0}^{\infty} \delta_t^{(r)} \qquad 1 \le t \le n-2$$

$$(495.6)\quad 0 < \delta_t^{(r)} < \delta_t^{(0)}\left(\frac{5}{n}\right)^r \qquad 1 \le r \qquad 1 \le t \le n-4 .$$

The constant $\left(\frac{5}{n}\right)$ is uniform for $1 \le t \le n-4$, and is not very well optimized. As we shall see later, (4.5.6) does not hold for $t = (n-2)$ or $(n-3)$.

Proposition (4.5.2) below gives, for each $t = 0,1,2,\ldots,n-2$, ρ_t as a linear function of $\rho_1,\rho_2,\ldots,\rho_{t-1}$.

Proposition 4.5.2.

$$\rho_t = \frac{\binom{n}{a+1}}{\binom{n}{a+1}-1}\left(t + \sum_{u=0}^{t-1} \rho_u \beta_{a,u}\right) \qquad 0 \le t \le n-2 \tag{4.5.7}$$

where

$$\alpha_t = \frac{1}{3}\ \frac{2}{(n-t)}\ \frac{1}{(n-t+1)} + \frac{1}{n(n-1)} \qquad 0 \le t \le n-2$$

and

$$\beta_{t,u} = \frac{\binom{t}{u}}{\binom{n}{u+1}} + \frac{\binom{t}{u+1}(u+2)}{\binom{n}{u+2}(n-u-2)}$$

$$- \frac{(n+2)}{n}(u+1)(u+2)\sum_{r=u+1}^{t} \frac{\binom{t}{r}}{r\cdot(r+1)(r+2)\binom{n-1}{r+1}}$$

$$(0 \le u \le t-1,\ 1 \le t \le (n-2)).$$

Solutions of equations like (4.4.2) often involve one or more cleverly selected substitutions. In our case, the following sequence of substitutions are not unnatural choices:

$$(4.5.8)\quad \begin{cases} c_t = (n-t-1)\rho_t + t\ \rho_t \\ d_t = (t+2)c_t - t\ c_{t\ 1} \\ e_t = \sum_{j=0}^{t} d_j(-1)^{t-j}\binom{t}{j}. \end{cases}$$

The direct correspondence between the e_t's and the ρ_t's is seen to be:

$$e_t = \sum_{j=0}^{t} (-1)^{t-j} \binom{t}{j} (j+2) c_j - \sum_{j=0}^{t} (-1)^{t-j} \binom{t}{j} j \; c_{j \; 1}$$

$$= \sum_{j=0}^{t} (-1)^{t-j} \left(\binom{t}{j} (j+2) + \binom{t}{j+1} (j+1) \right) c_j$$

$$= (t+2) \sum_{j=0}^{t} (-1)^{t-j} \binom{t}{j} c_j$$

$$= (t+2) \left[\sum_{j=0}^{t} (-1)^{t-j} \binom{t}{j} (n-j-1) \rho_j + \sum_{j=0}^{t} (-1)^{t-j} \binom{t}{j} j \, \rho_{j-1} \right]$$

$$= (t+2) \sum_{j=0} (-1)^{t-j} \rho_j \binom{t}{j} ((n-j-1) - (t-j))$$

so

(4.5.9) $$e_t = (t+2)(n-t-1) \sum (-1)^{t-j} \binom{t}{j} \rho_j \; .$$

From (4.5.9) we easily deduce

(4.5.10) $$\rho_t = \sum_{j=0}^{t} e_j \frac{\binom{t}{j}}{(j+2) \; (n-j-1)} \qquad 0 \le t \le n-2 \; .$$

Inserting (4.5.8) into (4.4.2) we obtain

$$\sum_{r=0}^{n-2} (z^r (r+2) - z^{r+1} (r+1)) (\rho_r (n-4-1) + \rho_{r \; 1} \; r)$$

$$= \sum_{r=0}^{n-2} (z^r (r+2) - z^{r+1} (r+1)) c_r$$

$$= \sum_{r=0}^{n-2} z^r ((r+2) c_r - r \; c_{r-1}) - z^{n-1} (n-1) c_{n \; 2}$$

$$= \sum_{r=0}^{n-2} z^r d_r - z^{n-1}(n-1)c_{n\ 2}$$

$$= \sum_{r=0}^{n-2} z^r \sum_{t=o}^{r} \binom{t}{r} e_r - z^{n-1}(n-1)c_{n\ 2}$$

and from (4.5.9)

$$\sum_{s=o}^{r} \binom{n-1}{r+1}\binom{r}{s} n\rho_s(-1)^{r-s} = \frac{\binom{n-1}{r+1}n}{(r+2)(n-r-1} e_r$$

leading to

$$\sum_{r=0}^{n-2} (z^r(r+2) - z^{r+1}\ r)\binom{n+1}{r+2}\frac{1}{n+1} e_r$$

$$= 2\sum_{r=0}^{n-1} z^r + z^{n-1}((n-1)\rho_{n\ 2} - n - (n-1)c_{n-2}) + \sum_{r=0}^{n-2} z^r \sum_{t=0}^{r}\binom{r}{t}e_t \quad .$$

From which we obtain

$$(4.5.11) \quad \binom{n}{r+1}e_r - \frac{r-1}{r+1}\binom{n}{r}e_{r\ 1} = 2 + \sum_{t=o}^{r}\binom{r}{t}e_t \qquad 0 \le r \le n-2$$

when neglecting the terms z^{n-1} .

Multiplying each equation (4.5.11) with $r\cdot(r+1)$ and summing from 3 through s $(0 \le s \le n-2)$ we obtain

$$\sum_{r=0}^{s}\binom{n}{r+1}r(r+1)e_r - \sum_{r=0}^{s}\frac{(r-1)}{(r+1)}\binom{n}{r}e_{r-1}\ r(r+1) = \binom{n}{s+1}s(s+1)e_r$$

$$= 2\sum_{r=0}^{s} r(r+1) + \sum_{r=0}^{s}\sum_{t=0}^{r}\binom{r}{t}e_t\ r(r+1) \quad .$$

The first sum being $\frac{2}{3}\, s(s+1)(s+2)$ we see

$$(4.5.12)\quad \left| \begin{array}{l} e_s = \frac{2}{3} \frac{s+2}{\binom{n}{s+1}} + \frac{1}{s(s+1)\binom{n}{s+1}} \sum_{r=0}^{s} \sum_{t=0}^{r} \binom{r}{t} r(r+1) e_t \\ \text{if } (1 \le s \le n-2) \ . \end{array} \right.$$

As (4.5.9) gives

$$e_0 = \frac{2}{n-1}$$

we see that (4.5.12) leads to

$$(4.5.13)\quad e_s = \frac{2n}{3(n-1)} \frac{s+2}{\binom{n}{s+1}} + \frac{1}{s(s+1)\binom{n}{s+1}} \sum_{r=1}^{s} \sum_{t=1}^{r} \binom{r}{t} r(r+1) e_t \ .$$

From the definitions (4.5.2) and (4.5.3) we see that if we define

$$u_s^{(r)} = \sum_{a=0}^{r} \alpha_s^{(a)} \qquad (0 \le r)$$

we find

$$u_s^{(r+1)} = \alpha_0 + \sum_{a=1}^{r+1} a_s^{(")}$$

$$= \alpha_0 + \frac{1}{s\cdot(s+1)\binom{n}{s+1}} \sum_{k=1}^{t} \sum_{j=1}^{k} \binom{k}{j} k(k+1) \sum_{a=0}^{r} \alpha_j^{(a}$$

so

$$e_s - u_s^{(r+1)} = \frac{1}{s(s+1)\binom{n}{s+1}} \sum_{k=1}^{t} \sum_{j=1}^{t} \binom{k}{j} k(k+1)(e_j - u_j^{(r)}) \ .$$

As

$$e_s - u_s^{(0)} = e_s - \frac{2n}{3(n-1)} \frac{s+2}{\binom{n}{s+1}} > 0$$

induction shows

$$u_s^{(r)} < e_s \qquad (0 \le r) \qquad (1 \le s \le n-2) \tag{4.5.14}$$

$u_s^{(0)}, u_s^{(1)}, \ldots$ is hence an increasing bounded sequence and therefore converges for all $s = 1,2,\ldots,n-2$. The fact that

$$\lim_{r \to \infty} u_s^{(r)} = e_s \qquad (1 \le s \le n-2) \tag{4.5.15}$$

follows from the fact that $u_s^{(\infty)}$ satisfies (4.5.12):

$$\begin{aligned} u_s^{(\infty)} &= \sum_{r=0}^{\infty} \alpha_s^{(r)} \\ &= \alpha_0 + \sum_{r=0}^{\infty} \alpha_s^{(r+1)} \\ &= \alpha_0 + \frac{1}{s(s+1)\binom{n}{s+1}} \sum_{r=1}^{s} \sum_{t=1}^{r} \binom{r}{t} r(r+1) u_s^{(\infty)} . \end{aligned}$$

From (4.5.10) we find for $1 \le t \le n-2$:

$$\begin{aligned} \rho_t &= \sum_{j=0}^{t} e_j \frac{\binom{t}{j}}{(j+2)(n-j-1)} \\ &= \frac{2}{n-1} \frac{1}{2\cdot(n-1)} + \sum_{=1}^{t} \frac{\binom{t}{j}}{(j+2)(n-j-1)} \cdot \sum_{a=0}^{\infty} \alpha_t^{(a)} \\ &= \frac{1}{(n-1)^2} + \sum_{a=0}^{\infty} \delta_t^{(a)} \end{aligned}$$

proving (4.5.5) of Proposition 4.5.1.

Now assume: $1 \le t \le n-2$, we find from (4.5.3)

$$\alpha_t^{(r+1)} = \frac{1}{t(t+1)\binom{n}{t+1}} \sum_{j=1}^{t} \alpha_j^{(r)} \sum_{k=j}^{t} \left(\binom{k-1}{j} + \binom{k-1}{j-1}\right)k(k+1)$$

$$= \frac{1}{t(t+1)\binom{n}{t+1}} \sum_{j=1}^{t} \alpha_j^{(r)} \left(\binom{t+2}{j+3}(j+1)(j+2) . \binom{t+2}{j+2} j(j+1)\right)$$

$$= \frac{t+2}{\binom{n}{t+1}} \sum_{j=1}^{t} a_j^{(r)} \left(\frac{1}{j+3}\binom{t-1}{j} + \frac{1}{j+2}\binom{t-1}{j-1}\right)$$

hence

$$(4.5.16) \quad \alpha_t^{(r+1)} < \frac{t+2}{\binom{n}{t+1}} \sum_{j=1}^{t} \alpha_j^{(r)} \binom{t}{j} \frac{1}{j+2} \qquad (1 \le t \le n-2 \; ; \; 0 \le r) \; .$$

Now, from (4.5.4) we easily deduce

$$(4.5.17) \quad \alpha_t^{(r)} = (t+2)(n-t-1) \sum_{j=1}^{t} \binom{t}{j} (-1)^{t-j} \delta_j^{(r)}$$

so

$$\sum_{j=1}^{t} \alpha_j^{(r)} \binom{t}{j} \frac{1}{j+2} = \sum_{k=1}^{t} \sum_{j=k}^{t} \binom{t}{j}\binom{j}{k} \frac{(j+2)(n-j-1)}{(j+2)} (-1)^{j-k} \delta_k^{(r)}$$

$$= \sum_{k=1}^{t} \delta_k^{(r)} \binom{t}{k} \sum_{j=0}^{t-k} \binom{t-k}{j} (-1)^j (n-k-1-j)$$

$$= \sum_{k=1}^{t} \delta_k^{(r)} \binom{t}{k} \left[\sum_{j=0}^{t-k} (n-k-1)\binom{t-k}{j}(-1)^j - \sum_{j=0}^{t-k} (t-k)\binom{t-k-1}{j-1}(-1)^j \right]$$

$$= \delta_t^{(r)} (n-t-1) + t\, \delta_{t-1}^{(r)}$$

provided $1 \le t$.

From (4.5.4) we easily see

$$\delta_{t-1}^{(r)} < \delta_t^{(r)} \qquad t = 2,3,\ldots,n-2$$

so for $2 \le t \le n-2$ we find from (4.5.16):

$$(4.5.18) \quad \alpha_t^{(r+1)} \le \frac{t+2}{\binom{n}{t+1}} \delta_t^{(r)}(n-1) \qquad (1 \le t \le n-2)$$

(the latter formula easily being checked for validity when $t = 1$).

*From (4.5.2) and (4.5.4) we find

$$\delta^{(0)} = \sum_{j=1}^{t} \frac{2n}{3(n-1)} \frac{j+2}{\binom{n}{j+1}} \frac{\binom{t}{j}}{(j+2)(n-j-1)}$$

$$= \frac{2n}{3(n-1)} \frac{1}{n(n-1)} \frac{1}{\binom{n-2}{t}} \left[\sum_{j=1}^{t} \frac{\binom{n-2}{j}\binom{n-2-j}{n-2-t}\binom{j+1}{1}}{\binom{n-2}{j}} \right.$$

$$= \frac{2n}{3(n-1)} \frac{1}{n(n-1)} \frac{1}{\binom{n-2}{t}} \left[\sum_{j=0}^{t} \binom{n-2-j}{n-2-t}\binom{j+1}{1} - \binom{n-2}{t} \right]$$

$$= \frac{2n}{3(n-1)} \frac{1}{n(n-1)} \left[\frac{\binom{n}{t}}{\binom{n-2}{t}} - 1 \right]$$

and hence

$$(4.5.19) \quad \delta_t^{(0)} = \frac{2n}{3(n-1)} \left(\frac{1}{(n-t)(n-t-1)} - \frac{1}{n(n-1)} \right) .$$

Using (4.5.18) in (4.5.4) we get, when $1 \le t \le n-4$:

$$\delta_t^{(1)} < \sum_{j=1}^{t} \frac{j+2}{\binom{n}{j+1}} \delta_j^{(0)} (n-1) \frac{\binom{t}{j}}{(j+2)(n-j-1)}$$

$$= \frac{2}{3} n \left[\sum_{j=0}^{t} \frac{\binom{t}{j}}{(n-j-1)(n-j)(n-j-1)\binom{n}{j+1}} - \sum_{j=0}^{t} \frac{\binom{t}{j}}{\binom{n}{j+1}(n-j-1)n(n-1)} \right]$$

$$< \frac{2}{3} n \left[\sum_{j=0}^{t} \frac{\binom{t}{j}}{\binom{n}{j+1}(n-j-1)(n-j-2)(n-j-3)} \right] - \frac{2}{3(n-1)} \left[\frac{1}{(n-t-1)(n-t)} \right]$$

$$= \frac{2}{3} n \frac{1}{\binom{n-4}{t}} \sum_{j=0}^{t} \frac{\binom{n-4}{\cdot}\binom{n-4-j}{n-4-t}\binom{j+1}{1}}{n(n-1)(n-2)(n-3)\binom{n-4}{j}} - \frac{1}{3(n-1)} \frac{1}{(n-t-1)(n-t)}$$

$$= \frac{2}{3} n \frac{\binom{n-2}{t}}{n(n-1)(n-2)(n-3)\binom{n-4}{t}} - \frac{1}{3(n-1)} \frac{1}{(n-t-1)(n-t)}$$

$$= \frac{2}{3(n-1)} \left[\frac{1}{(n-t-3)(n-t-2)} - \frac{1}{(n-t-1)(n-t)} \right] .$$

Hence we find

(4.5.20) $$\delta_t^{(1)} < \frac{5}{nt} \delta^{(0)} .$$

We will use this as a starting point in an inductive proof of (4.5.6). (4.5.20) shows (4.5.6) to be true for $r = 1$. Suppose it is true for $r = x$. Then, from (4.5.4) and (4.5.18) we find

$$\delta_t^{(x+1)} \leq \sum_{j=1}^{t} \frac{(j+2)}{\binom{n}{j+1}} (n-1) \; \delta_j^{(x)} \frac{\binom{t}{j}}{(j+2)(n-j-1)}$$

$$< \frac{5}{n}^{x} \sum_{j=1}^{t} \frac{(j+2)}{\binom{n}{j+1}} (n-1) \frac{\binom{t}{j}}{(j+2)(n-j-1)} \delta_j^{(0)}$$

$$< \left(\frac{5}{n}\right)^{x} \left(\frac{5}{n}\right) \delta_t^{(0)} = \left(\frac{5}{n}\right)^{x+1} \delta_t^{(0)}$$

as in the proof of (4.5.20). This is (4.5.6) and hence Proposition 4.5.1 is proven.

We proceed to prove Proposition 4.5.2.

Inserting (4.5.12) in (4.5.10), using (4.5.9):

$$\rho_a = \frac{e_0}{2(n-1)} + \sum_{r=1}^{a} e_r \frac{\binom{t}{r}}{(r+2)(n-r-1)}$$

$$= \frac{e_0}{2(n-1)} + \sum_{r=1}^{a} \frac{\binom{a}{r}}{(r+2)(n-r-1)} \frac{2}{3} \frac{r+2}{\binom{n}{r+1}}$$

$$+ \sum_{r=1}^{a} \sum_{s=0}^{r} \sum_{t=0}^{s} \sum_{u=0}^{t} \frac{\binom{a}{r}}{(r+2)(n-r-1)} \frac{1}{(r+1)r\binom{n}{r+1}} \binom{s}{t} s(s+1)$$

$$\cdot (t+2)(n-t-1)(-1)^{t-u} \binom{t}{u} \rho_u$$

$$(0 \leq a \leq n-2) .$$

From (4.5.10) we see

$$\rho_0 = \frac{e_0}{2(n-1)} = \frac{1}{(n-1)^2}$$

and from the proof of (4.5.19) we have:

$$\sum_{r=1}^{a} \frac{r}{(r+2)(n-r-1)\binom{}{}} \;\tfrac{2}{3}\; \frac{(r+2)\ n}{\binom{}{r+1}} = \tfrac{2}{3}\left(\frac{1}{(n-a)(n-a-1)} - \frac{1}{n(n-1)}\right)$$

so we obtain:

$$(4.5.21) \qquad \rho_a = \frac{2}{3}\left(\frac{1}{(n-a)(n-a-1)} - \frac{1}{n(n-1)}\right) + \frac{1}{(n-1)^2} + T_a$$

where T_a is the last sum in the previous formula for ρ_a. As $T_a = 0$ when $a = 0$ (sum being empty), we see that (4.5.21) is valid for $a = 0$ also.

To evaluate T_a we shall consider the sums

$$\mu_{u,r} = \sum_{s=u}^{r} \sum_{t=u}^{s} \binom{s}{t}(s(s+1))(t+2)(n-t-1)(-1)^{t-u}\binom{t}{u}$$

so that

$$T_a = \sum_{r=1}^{a} \sum_{u=0}^{r} \frac{\binom{a}{r}}{r(r+1)(r+2)n\binom{n-1}{r+1}} \rho_a \mu_{u,r} .$$

Now :

$$\sum_{t=u}^{s} (t+2)(n-t-1)(-1)^{t-u}\binom{t}{u}\binom{s}{t}$$

$$= \binom{s}{u} \sum_{t=u}^{s} (t+2)(n-t-1)\binom{s-u}{t-u}(-1)^{t-u}$$

$$= \binom{s}{u} \sum_{t=0}^{s-u} ((n-u-1)(u+2) + t(n-2u-4) - t(t-1))\binom{s-u}{t}(-1)^{t}$$

$$= \binom{s}{u}\Big[(n-u-1)(u+2)\sum_{t=0}^{s-u}\binom{s-u}{t}(-1)^t - (n-2u-4)(s-u)\sum_{t=0}^{s-u}\binom{s-u-1}{t-1}(-1)^{t-1}$$

$$- (s-u)(s-u-1)\sum_{t=0}^{s-u}\binom{s-u-2}{t-2}(-1)^{t-2}\Big]$$

$$= \begin{cases} (n-u-1)(u+2) & \text{if } s = u \\ -(n-2u-4)(u+1) & \text{if } s = u+1 \\ -(u+2)(u+1) & \text{if } s = u+2 \end{cases}$$

and hence

$$T_a = \sum_{r=1}^{a} \rho_r \frac{\binom{a}{r}}{r(r+1)(r+2)n\binom{n-1}{r+1}} \mu_{r,r}$$

$$+ \sum_{r=1}^{a} \rho_{r-1} \frac{\binom{a}{r}}{r(r+1)(r+2)n\binom{n-1}{r+1}} \mu_{r-1,r}$$

$$+ \sum_{r=2}^{a} \frac{\binom{a}{r}}{r(r+1)(r+2)n\binom{n-1}{r+1}} \sum_{u=0}^{r-2} \rho_a \ \mu_{u,r}$$

$$= \sum_{r=1}^{a} \rho_r \frac{\binom{a}{r}}{r(r+1)(r+2)n\binom{n-1}{r+1}} r(r+1)(n-r-1)(r+2)$$

$$+ \sum_{r=1}^{a} \rho_{r-i} \frac{\binom{a}{r}}{r(r+1)(r+2)n\binom{n-1}{r+1}} ((r+1)(r+2)(r+1)(r))$$

$$+ \sum_{r=1}^{a} \frac{\binom{a}{r}}{r(r+1)(r+2)n\binom{n-1}{r+1}} \Big[\sum_{u=0}^{r-1} \rho_u [u(u+1)(n-u-1)(u+2)$$

$$- (u+1)(u+2)(n-2u-4)(u+1) - (u+2)(u+3)(u+2)(u+1)] \Big]$$

$$= \sum_{r=1}^{a} \rho_r \frac{\binom{a}{r}}{\binom{n}{r+1}} + \sum_{r=1}^{a} \rho_{r-1} \frac{\binom{a}{r}(r+1)}{\binom{n}{r+1}(n-r-1)}$$

$$- \sum_{u=o}^{a-1} \rho_u (u+1)(u+2)(n+2) \sum_{r=u+1}^{a} \frac{\binom{a}{r}}{r(r+1)(r+2)\binom{n-1}{r+1} n} .$$

Inserting this in (4.5.21) yields

$$\rho_a = \frac{2}{3}\left(\frac{1}{(n-a-1)\ \ n-a)} - \frac{1}{n(n-1)} \right) + \frac{1}{(n-1)^2} + \rho_a \frac{1}{\binom{n}{a+1}} - \frac{1}{n(n-1)^2}$$

$$+ \sum_{r=0}^{a-1} r\ \rho \frac{\binom{a}{r}}{\binom{n}{r+1}} + \sum_{r=0}^{a-1} \rho_r \frac{\binom{a}{r+1}(r+2}{\binom{n}{r+2}(n-r-2)}$$

$$- \sum_{u=o}^{a-1} \rho_u (u+1)(u+2) \frac{(n+2)}{n} \sum_{r=u+1}^{a} \frac{\binom{a}{r}}{r(r+1)(r+2)\binom{n-1}{r+1}}$$

and we easily see that we have proven Proposition 4.5.2.

4.6 Proof of Proposition 4.1.1.

From (4.5.6) we find

$$0 < \sum_{r=1}^{\infty} \delta_t^{(0)} < \frac{5}{n-5} \delta_t^{(0)} \qquad 1 \le t \le n-4 .$$

So, bringing in (4.5.19) together with (4.5.5) we find

$$\rho_t = \frac{2n}{3(n-1)} \frac{1}{(n-t)(n-t-1)} - \frac{1}{3(n-1)^2} + \varepsilon_t \qquad (1 \le t \le n-4)$$

where

$$0 < \varepsilon_t < \frac{2n}{3(n-1)}\left(\frac{1}{(n-t)(n-t-1)} - \frac{1}{n(n-1)}\right)\frac{5}{(n-5)}$$

so, as $\rho_t = \eta^{(n)}_{n-t-2,1}$ we find

$$\eta^{(n)}_{t,1} = \frac{2}{3t(t+1)} + \frac{1}{3n(n-1)} + \eta^{(n)}_{t,1}\ \frac{1}{n} \cdot M_t \qquad 4 \le t \le n-2\ ,\ 0 < M_t < M$$

where M is some uniform positive constant (at least less than 26).

This proves the first statement of Proposition 4.1.1, as the formula for t = n-2 is trivial.

(4.5.6) is not valid for t = n-3 or t = n-2 , so we have to treat these two cases separately.

We introduce the notations:

(4.6.1) $$S^{(r)}_t = \sum_{t=1}^{n-2} \delta^{(r)}_t \qquad (0 \le r) \quad 1 \le t \le n-2$$

and shall concentrate on $S^{(r)}_{n-2}$ first.

We have

$$S^{(r)}_{n-2} = \sum_{t=1}^{n-2} \sum_{j=1}^{t} \alpha^{(r)}_j \frac{\binom{t}{j}}{(j+2)(n-j-1)}$$

$$= \sum_{j=1}^{n-2} \alpha^{(r)}_j \frac{\binom{n-1}{j+1}}{(j+2)(n-j-1)} .$$

(4.6.2) $$S^{(r)}_{n-2} = \sum_{j=1}^{n-2} \alpha^{(r)}_j \binom{n+1}{j+2} \frac{1}{n(n+1)} .$$

We find from (4.5.3)

$$S_{n-2}^{(r+1)} = \sum_{j=1}^{n-2} \frac{\binom{n+1}{j+2}}{(n+1)\, n\, j(j+1)\binom{n}{j+1}} \sum_{k=1}^{j} \alpha_k^{(r)} [k(k+1)\binom{j+2}{k+2} + (k+1)(k+2)\binom{j+2}{k+3}$$

$$= \sum_{k=1}^{n-2} \frac{1}{n} \alpha_k^{(r)} \left[\frac{1}{k+2} \sum_{j=k}^{n-2} \binom{j-1}{k-1} + \frac{1}{k+3} \sum_{j=k}^{n-2} \binom{j-1}{k} \right] .$$

$$(4.6.3) \qquad S_{n-2}^{(r+1)} \text{ a } \sum_{k=1}^{n-2} \frac{\alpha_k^{(r)}}{n} \left[\frac{1}{k+2} \binom{n-2}{k} + \frac{1}{k+3} \binom{n-2}{k+1} \right]$$

Inserting (4.5.2) in (4.6.3) we find

$$S_{n-2}^{(1)} = \sum_{k=1}^{n-2} \frac{1}{n} \frac{2n}{3(n-1)} \frac{k+2}{\binom{n}{k+1}} \left[\frac{1}{(k+2)} \binom{n-2}{k} + \frac{1}{(k+3)} \binom{n-2}{k+1} \right]$$

$$= \frac{2}{3(n-1)} \left[\sum_{k=1}^{n-2} \frac{(k+1)(n-k-1)}{n(n-1)} + \frac{(k+2)(n-k-1)(n-k-2)}{(k+3)n(n-1)} \right]$$

and eventually

$$(4.6.4) \qquad S_{n-2}^{(1)} = \frac{2}{3} \frac{(n+1)(n+2)}{(n-1)n} \left[\frac{1}{2} - \frac{H_n}{n-1} + \frac{1}{n-1} \right] - \frac{4}{9} \frac{(n-2)}{n(n-1)} .$$

Similarly, as

$$\alpha_t^{(1)} = \frac{1}{t(t+1)\binom{n}{t+1}} \sum_{j=1}^{t} \sum_{k=1}^{j} \binom{j}{k} j(j+1) \frac{2n}{3(n-1)} \frac{k+2}{\binom{n}{k+1}}$$

$$= \frac{2n}{3(n-1)t(t+1)\binom{n}{t+1}} \sum_{j=1}^{t} \frac{j(j+1)}{\binom{n-1}{j}} \left(\sum_{k=0}^{j} \frac{\binom{n-k-1}{n-j-1} 2 \binom{k+2}{2}}{n} - \frac{2}{n} \binom{n-1}{j} \right)$$

$$= \frac{4n}{3(n-1)} \frac{(n+1)(n+2)}{t(t+1)\binom{n}{t+1}} \sum_{j=1}^{t} \frac{j(j+1)}{(n-j)(n-j+1)(n-j+2)} - \frac{2}{3n} \alpha_t^{(0)} ,$$

we find from (4.6.3)

$$S_{n-2}^{(2)} = \frac{4(n+1)(n+2)}{3(n-1)n} \left(\sum_{t=1}^{n-2} \left(\frac{1}{t+2} \binom{n-2}{t} + \frac{1}{t+3} \binom{n-2}{t+1} \right) \frac{1}{t(t+1)\binom{n}{t+1}} \right.$$

$$\left. \cdot \sum_{j=1}^{t} \frac{j(j+1)}{(n-j)(n-j+1)(n-j+1)} \right) - \frac{2}{3n} S_{n-2}^{(1)}$$

$$= \frac{4(n+1)(n+2)}{3(n-1)^2 n} \left(\sum_{t=1}^{n-2} \sum_{j=1}^{t} \frac{j(j+1)}{(n-j)(n-j+1)(n-j+1)} \right.$$

$$\left. \cdot \left(\frac{(n-t-1)}{t(t+2)} + \frac{(n-t-1)(n-t-2)}{t(t+1)(t+3)} \right) \right) - \frac{2}{3n} S_{n-2}^{(1)} .$$

After tedious computations we find

$$(4.6.5) \quad S_{n-2}^{(2)} = - \frac{17}{9(n-1)} + \frac{1}{n(n-1)^2} \left(- \frac{85}{27} n - \frac{55}{9} - \frac{2}{27n} - \frac{16}{3(n+3)} + \frac{88}{9(n+4)} \right)$$

$$+ \frac{H_n}{n(n-1)^2} \left(\frac{2}{3} n^2 + \frac{4}{3} n + \frac{14}{3} + \frac{4}{9n} + \frac{32}{9(n+3)} - \frac{16}{n+1} \right) .$$

We shall, however use approximations and write

$$(4.6.6) \quad S_{n-2}^{(1)} = \frac{1}{3} - \frac{2}{3} \frac{H_n}{n} + \frac{14}{9n} + 0\left(\frac{H_n}{n^2} \right)$$

$$(4.6.7) \quad S_{n-2}^{(2)} = \frac{2}{3} \frac{H_n}{n} - \frac{17}{9n} + 0\left(\frac{H_n}{n^2} \right) .$$

In order to find approximations for $\delta^{(1)}_{n-2}$ and $\delta^{(2)}_{n-2}$ we will use the following formula

$$(4.6.8) \qquad \delta^{(r+1)}_{n-2} = \delta^{(r)}_{n-2} - \frac{n}{(n-1)} S^{(r)}_{n-2} + \frac{2n-1}{n-1} S^{(r+1)}_{n-2} \quad .$$

From (4.5.4) we have

$$\delta^{(r)}_{n-2} = \sum_{j=1}^{n-2} \alpha^{(r)}_j \frac{\binom{n-2}{j}}{(j+2)(n-j-1)} = \sum_{j=1}^{n-2} \alpha^{(r)}_j \binom{n-1}{j} \frac{1}{(j+2)(n-1)}$$

and similar to the proof of (4.6.3) we find

$$\delta^{(r+1)}_{n-2} = \frac{1}{n(n-1)} \sum_{j=1}^{n-2} \alpha^{(r)}_j \left[\frac{k}{k+2}\binom{n-1}{k+1} + \frac{1}{k+2}\binom{n-2}{k} + \frac{k+1}{k+3}\binom{n-1}{k+2} + \frac{1}{k+3}\binom{n-2}{k+1} \right] .$$

Now

$$\binom{n-1}{k} \frac{1}{(k+2)(n-1)} - \frac{n}{(n-1)} \binom{n+1}{k+2} \frac{1}{n(n+1)} + \frac{2n-1}{(n-1)} \frac{1}{n} \left(\frac{1}{k+2}\binom{n-2}{k} + \frac{1}{k+3}\binom{n-2}{k+1} \right)$$

$$\frac{1}{n(n-1)} \left[\frac{k}{k+2}\binom{n-1}{k+1} + \frac{1}{k+2}\binom{n-2}{k} + \frac{k+1}{k+3}\binom{n-1}{k+2} + \frac{1}{k+3}\binom{n-2}{k+1} \right]$$

$$= \binom{n-1}{k} \frac{1}{(k+2)} \left[\frac{1}{(n-1)} \quad \frac{n}{(n-1)(k+1)} + \frac{2n-1}{n(n-1)} \frac{n-k-1}{(n-1)} - \frac{k(n-k-1)}{(k+1)n(n-1)} - \frac{(n-k-1)}{n(n-1)^2} \right]$$

$$+ \binom{n-2}{k+1} \frac{1}{k+3} \left[\frac{2n-1}{(n-1)n} - \frac{n-1}{(k+2)} \frac{k+1}{n(n-1)} - \frac{1}{n(n-1)} \right]$$

$$= \binom{n-1}{k} \frac{1}{k+2} \frac{(n-k-1)(n-k-2)}{n(n-1)(k+1)} - \binom{n-2}{k+1} \frac{1}{n(k+2)}$$

$$\equiv 0 \quad .$$

So, according to (4.6.2),(4.6.3) and the two formulae above for $\delta^{(r)}_{n-2}$ and $\delta^{(r+1)}_{n-2}$ we have (4.6.8).

From (4.5.19) we have

$$\delta_t^{(0)} = \frac{2n}{3(n-1)} \frac{1}{(n-t-1)(n-t)} - \frac{2}{3(n-1)^2} \qquad 1 \le t \le n-2$$

so

(4.6.9) $$\delta^{(0)} = \frac{3}{} + \frac{1}{} + 0\left(\frac{3n}{n^2}\right)$$

and also

(4.6.10) $$S_{n-2}^{(0)} = \frac{2}{3} - \frac{2}{} + 0\left(\frac{3n\, H_n}{n^2}\right)$$

From (4.6.6) - (4.6.8) we then find

$$\delta_{n-2}^{(1)} = \delta_{n-2}^{(0)} - \frac{n}{n-1} S_{n-2}^{(0)} + \frac{2n-1}{n-1} S_{n-2}^{(1)}$$

giving

(4.6.11) $$\delta_{n-2}^{(1)} = \frac{1}{3} - \frac{4H_n}{3n} + \frac{34}{9n} + 0\left(\frac{H_n}{n^2}\right)$$

and similarly

$$\delta_{n-2}^{(2)} = \delta_{n-2}^{(1)} - \frac{n}{n-1} S_{n-2}^{(1)} + \frac{2n-1}{n-1} S_{n-2}^{(2)}$$

giving

(4.6.12) $$\delta_{n-2}^{(2)} = \frac{2}{3} \frac{H_n}{n} - \frac{17}{9n} + 0\left(\frac{H_n}{n^2}\right) .$$

From the above formulae (4.6.6), (4.6.7), (4.6.9) - (4.6.11) we easily obtain

(4.6.13) $$S_{n-3}^{(0)} = \frac{1}{3} - \frac{1}{n}$$

(4.6.14) $$S_{n-3}^{(1)} = \frac{2}{3} \frac{H_n}{n} - \frac{20}{9n} + 0\left(\frac{H_n}{n^2}\right)$$

$$(4.6.15)\qquad S^{(2)}_{n-3} = O\left(\frac{H_n}{n^2}\right) .$$

We already know $\delta^{(0)}_{n-3}$ from (4.5.19)

$$(4.6.16)\qquad \delta^{(0)}_{n-3} = \frac{1}{9} + \frac{1}{9n} + O\left(\frac{H_n}{n^2}\right)$$

and (4.6.15) implies

$$(4.6.17)\qquad \delta^{(2)}_{n-3} = O\left(\frac{H_n}{n^2}\right) .$$

To find $\delta^{(1)}_{n-3}$ we inspect $S^{(1)}_{n-4}$. We use (4.6.1), (4.5.4) and the formula for $\alpha^{(1)}_t$ established below (4.6.4) above to obtain:

$$S^{(1)}_{n-4} = \frac{4(n+1)(n+2)n}{3(n-1)} \sum_{t=1}^{n-4} \frac{\binom{n-3}{t+1}}{(t+2)(n-t-1)t(t+1)\binom{n}{t+1}} \sum_{j=1}^{t} \frac{j(j+1)}{(n-j)(n-j+1)(n-j+2)}$$

$$- \frac{2}{3n} \sum_{t=1}^{n-4} \alpha^{(0)}_t \frac{\binom{n-3}{t+1}}{(t+2)(n-t-1)} .$$

So

$$S^{(1)}_{n-4} + \frac{2}{3n} S^{(0)}_{n-4} = \frac{4(n+1)(n+2)}{3n(n-1)(n-2)} \sum_{t=1}^{n-4} \sum_{j=1}^{t} \frac{j(j+1)}{(n-j)(n-j+1)(n-j+2)} \frac{(n-t-3)(n-t-2)}{t(t+1)(t+2)}$$

$$= \frac{4}{3} H^{(2)}_n - 3n + O\left(\frac{H_n}{n^2}\right) .$$

Now, (4.6.13) and (4.6.14) give

$$(4.6.18)\qquad S^{(0)}_{n-4} = \frac{2}{9} - \frac{10}{9n} + O\left(\frac{H_n}{n^2}\right)$$

we find

(4.6.19) $$S^{(1)}_{n-4} = \frac{4}{3} H^{(2)} - \frac{49}{27}\frac{1}{n} + O\left(\frac{H_n}{n^2}\right) .$$

Together with (4.6.14) we arrive at

(4.6.20) $$\delta^{(1)}_{n-3} = \frac{2}{3}\frac{H_n}{n} - \left(\frac{11}{27} + \frac{4}{3} H^{(2)}_n\right)\frac{1}{n} + O\left(\frac{H_n}{n^2}\right) .$$

From (4.5.1) and (4.5.5) we see

$$1 = \frac{1}{n-1} + \sum_{r=0}^{\infty} S^{(r)}_{n-2}$$

so that from (4.6.10), (4.6.6) and (4.6.7) we see

(4.6.21) $$\sum_{r=3}^{\infty} S^{(r)}_{n-2} = 1 - \frac{1}{n-1} - S^{(0)}_{n-2} - S^{(1)}_{n-2} - S^{(2)}_{n-2} = O\left(\frac{H_n}{n^2}\right)$$

proving

(4.6.22) $$\rho_{n-2} = \frac{1}{(n-1)^2} + \delta^{(0)}_{n-2} + \delta^{(1)}_{n-2} + \delta^{(2)}_{n-2} + O\left(\frac{H_n}{n^2}\right)$$

leading to the value for $\eta^{(n)}_{2,1}$ stated in Proposition 4.1.1.

From (4.6.15) we see that

$$\delta^{(2)}_{n-3} = O\left(\frac{H_n}{n^2}\right)$$

and from (4.6.21) we see

$$\sum_{r=2}^{\infty} \delta^{(r)}_{n-3} = O\left(\frac{H_n}{n^2}\right)$$

so that

$$\rho_{n-3} = \frac{1}{(n-1)^2} + \delta^{(0)}_{n-3} + \delta^{(1)}_{n-2} + O\left(\frac{H_n}{n^2}\right) .$$

Referring to (4.6.16) and (4.6.20) we have then proven the value of $\eta^{(n)}_{3,1}$ in Proposition 4.1.1.

5. Measures of Efficiency in $S_1^{(n)}$

5.1 General Formulae for Basic Probabilities.

In order to obtain the measures for $S_1^{(n)}$:

$L^* = L_{S_1^{(n)}}$ -- the expected left path length

$S^* = S_{S_1^{(n)}}$ -- the expected number of key comparisons

$R^* = R_{S_1^{(n)}}$ -- the expected right path length

$RL^* = RL_{S_1^{(n)}}$ -- the expected length of the last right subtree

$C^* = C_{S_1^{(n)}}$ -- the expected recursion depth

(see Section 2.4) we need knowledge of some properties of the CLPP of $S_1^{(n)}$:

$$A_1^{(n)}(z,w) = \sum_{a=2}^{n} \sum_{b=1}^{a-1} \eta_{a,b}^{(n)} z^a w^b .$$

Formula (3.4.1), together with the approximate values for $\eta_{a,1}^{(n)}$ proven in Chapter 4 could give us values of $\eta_{a,b}^{(n)}$ for general $1 \leq b < a \leq n$. However, it turns out that we may express all the quantities needed in terms of $\eta_{a,1}^{(n)}$'s, without knowing the $\eta_{a,b}^{(n)}$'s in general.

To establish the measures above we need formulae for

$$(5.1.1) \qquad \lambda_r = \sum_{b=1}^{n-1-r} \eta_{r+b+1,b}^{(n)} \qquad 0 \leq r \leq n-2$$

$$(5.1.2) \qquad \mu_a = \sum_{b=1}^{a-1} \eta_{a,b}^{(n)} \qquad 2 \leq a \leq n$$

$$(5.1.3) \qquad \tau_b = \eta_{n,b}^{(n)} \qquad 1 \le b \le n-1 \ .$$

Knowing that

$$(5.1.4) \qquad A_1^{(n)}(1,1) = \sum_{r=0}^{n-2} \lambda_r = \sum_{a=2}^{n} \mu_a$$

$$(5.1.5) \qquad \frac{1}{z} A_1^{(n)}\left(z \, , \, \frac{1}{z}\right) = \sum_{r=0}^{n-2} \lambda_r \, z^r$$

$$(5.1.6) \qquad \left[\frac{\partial \ A_1^{(n)}(z,w)}{\partial z} \right]_{z=w=1} = \sum_{a=2}^{n} a \ \mu_a$$

we see that we then will have the sufficient knowledge to establish the measures needed (see Section 2.4).

We will use the notation

$$(5.1.7) \qquad \beta_a = (a-1)\eta_{a,1}^{(n)} + (n-a)\eta_{a+1,1}^{(n)} \qquad 1 \le a \le n \ .$$

First we prove

$$(5.1.8) \quad \lambda_r = \frac{1}{r+1} \sum_{k=r+2}^{n} \beta_k \left(\frac{3k-2r-5}{k-1} - 2(H_{k-1} - H_{r+1}) \right) \qquad 0 \le r \le n-2 \ .$$

Using $w = 1/z$ in (3.4.1) we obtain

$$\sum_{a=2}^{n} \sum_{b=1}^{a-1} \eta_{a,b}^{(n)} \, z^{a-b}$$

$$= \frac{1}{n+1} \sum_{a=2}^{n} \sum_{b=1}^{a-1} z^{a-b}\left(b \ \eta_{a,b}^{(n)} + (a-b-1)\eta_{a,b+1}^{(n)} + (n-a)\eta_{a+1,b+1}^{(n)}\right)$$

$$+ \frac{1}{n+1} \sum_{a=2}^{n} \beta_a \left(z + H_0^{(a-1)}\left(\cdot \quad \cdot \ \frac{1}{z}\right)\right) \cdot \frac{z}{n+1} \ .$$

We have

$$(5.1.9)\qquad H_0^{(a-1)}\left(z\,,\,\frac{1}{z}\right) = z\sum_{r=0}^{a-2}\left(\frac{2}{r+1}-\frac{2}{a-1}\right)z^r \qquad (2\le a)\ .$$

Rearranging the general equation we find

$$\sum_{j=0}^{n-2}(j+1)\lambda_j z^j(1-z) = 1-(n-1)\eta_{2,1}^{(n)} + \sum_{a=2}^{n}\beta_a^{(n)}\left(1-z^{a-1}+\frac{1}{z}H_0^{(a-1)}\left(z\,,\,\frac{1}{z}\right)\right).$$

From this we see

$$(j+2)\lambda_{j+1}-(j+1)\lambda_j = -\beta_{j+2}+\sum_{a=j+3}^{n}\left(\frac{2}{j+2}-\frac{2}{a-1}\right)\beta_a \qquad (0\le j\le n-2)$$

when regarding the coefficients of $z, z^1, \ldots, z^{n-1}$. Summing these equations from $j = r$ to $j = n-2$ eventually proves (5.1.8).

For the μ_a 's we will find

$$(5.1.10)\qquad \mu_r = \frac{1}{(n+1-r)}\left[1+\sum_{k=r+1}^{n}\beta_k\left(H_{k-r}+H_{k-1}-H_{r-1}-\frac{k-r}{k-1}\right)\right] \qquad 2\le r\le n\ .$$

Putting $w = 1$ in (3.4.1) we have

$$\sum_{a=2}^{n}\mu_a z^a = \frac{1}{n+1}z^n + \frac{1}{n+1}\sum_{a=2}^{n}z^a\sum_{b=1}^{a-1}b\,\eta_{a,b}^{(n)} + (a-b-1)\eta_{a,b+1}^{(n)} + (n-a)\eta_{a+1,b+1}^{(n)}$$

$$+\frac{1}{n+1}\sum_{a=2}^{n}\beta_a\left(z^a + H_0^{(a-1)}(z,1)\right)$$

where

$$H_0^{(a-1)} = \sum_{r=2}^{a-1}z^r\sum_{s=1}^{r-1}\frac{1}{(a-1-s)(a-s)}+\frac{1}{(r-1)r}$$

$$= \sum_{r=2}^{a-1}z^r\left(\frac{1}{a-r}-\frac{1}{a-1}+\frac{1}{r}\right) \qquad (2\le a)$$

giving

$$\sum_{a=2}^{n} (\mu_a(n+1-a) - \mu_{a+1}(n-a))z^a = z^n + \sum_{a=2}^{n} \beta_a \, H_0^{(a-1)}(z,1) \quad .$$

As for λ_r above we obtain for $2 \le r \le n$

$$(n+1-r)\mu_r = 1 + \sum_{k=r}^{n-1} \sum_{a=k+1}^{n} \beta_a \left(\frac{1}{a-k} - \frac{1}{a-1} + \frac{1}{k} \right)$$

leading to (5.1.10).

The τ_b 's turn out to be

$$(5.1.11) \qquad \tau_b = \frac{n}{(n-1)(n+1-b)(n-b)} \qquad 1 \le b \le n-1$$

as proven by isolating the terms in (3.4.1) having z to the power (n-1) :

$$\sum_{b=1}^{n-1} \tau_b w^b = \frac{1}{n+1} \sum_{b=1}^{n-1} w^b (b\,\tau_b + (n-1-b)\tau_{b+1}) + \frac{1}{n+1} w^{n-1}(n-1)\tau_1 + \frac{w^{n-1}}{n+1}$$

yielding

$$2\,\tau_{n-1} = 1 + (n-1)\tau_1$$

and

$$\tau_b(n+1-b) - (n-1-b)\tau_{b+1} = 0 \qquad 1 \le b \le n-2 \; .$$

As $\tau_1 = \eta_{n,1}^{(n)} = \frac{1}{(n-1)^2}$ is found earlier we easily see (5.1.11).

We will also prove the useful relation

$$(5.1.12) \qquad \eta_{2,1}^{(n)} = \frac{3}{n+1} + \frac{2n}{n+1} \sum_{a=3}^{n} \eta_{a,1}^{(n)}(H_{a-2} - 1) \quad .$$

This is seen from (5 .1.10) and the fact that

$$\mu_2 = \eta_{2,1}^{(n)} \quad .$$

We find

$$\eta^{(2)}_{2,1}(n-1) = 1 + \sum_{k=3}^{n} ((k-1)\eta^{(n)}_{k,1} + (n-k)\eta^{(n)}_{k+1,1})(2H_{k-1} - 2)$$

$$= 1 + \sum_{k=3}^{n} (k-1)\eta^{(n)}_{k,1}\left(2H_{k-2} - 2 + \frac{2}{k-1}\right) + \sum_{k=4}^{n} (n-k+1)\eta^{(n)}_{k,1}(2H_{k-2} - 2)$$

$$= 1 + 2n \sum_{k=3}^{n} \eta_{k,1}(H_{k-2} - 1) + 2(1 - \eta^{(n)}_{2,1})$$

from which 5.1.12) follows easily.

5.2 The Expected Left Path Length.

The β_a 's defined in statement (5.1.7) are approximated from Proposition 4.1.1 by

$$(5.2.1)\quad \begin{cases} \beta_1 = \frac{2}{3}n - \frac{2}{3}H_n + \frac{14}{9} + O\left(\frac{H_n}{n}\right) \\ \beta_2 = \frac{1}{9}n + \frac{2}{3}H_n + \frac{4}{27} - \frac{4}{3}H_n^{(2)} + O\left(\frac{H_n}{n}\right) \\ \beta_t = \frac{2}{3}\frac{(n+1)}{t(t+1)} + \frac{1}{3n} + O\left(\frac{1}{n}\right)\beta_t^{(n)} \qquad 3 \le t \le n . \end{cases}$$

Inserting (5.1.10) in (5.1.4) gives

$$(5.2.2)\quad A_1^{(n)}(1,1) = \sum_{a=2}^{n} \frac{1}{n+1-a}\left(1 + \sum_{k=a+1}^{n} \beta_k\left(\sum_{t=a}^{k-1} \frac{1}{t-a+1} + \frac{1}{t} - \frac{1}{k-1}\right)\right)$$

$$= H_{n-1} + N$$

where, according to (5.2.1)

$$N = \left[\sum_{a=2}^{n-1} \sum_{k=a+1}^{n} \sum_{t=a}^{k-1} \left(\frac{1}{t-a+1} + \frac{1}{t} - \frac{1}{k-1} \right) \frac{1}{n+1-a} \left(\frac{2}{3} \frac{(n+1)}{k(k+1)} + \frac{1}{3n} \right) \right] + N \cdot O\left(\frac{1}{n}\right) .$$

Straightforward calculations lead to

(5.2.3) $$N = \frac{1}{3} H_n^2 + \frac{2}{3} H_n - H_n^{(2)} - \frac{7}{3} + O\left(\frac{H_n^2}{N}\right)$$

and hence from (2.4.6)

(5.2.4) $$L^* = \frac{1}{3} H_n^2 + \frac{5}{3} H_n - H_n^{(2)} - \frac{4}{3} + O\left(\frac{H_n^2}{n}\right) .$$

The expected length of the left path has increased from

$$2 H_n - 1$$

in the normal p-tree forest to the value given in (5.2.4).

5.3 The Average Number of Key Comparisons.

The formula for the expected number of key comparisons in the stationary p-tree forest is found from (2.4.10), (5.1.6) and (5.1.1) to be

(5.3.1) $$S^* = 1 + \frac{1}{n+1} \sum_{r=2}^{n} a \mu_a + \sum_{k=0}^{n-2} \frac{k+1}{n+1} S_{F_0}(k) \lambda_k$$

where $S_{F_0}(k)$ is the corresponding value for the normal p-tree forest, defined in (2.4.13).

To establish a formula for

$$T = 1 + \frac{1}{(n+1)} \sum_{r=2}^{n} r \mu_r$$

we see

$$T = 1 + \sum_{r=2}^{n} \left(1 - \frac{n+1-r}{n+1} \right) \mu_r = 1 + A_1^{(n)}(1,1) - K$$

where

$$(5.3.2) \quad K = \frac{1}{n+1} \left[\sum_{r=2}^{n} \left(1 + \sum_{k=r+1}^{n} \beta_k \left(H_{k-r} + H_{k-1} - H_{r-1} - \frac{k-r}{k-1} \right) \right) \right]$$

$$= \frac{n-1}{n+1} + \frac{1}{n+1} \sum_{k=3}^{n} \beta_k (k-2) \cdot H_{k-1} - \frac{1}{2} \quad .$$

Defining

$$(5.3.3) \quad U = \sum_{k=0}^{n-2} \frac{k+1}{n+1} \; S_{F_0^{(k)}} \; \lambda_k$$

we have from (5.3.1) and (2.4.6)

$$(5.3.4) \quad S^* = L^* - K + U \; .$$

To evaluate U we use (5.1.8) and (2.4.13):

$$u = \frac{2}{n+1} \lambda_1$$

$$+ \sum_{k=4}^{n} \frac{1}{n+1} \beta_k \sum_{r=2}^{k-2} \left(\frac{3k-2r-5}{k-1} - 2(H_{k-1} - H_{r+1}) \right) \left(\frac{1}{3} (H_{r+1}^2 - H_{r+1}^{(2)}) + \frac{10}{9} H_{r-1} - \frac{28}{27} \right) .$$

The latter inner sum simplifies nicely and we obtain eventually

$$U = \frac{2}{n+1} \lambda_1 + \sum_{k=4}^{n} \frac{1}{n+1} \beta_k \left(k H_{k-1} - \frac{1}{2} k - 5 + \frac{4}{k-1} \right)$$

$$= \sum_{k=3}^{n} \frac{\beta_k}{n+1} \cdot \left(k H_{k-1} - \frac{1}{2} k - 5 + \frac{4}{k-1} + \frac{3k-7}{k-1} - 2(H_{k-1} - H_2) \right)$$

from (5.1.8) with $r = 1$.

Rearranging the terms yields

$$u= \sum_{k=3}^{n} \frac{\beta_k}{n+1} \quad (k-2)\left(H_{k-1} - \frac{1}{2} \right)$$

Referring to (5.3.2), we see that

(5.3.5) $$U = K - \frac{n-1}{n+1}$$

and hence, by insertion in (5.3.4)

(5.3.6) $$S^* = L^* - \frac{n-1}{n+1} \quad .$$

Using the approximate value of L^* from the previous section gives us

(5.3.7) $$S^* = \frac{1}{3} H_n^2 + \frac{5}{3} H_n - H_n^{(2)} \; \underline{7}\;_3 + O\left(\frac{H_n^2}{n} \right) \quad .$$

The expected number of key comparisons is hence slightly less than the expected left path length, and has the same dominating term as the corresponding quantity of the normal p-tree forest, being

$$S = \frac{1}{3} H_n^2 + \frac{-}{9} H_n \, 10 - \frac{1}{3} H_n^{(2)} \; - \overline{27} \quad .$$

Formula (5.3.6) is surprisingly simple, indicating that there should be an easier way to prove it than the one we have been using here.

5.4 The Expected Length of the Right Path.

From (2.4.14) and (5.1.11) we find the expected length of the right path to be

(5.4.1) $$R^* 1 + \sum_{k=0}^{n-2} R_{F_0(k)} \frac{n}{(n-1)(k+1)(k+2)} \, .$$

In [2] is quoted the recursion formula for $R_{F_0}(n)$ being

$$(5.4.2) \qquad R_{F_0}(n) = 1 + \sum_{k=0}^{n-2} R_{F_0}(k) \left(\frac{1}{(k+1)(k+2)} + \frac{1}{n(n-1)} \right) .$$

From these two equations we find

$$(5.4.3) \qquad R^* = R_{F_0}(n) + \frac{1}{n-1} (R_{F_0}(n) - 1) - \frac{1}{(n-1)^2} \sum_{k=0}^{n-2} R_{F_0}(k) .$$

From (5.4.1) we find

$$(5.4.4) \qquad R^*_{n+1} - R^*_n = \frac{1}{n^2} (R_{F_0}(n) + 1 - R^*_n) .$$

$R_{F_0}(n)$ is known to be a nondecreasing sequence of positive real numbers, approaching the limit

$$R_\infty = \sum_{j=0}^{\infty} \frac{2^j}{((j+1)!)^2} = 1.6261\ldots .$$

(5.4.3) and (5.4.4) show that the R^*_n have the same properties as $R_{F_0}(n)$.

5.5 The Expected Length of the Left Path of the Last Right Subtree.

From (2.4.15) and (2.4.21) we find the expected length of the left path of the last right subtree in the stationary p-tree forest to be:

$$(5.5.1) \qquad RL^* = \sum_{k=1}^{n-1} \eta'^{(n)}_{k+2,\,1} (2H_k - 1) .$$

Referring to (5.1.12) we find

$$RL^* = 2\sum_{k=3}^{n} \eta_{k,1}^{(n)}(H_{k-2} - 1) + \sum_{k=3}^{n} \eta_{k,1}^{(n)}$$

$$= \frac{n+1}{n}\eta_{2,1} - \frac{3}{n} + (1 - \eta_{2,1})$$

(5.5.2) $$RL^* = 1 - \frac{1}{n}(3 - \eta_{2,1}) \quad .$$

Inserting the approximate value for $\eta_{2,1}$ in Proposition 4.1.1 we find

(5.5.3) $$RL^* = 1 - \frac{7}{3n} + O\left(\frac{H_n}{n^2}\right) \quad .$$

5.6 <u>The Expected Recursion Depth</u>.

Inserting the values of the expected recursion depth in the normal p-tree forest:

(5.6.1) $$C_{F_0}(n) = \frac{2}{3}H_{n+1} + \frac{1}{9} \qquad (n \geq 2)$$

$$C_{F_0}(0) = C_{F_0}(1) = 1$$

in (2.4.16), yields

(5.6.2) $$C^* = 1 + \frac{1}{n+1}\left(\lambda_0 + 2\lambda_1 + \sum_{k=2}^{n-2}(k+1)\lambda_k\left(\frac{2}{3}H_{k+1} + \frac{1}{9}\right)\right) \quad .$$

using (5.1.8) the latter sum becomes:

$$\sum_{k=2}^{n-2} (k+1)\lambda_k\left(\frac{2}{3} H_{k+1} + \frac{1}{9}\right)$$

$$= \sum_{k=4}^{n} \beta_k\left(\sum_{r=2}^{k-2} \left(\frac{3k-2r-5}{k-1} - 2(H_{k-1} - H_{r+1})\right)\left(\frac{2}{3} H_{r+1} + \frac{1}{9}\right)\right)$$

$$= \sum_{k=3}^{n} \beta_k\left(4 H_{k-1} + k - 12 + \frac{6}{k-1}\right) \quad .$$

Again using (5.1.8) for $r = 0$ and 1 we see

$$C^* = 1 + \frac{1}{n+1}\beta_2 + \frac{1}{n+1} \sum_{k=3}^{n} \beta_k\left(4 H_{k-1} + k - 12 + \frac{6}{k-1} + \frac{3k-5}{k-1} - 2 H_{k-1} + 2 H_1 + \frac{3k-7}{k-1} - 2 H_{k-1} + 2 H_2\right)$$

and eventually

$$(5.6.3) \qquad C^* = 1 + \frac{1}{n+1} \sum_{k=2}^{n} \beta_k(k-1) \quad .$$

Inserting the values from (5.2.1) we find:

$$C^* = \frac{2}{3} H_n - \frac{1}{6} + O\left(\frac{H_n}{n}\right) \quad .$$

References

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