DISTANCES IN ORIENTATIONS OF GRAPHS

by

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STAN-CS-75-517
AUGUST 1975

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School of Humanities and Sciences
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Abstract

We prove that there is a function $h(k)$ such that every undirected graph G admits an orientation H with the following property: if an edge uv belongs to a cycle of length k in G then uv or vu belongs to a directed cycle of length at most $h(k)$ in H. Next, we show that every undirected bridgeless graph of radius r admits an orientation of radius at most r^2+r, and this bound is best possible. We consider the same problem with radius replaced by diameter. Finally, we show that the problem of deciding whether an undirected graph admits an orientation of diameter (resp. radius) two belongs to a class of problems called NP-hard.

This research was supported in part by National Science Foundation grant DCR72-03752 A02 and by the Office of Naval Research contract NR 044-402.

0. Introduction.

In 1939, H. E. Robbins [12] proved that an undirected graph G admits a strongly connected orientation if and only if G is connected and bridgeless. If G is thought of as the system of two-way streets in a city, then the theorem gives necessary and sufficient conditions for being able to make every street in the city one-way and still get from every point to every other point. The theorem, however, asserts nothing about the distance one has to travel from x to y in the one-way system as compared to the distance between x and y in the two-way system. Actually, the comparison may be quite discouraging: if G is a cycle of length k then for each strongly connected orientation H of G, there are vertices x and y such that x and y are adjacent in G but it takes k-1 edges to get from x to y in H.

J. A. Bondy and U. S. R. Murty proposed to study quantitative variations on Robbins' theorem. In particular, they conjectured the existence of a function f such that every bridgeless graph of diameter d admits an orientation of diameter f(d). We shall prove their conjecture as a corollary to a rather general theorem. The theorem asserts that every undirected graph G admits an orientation H with the following property: if an edge uv belongs to a cycle of length k in G then uv or vu belongs to a cycle of length at most

$$(k-2)2^{[(k-1)/2]}+2$$

in H. It is an easy exercise to prove that, in a bridgeless graph G of diameter d, every edge belongs to a cycle of length at most 2d+1.

Thus our theorem implies that the conjecture is true; in fact, it implies that

$$f(d) \leq d((2d-1)2^d + 1) \quad .$$

This bound may be drastically improved. Indeed, we shall prove that every bridgeless graph of radius r admits an orientation of radius at most r^2+r (and this bound is best possible). It follows immediately that

$$f(d) \leq 2d^2 + 2d \quad .$$

On the other hand we shall show that

$$f(d) \geq \frac{1}{2} d^3 - i - d$$

and we shall describe graphs of diameter d and arbitrary high (possibly infinite) connectivity such that every orientation has diameter at least

$$\frac{1}{4} d^2 + d \quad .$$

Thus the order of growth of f is established; however, to find the exact values of f seems to be difficult. As a first step in this direction, we show that $f(2) = 6$; the Petersen graph provides the lower bound. Finally, we shall turn our attention to the general problem of finding, for an undirected graph G , its orientation with a minimum diameter (resp. radius). We shall show that this problem is very difficult: in a sense, it is as difficult as the problem of deciding whether G has a hamiltonian cycle or the problem of finding the chromatic number of G .

In general, we follow the standard graph-theoretical notation and terminology, see Berge [2] or Harary [7]. In an undirected (resp.

directed) graph G , the distance dist(u,v;G) from u to v is the number of edges in a shortest path (resp. directed path) from u to v . Note that for an undirected graph G , the function dist(u,v;G) is a metric whereas for directed graphs G , we often have dist(u,v;G) $\neq$ dist(v,u;G) . Unlike Moon [11], we postulate dist(u,u;G) = 0 . The diameter of a graph G is the longest distance in G ; the radius of G is

$$\min_u \max_v (\max\{dist(u,v;G) , dist(v,u;G))) .$$

Thus the diameter of G is at least the radius and at most twice the radius of G . Note that the diameter and the radius are defined only for connected undirected graphs and for strongly connected directed graphs. Also, if there is no finite upper bound on the distances in G then the diameter and radius are undefined.

1. From Cycles to Directed Cycles.

In the theorem below, we set

$$h(k) = (k2)2^{[(k-1)/2]} + 2$$

for every integer k such that $k \geq 3$.

THEOREM 1. Every graph G admits an orientation H with the following property: if an edge uv belongs to a cycle of length k in G then uv or vu belongs to a directed cycle of length at most h(k) in H.

PROOF. Let H_3 be a maximal directed graph such that H_3 is an orientation of some subgraph of G and such that every edge of H_3 is in a directed cycle of length three. (If G is infinite then H_3 exists by Zorn's lemma.) When H_{i-1} has been chosen for some i, let H_i be a maximal directed graph such that $H_{i-1} \subset H_i$ and H_i is an orientation of some subgraph of G and every edge of H_i is in a directed cycle of length at most i. The graph

$$H = \bigcup_{i=3}^{\infty} H_i$$

is not necessarily an orientation of G: the bridges of G do not belong to H. However, we shall prove that H has the other desired property: if an edge uv belongs to a cycle of length k in G then uv or vu belongs to a directed cycle of length at most h(k) in H. Clearly, that is all we need: the edges belonging to no cycles of G (that is, the bridges of G) may be directed quite arbitrarily.

Let us consider a cycle $u_1, u_2, \ldots, u_k, u_1$ in G such that, for some n, neither $u_1 u_k$ nor $u_k u_1$ belongs to H_n; all we have to prove is that $n < h(k)$.

For each i with $3 < i \leq n$, let x_i (resp. y_i) denote the number of those directed edges $u_{j+1} u_j$ (resp. $u_j u_{j+1}$) that belong to H_i but not to H_{i-1}. For each $m = 3, 4, \ldots, n$, we shall prove that

$$\sum_{i=3}^{m} (i-2) x_i \geq m+1-k \quad . \tag{1}$$

For this purpose, consider the graph H_m^* obtained from H_m by adding the directed edge $u_k u_1$ and all the directed edges $u_j u_{j+1}$ such that

$u_{j+1}u_j \notin H_m$. Clearly, all the new edges of H_m^* lie on a directed cycle of length at most

$$1 + \sum_{j=1}^{k-1} \text{dist}(u_j, u_{j+1}; H_m^*) = \left(k - \sum_{i=3}^{m} x_i\right) + \sum_{i=3}^{m} (i-1)x_i \quad .$$

By the **maximality** of H_m , this number is at least m+1 and so (1) follows.

Next, define $m(0) = 2$ and, for every positive integer t , $m(t) = (k-2)2^{t-1} + 2$. For each t such that $m(t) \le n$, we shall prove that

$$\sum_{i=3}^{m(t)} x_i \geq t \quad . \tag{2}$$

Let t be the smallest nonnegative integer such that $m(t) \le n$ and such that (2) fails. Trivially, $t \geq 1$; by the minimality of t , we have

$$t-1 \le \sum_{i=3}^{m(t-1)} x_i \le \sum_{i=3}^{m(t)} x_i \le t-1$$

and so $x_i = 0$ for $m(t-1) < i \le m(t)$. Consequently,

$$\sum_{i=3}^{m(t)} (i-2)x_i = \sum_{i=3}^{m(t-1)} (i-2)x_i$$

$$\le \sum_{s=1}^{t-1} (m(s)-2) \sum_{i=m(s-1)+1}^{m(s)} x_i = \sum_{s=1}^{t-1} (m(s) - m(s-1)) \sum_{i=m(s-1)+1}^{m(t-1)} x_i$$

$$\le \sum_{s=1}^{t-1} (m(s) - m(s-1))(t-s) = \sum_{s=1}^{t-1} m(s) - (t-1)m(0)$$

$$= m(t) - k \ ,$$

contradicting (1). The same argument shows that, for each t such that $m(t) \leq n$, we have

$$\sum_{i=3}^{m(t)} y_i \geq t \quad .$$

Now, we cannot have $n \geq h(k) = m([(k+1)/2])$: indeed, this would imply

$$\sum_{i=3}^{n} x_i + \sum_{i=3}^{n} y_i \geq 2\left[\frac{k+1}{2}\right] \geq k$$

which is clearly a contradiction.

<u>COROLLARY 1</u>. - Let G be a graph such that every edge of G belongs to a cycle of length at most k. Then there is an orientation H of G such that

$$\text{dist}(u,v;H) \leq (h(k)-1)\text{dist}(u,v;G)$$

for every two vertices u and v.

A particular instance of Corollary 1 (to be used in Section 3) asserts the following: if every edge of G belongs to a triangle then there is an orientation H of G such that

$$\text{dist}(u,v;H) \leq 3\cdot\text{dist}(u,v;G)$$

for every two vertices u and v.

Note that $h(3) = 4$ and $h(4) = 6$. If h' is any function such that Theorem 1 holds with h' instead of h, then we must have $h'(3) \geq 4$ (as demonstrated by a wheel with an odd number of spokes) and $h'(4) \geq 6$ (as demonstrated by the pentagonal prism) so for $k = 3, 4$ the result of Theorem 1 is best possible.

However, we do not know if h' can be chosen to be a polynomial or even a linear function.

2. From Radius to Directed Radius.

THEOREM 2. Every bridgeless graph of radius r admits an orientation of radius at most r^2+r .

PROOF. We shall find it useful to work with orientations of multigraphs: in such orientations, the multiple edges may be directed both ways (whereas the single edges must be directed only one way). By induction on r , we shall prove the following statement: "if G is a bridgeless multigraph and if u is a vertex of G such that $\mathrm{dist}(u,v;G) \le r$ for every vertex v then there is an orientation H of G such that $\mathrm{dist}(u,v;H) \le r^2+r$ and $\mathrm{dist}(v,u;H) \le r^2+r$ for every vertex v ".

For every neighbor v of u , the edge uv is contained in some cycle; let $k(v)$ denote the length of a shortest such cycle. It is important to note that

$$k(v) \le 2r+1 \quad \text{for every } v ; \tag{3}$$

the proof of this fact is left to the reader. An orientation A of some subgraph of G will be called admissible if there is a set S of neighbors of u , together with a directed cycle C_v for each $v \in S$, such that

(i) each C_v has length $k(v)$ and contains either the edge uv or the edge vu ,

(ii) **A** is the union of all these cycles C_v ($v \in S$) .

Note that by (3) and this definition, we have

$$\text{dist}(u,w;A) \le 2r \quad \text{and} \quad \text{dist}(w,u;A) \le 2r \tag{4}$$

for every vertex w of A . Furthermore, we shall prove that

every maximal admissible graph contains all the neighbors of u . (5)

Assume the contrary, so that $w \notin A$ for some maximal admissible graph A and for some neighbor w of u . There is a cycle $w_1, w_2, \ldots, w_k, w_1$ in G such that $w_1 = u$, $w_2 = w$ and $k = k(w)$. If none of the vertices $w_2, w_3, \ldots, w_k$ belongs to A then adding the directed circuit $w_1 \to w_2 \to \ldots \to w_k \to w_1$ to A we obtain a larger admissible graph: a contradiction. Thus we may assume that at least one of the vertices $w_2, w_3, \ldots, w_k$ belongs to A ; let w_i be such a vertex with the smallest subscript. Since $w_i \in A$, there is some $v \in S$ such that $w_i \in C_v$. Writing $u = v_1 \to v_2 \to \ldots \to v_m \to v_1$ for C_v , we have $m = k(v)$ and either $v = v_2$ or $v = v_m$; there is no loss of generality in assuming that $v = v_2$. We also have $w_i = v_j$ for some j . Now, we shall distinguish between two cases.

<u>Case</u> 1. $w_k = v$. In this case, define C_w to be the directed cycle

$$u \to v_2 \to v_3 \to \ldots \to v_j \to w_{i-1} \to w_{i-2} \to \ldots \to w_2 \to u$$

and note that C_w has length $k(w)$. (Indeed, if C_w had more than $k(w)$ edges then the path $u, v, v_3, \ldots, v_j$ would be longer than the path $u, v, w_{k-1}, \ldots, w_i$. In that case, the closed walk $u, v, w_{k-1}, \ldots, w_i, v_{j+1}, \ldots, v_m, u$ would produce a cycle in G of length less than $k(v)$ and yet containing the edge uv : a contradiction.) Adding C_w to A we obtain a larger admissible graph: a contradiction.

<u>Case2</u>. $w_k \neq v$. In this case, define C_w to be the directed cycle

$$u \to w_2 \to w_3 \to \dots \to w_i \to v_{j+1} \to v_{j+2} \to \dots \to v_m \to u$$

and note that C_w has length $k(w)$. (Indeed, if C_w had more than $k(w)$ edges then the path $v_j, v_{j+1}, \dots, v_m, u$ would be longer than the path $w_i, w_{i+1}, \dots, w_k, u$. In that case, the closed walk $u, v, v_3, \dots, w_k, u$ would produce a cycle in G of length less than $k(v)$ and yet containing the edge uv : a contradiction.) Adding C_w to A we obtain a larger admissible graph: a contradiction.

Now, (5) is proved and the rest is fairly straightforward. Consider a maximal admissible graph A . (If G is infinite then the existence of A follows by Zorn's lemma.) In G , contract all the vertices of A into a new vertex u^* (this may create new multiple edges) and call the resulting graph G^* . Note that G^* is bridgeless and that by (5), we have

$$\text{dist}(u^*, v; G) \leq r-1$$

for every vertex of G^* . By the induction hypothesis, there is an orientation H^* of G^* such that

$$\text{dist}(u^*, v; H^*) \leq r^2-r \quad \text{and} \quad \text{dist}(v, u^*; H^*) \leq r^2-r \qquad (6)$$

for every vertex v of G^* . We may think of H^* as an orientation of some subgraph of G . Continuing this orientation with A (and directing all the remaining edges of G arbitrarily) we obtain an orientation H of G . By (4) and (6), we have

$$\mathrm{dist}(u,v;H) \le r^2+r \quad \text{and} \quad \mathrm{dist}(v,u;H) \le r^2+r$$

for every vertex v of G .

<u>THEOREM 3</u>. For every positive integer r there is a bridgeless graph G_r of radius r such that every orientation of G_r has radius r^2+r .

<u>PROOF</u>. We shall construct a certain sequence $H_1, H_2, \ldots$ of rooted graphs. H_3 is simply a triangle with one of its vertices designated as the root. To construct H_r , take a cycle $u_0, u_1, \ldots, u_{2r}, u_0$ and two disjoint copies of H_{r-1} . Then identify the root of the first (resp. second) copy of H_{r-1} with u_1 (resp. u_{2r}). The resulting graph, rooted at u_0 , is H_r . Finally, G_r is obtained by taking two disjoint copies of H_r and identifying their roots. The graph G_3 is shown in Figure 1; we leave it to the reader to verify that G_r has the desired property.

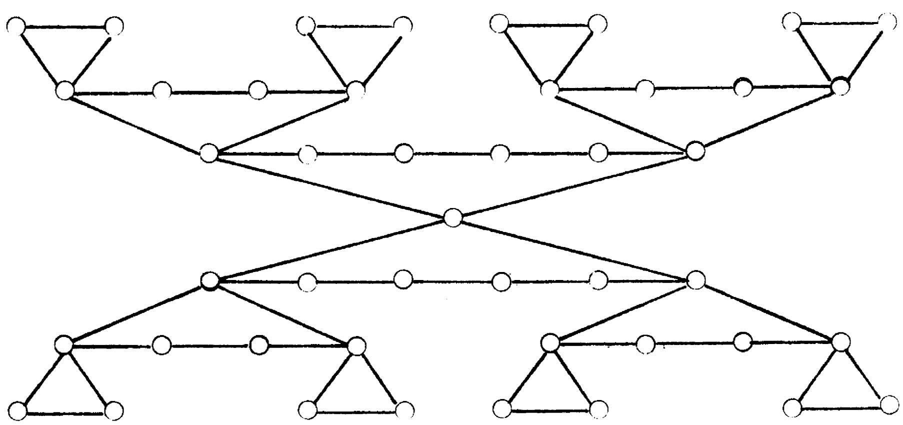

Figure 1

REMARK. The graphs G_r constructed above are so easy to study because of their simple structure and numerous cutpoints. We do not know if there are undirected graphs G of arbitrarily high connectivity and radius r such that every orientation of G has radius at least r^2+r. Nevertheless, we can construct undirected graphs G of arbitrarily high connectivity and radius r such that every orientation of G has radius at least $r^2/2+r$. These are the graphs $G_{2r,k}$ constructed in Theorem 4.

3. <u>From Diameter to Directed Diameter</u>.

For each positive integer d, let f(d) be the smallest integer such that every bridgeless graph of diameter d admits an orientation of diameter at most f(d). By Theorem 2, we have $f(d) \leq 2d^2+2d$. On the other hand, the reader may verify that each of the graphs G_r of Theorem 3 has diameter $d = 2r$ and that every strongly connected orientation of G_r has diameter $2(r^2+r) = d^2/2+d$. Let G'_r denote the graph obtained from H_r and H_{r+1} (of Theorem 3) by identifying their roots. Then G'_r has diameter $d = 2r+1$ and every strongly connected orientation of G'_r has diameter $r^2+r+(r+1)^2+(r+1) = \frac{1}{2}d^2+d+\frac{1}{2}$. Hence $f(d) \geq \frac{1}{2}d^2+d$ for all $d \geq 2$.

<u>THEOREM 4</u>. For each pair d, k, where d is a positive integer and k is a finite or infinite cardinal, there is a k-connected undirected graph $G_{d,k}$ of diameter d such that every orientation of $G_{d,k}$ has diameter at least $\frac{1}{4}d^2+d$.

PROOF. Begin with disjoint sets of vertices $S_1, S_2, \ldots, S_m$ such that S_1 and S_m have cardinality 1, each of S_i $(2 \le i \le m-1)$ has cardinality k, and

$$m = \begin{cases} 1 + (d+1)^2/2 & (d \text{ odd}), \\ 1 + d(d+2)/2 & (d \text{ even}). \end{cases}$$

Then, for each $i = 1, 2, \ldots, m-1$, join every vertex from S_i to every vertex from S_{i+1}. The resulting graph is k-connected; by adding as few as d edges, we shall bring its diameter down to d. To do so, we shall first define

$$i(j) = \begin{cases} 1 + (j+1)j & \text{for } 0 \le j \le d/2, \\ m - (d-j+1)(d-j) & \text{for } d/2 \le j \le d. \end{cases}$$

Note that $3 = i(1) < i(2) < \ldots < i(d-1) = m-2$ and that the sequence of differences $i(j+1) - i(j)$ is 2, 4, 6, ... 6, 4, 2. For each $j = 0, 1, \ldots, d$ choose a vertex $u_j \in S_{i(j)}$. For each $j = 0, 1, 2, \ldots, d-1$ join u_j to u_{j+1}. Call the resulting graph $G_{d,k}$. The graph $G_{4,2}$ is shown in Figure 2. It is easy to see that $G_{d,k}$ has diameter d. Now consider any strongly connected orientation of $G_{d,k}$. Let P_1 (resp. P_2) denote a shortest directed path from u_0 to u_d (resp. from u_d to u_0). Let ℓ_s denote the length of P_s for $s = 1, 2$. For each j, $0 \le j \le d-1$, at least one of the paths P_s, $1 \le s \le 2$, contains a subpath from $S_{i(j)}$ to $S_{i(j+1)}$ avoiding the edge between u_j and u_{j+1}, so $\ell_1 + \ell_2 \ge \sum_{j=0}^{d-1} (i(j+1) - i(j) + 1) = m - 1 + d$. Hence the diameter is at least $\frac{1}{2}(m-1+d) \ge \frac{1}{4}d^2 + d$.

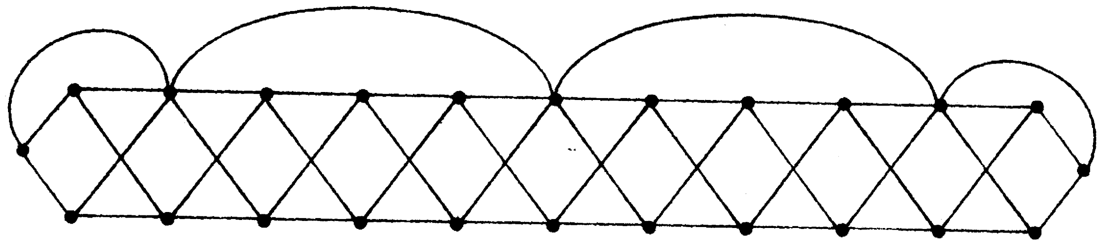

Figure 2

In the rest of this section, we shall prove $f(2) = 6$.

THEOREM 5. Every bridgeless graph of diameter two admits an orientation of diameter at most six.

PROOF. Let G be a bridgeless graph of diameter two. We may assume that some edge uv of G is contained in no triangle (otherwise the desired conclusion follows from Corollary 1). Let A (resp. B) denote the set of all the neighbors of u (resp. v) other than v (resp. u). Furthermore, let A_1 (resp. B_1) denote the set of all the vertices in A (resp. B) that have no neighbors in B (resp. A). Set $A_2 = A-A_1$, $B_2 = B-B_1$ and denote by C the set of all the vertices not in $\{u,v\} \cup A \cup B$. The reader may easily verify that the orientation of G , described simply by $u \to v \to B \to C \to A \to u$ and $B_1 \to B_2 \to A_2 \to A_1$, has diameter at most six. (Here $X \to Y$ means that every edge joining a vertex of x of X with a vertex y of Y is directed from x to y .)

Next, we shall prove that Theorem 5 is best possible. A part of our argument is of independent interest; therefore we shall state it on its own.

<u>LEMMA</u>. Every strongly connected orientation of the Petersen graph contains a directed cycle of length five.

<u>PROOF</u>. Let H be a strongly connected orientation of the Petersen graph; assume that H contains no directed cycle of length five. Since H is strongly connected, it contains some directed cycle; furthermore, the shortest directed cycle has no diagonals. In the Petersen graph, there are no cycles of length seven and every cycle of length greater than seven has a diagonal. Hence we may assume that H contains a directed cycle of length six. In the Petersen graph, every two cycles of length six are equivalent under some automorphism; hence we may assume that H contains the directed cycle 1 4 2 4 3 4 4 4 5 4 6 4 1 shown in Figure 3.

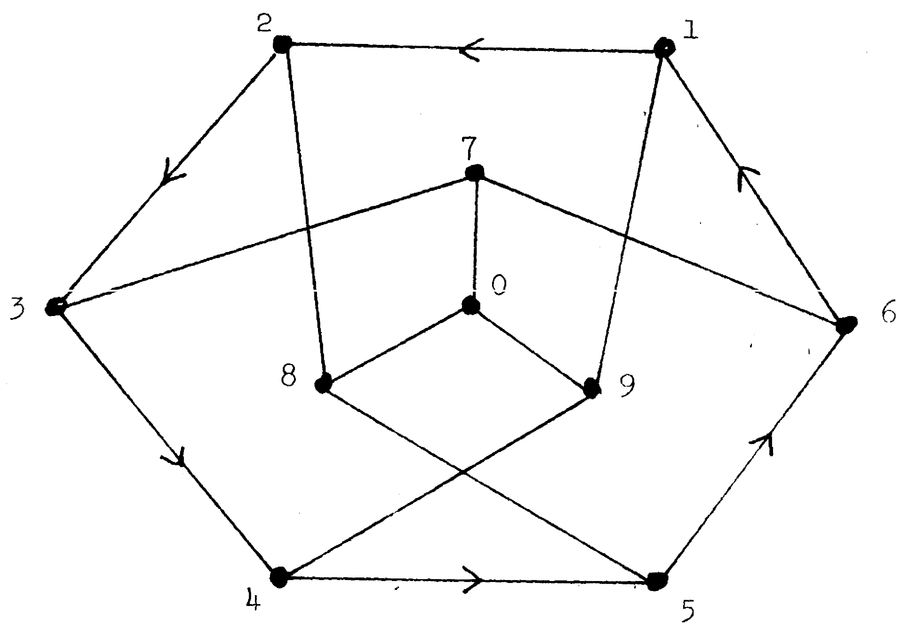

Figure 3

At least one edge of H enters 0 and at least one edge of H leaves 0 ; without loss of generality, we may assume $7 \to 0$ and $8 \to 0 \to 9$. Now, we must have $8 \to 2$ (otherwise $2 \to 8$ forces

creating the directed cycle $5 \to 6 \to 1 \to 2 \to 8 \to 5$). Similarly, we must have $3 \to 7$ (otherwise $7 \to 3$ forces $6 \to 7$ creating the directed cycle $3 \to 4 \to 5 \to 6 \to 7 \to 3$). But then $3 \to 7 \to 0 \to 8 \to 2 \to 3$ is a directed cycle of length five: a contradiction.

(Let us digress in order to mention a problem suggested by the lemma. Which bridgeless graphs G have the property that every strong orientation of G contains a directed cycle whose length equals the girth of G ? The Petersen graph has this property and so does every bridgeless graph of radius one.)

THEOREM 6. Every orientation of the Petersen graph has diameter at least six.

PROOF. Let us assume that some orientation H of the Petersen graph has diameter at most five. By the lemma, H contains a directed cycle of length five. In the Petersen graph, every two cycles of length five are equivalent under some automorphism; hence we may assume that H contains the cycle $1 \to 2 \to 3 \to 4 \to 5 \to 1$ shown in Figure 4.

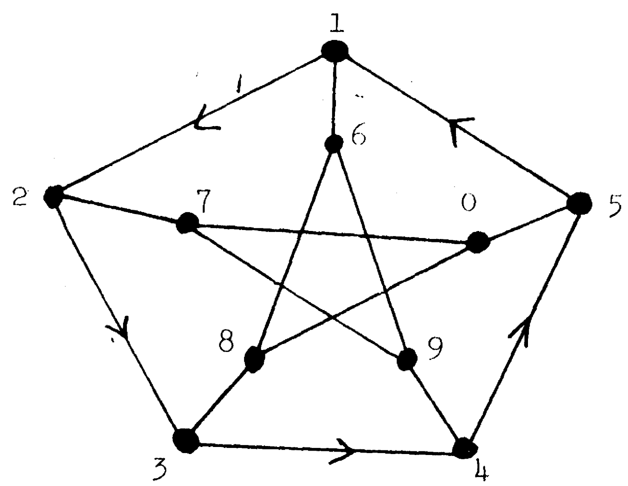

Figure 4

Consider the "cross edges" 16, 27, 38, 49, 50 ; each of them may be directed towards the pentagon or away from it.

<u>Case 1</u>. Three consecutive cross edges are directed in the same sense. Without loss of generality, we may assume that $0 \to 5$, $6 \to 1$ and $7 \to 2$. Here, the contradiction is immediate: $\mathrm{dist}(5,7;H) \geq 6$.

<u>Case 2</u>. No three consecutive edges are directed in the same sense. We may assume $8 \to 3$ and $9 \to 4$, forcing $5 \to 0$, $2 \to 7$ and, in turn, $6 \to 1$. Now, $\mathrm{dist}(3,6;H) \leq 5$ forces $0 \to 8 \to 6$ and $\mathrm{dist}(3,9;H) < 5$ forces $0 \to 7 \to 9$. But then $\mathrm{dist}(1,8;H) \geq 6$, a contradiction.

We do not know any other bridgeless graph of diameter 2 which can play the role of the Petersen graph in Theorem 6. Perhaps the Moore graph of diameter two and degree seven, constructed by Hoffman and Singleton [8], is another example.

4. Finding Optimum Orientatiops.

Given an undirected graph G , let us ask for its orientation with smallest possible diameter (resp. radius). Clearly, such an orientation can be found in a finite time: the diameter and the radius of a directed graph on n vertices can be found in $O(n^3)$ steps [6] and an undirected graph with m edges has 2^m distinct orientations. In the spirit of Edmonds [5], we shall ask for a "better-than-finite" algorithm finding optimum orientations; more specifically, we shall ask for such an algorithm terminating within p(n) steps for some fixed polynomial p . Our results are rather discouraging; "they suggest that no such an algorithm exists.

The key notion is that of an "NP-hard" problem. A certain class of problems is called NP. This class is very wide: it consists of all the problems for which the correctness of a proposed solution may be checked in a polynomial time (relative to the size of the problem). That is, NP consists of all the problems with "good characterizations" (this notion is due to Edmonds [4]). For example, the problem "Is a graph G k-colorable?" belongs to NP. Now, let X be a problem with the following property: if X can be solved in a polynomial time then every problem in NP can be solved in a polynomial time. Such a problem is called NP-hard. (To many, it seems unlikely that every problem in NP can be solved in a polynomial time; such a belief implies that no NP-hard problem can be solved in a polynomial time.) In a pioneering paper [3], Cook proved that it is NP-hard to find the largest clique in a graph. Since then, many other people have shown many other problems to be NP-hard; as a rule, this is done by "reducing" the problem of finding the largest clique in a graph (or another problem which already has been shown to be NP-hard) into the problem in question. (For more information on the subject, the reader is referred to [1].)

In particular, Lovász [10] has shown that it is NP-hard to decide if a hypergraph is 2-colorable; it is implicit in his proof that the same problem remains NP-hard even when the input is restricted to hypergraphs of rank three. The relevant definitions may be found in Berge's monograph [2]; for the sake of completeness, we shall repeat them here. A _hypergraph_ is an ordered pair $H = (V,E)$ such that V is a set and such that E is a family of subsets of V. The elements of V are called the _vertices_ of H, the elements of E are called the _edges_ of H. The number of vertices of H is called the _order_ or H, the cardinality of the largest edge of' Ii is called the _rank_ of H. A hypergraph is called _2-colorable_ if its vertices can be colored red and blue in such a way that every edge includes at least one vertex of each color.

THEOREM 7. Given a hypergraph H of rank three and order n, we can construct in $O(n^6)$ steps a graph G with the following property: G admits an orientation of diameter two if and only if H is 2-colorable.

PROOF. Let k be the integer satisfying $10 \leq k \leq 13$ and $n+k \equiv 2 \pmod 4$. We shall find it convenient to work with the hypergraph H_0, obtained from H by adding k new vertices $v_1, v_2, \ldots, v_k$ and, if H has an even number of edges, adding new edge $\{v_1, v_2\}$. Note that H_0 has an odd number of edges. To construct G , take disjoint sets P and Q such that the elements of P (resp. Q) are in a one-to-one correspondence with the vertices (resp. the edges) of H_0 ; for simplicity, we shall label each element of P (resp. Q) by the corresponding vertex (resp. edge) of H_0 . Join by an edge every two vertices in P and every two vertices in Q ; join a vertex $v \in P$ to a vertex $e \in Q$ if and only if $u \in e$ in H_0 . Then add four vertices w_1 , w_2 , w_3 , w_4 and join each of them to all the vertices in $P \cup Q$. Finally, add a new vertex x and join it to all the vertices in P . We shall show that the resulting graph G has the desired property. (Note that the number of edges of G may be of the order n^6 .)

Firstly, assume that G admits an orientation G^* of diameter two. Color a vertex u of H blue (resp. red) if in G^* , we have $x \to u$ (resp. $u \to x$). Since $dist(x,e;G^*) = 2$ (resp. $dist(e,x;G^*) = 2$) for every $e \in Q$, every edge of H includes at least one blue (resp. red) vertex. Thus H is 2-colorable.

Secondly, assume that H is 2-colorable. Then H_0 admits a 2-coloration such that the number of blue (and red) vertices is odd and at least five; this 2-coloration induces a partition $P = P_1 \cup P_2$. We are going to describe an orientation G^* of G ; before doing so, let us digress a little. By a <u>cyclic tournament</u> of order 2k+1 , we shall mean the tournament with vertices $u_1, u_2, \ldots, u_{2k+1}$ such that $u_i \to u_{i+j}$ for every $j = 1,2,\ldots,k$ (arithmetic modulo 2k+1). The <u>parity partition</u> of such a tournament is the partition $A \cup B$ defined by $A = \{u_1, u_3, \ldots, u_{2k+1}\}$ and $B = \{u_2, u_4, \ldots, u_{2k}\}$. If $k \geq 2$ then the parity partition has the following nice properties:

(i) if $u \notin A$ then there are $v_1, v_2 \in A$ such that $v_1 \to u \to v_2$,

(ii) if $u \notin B$ then there are $v_1, v_2 \in B$ such that $v_1 \to u \to v_2$.

Now, the orientation G^* of G may be described as follows. For each of the three sets P_1, P_2, Q, direct the edges of the complete graph induced by that set so as to obtain a cyclic tournament. Let $A_i \cup B_i$ be the parity partition of P_i $(i = 1,2)$ and let $A \cup B$ be the parity partition of Q. Direct

$$x \to P_1 \to P_2 \to x \ ,$$
$$P_1 \to Q \to P_2 \ ,$$
$$A_1 \cup A_2 \to w_1 \to A \ ,$$
$$B \to w_1 \to B_1 \cup B_2 \ ,$$
$$A_1 \cup A_2 \to w_2 \to B \ ,$$
$$A \to w_2 \to B_1 \cup B_2 \ ,$$
$$B_1 \cup B_2 \to w_3 \to A \ ,$$
$$B \to w_3 \to A_1 \cup A_2 \ ,$$
$$B_1 \cup B_2 \to w_4 \to B \ ,$$
$$A \to w_4 \to A_1 \cup A_2 \ .$$

We leave it to the reader to verify that G^* has diameter two.

<u>COROLLARY 2</u>. It is NP-hard to decide whether an undirected graph admits an orientation of diameter two.

<u>THEOREM 8</u>. Given a hypergraph H of rank three and order n, we can construct in $O(n^6)$ steps a graph G with the following property: G admits an orientation of radius two if and only if H is 2-colorable.

PROOF. Take disjoint sets P and Q such that the elements of P (resp. Q) are in a one-to-one correspondence with the vertices (resp. the edges) of H . Join by an edge every two vertices in P ; join a vertex $v \in P$ to a vertex $e \in Q$ if and only if $v \in e$ in H . Then add a new vertex x and join it to all the vertices in P ; call the resulting graph G_0 . To construct G , take two disjoint copies of G_0 and identify their vertices x . We leave it to the reader to verify that G has the desired property.

COROLLARY 3. It is NP-hard to decide whether an undirected graph admits an orientation of radius two.

REMARK. Easy modifications of our constructions show that (i) for every k with $k \geq 4$, it is NP-hard to decide whether an undirected graph admits an orientation with diameter at most k , (ii) for every k with $k \geq 4$, it is NP-hard to decide whether an undirected graph G admits an orientation with radius at most k . To prove (i), take the graph G as constructed in Theorem 7, add a cycle $u_0, u_1, \ldots, u_{k-2}, u_0$ and identify u_0 with x . To prove (ii), take two disjoint copies of the graph G_0 constructed in Theorem 8, add a tree T consisting of four paths of length k-2 starting at the same vertex. Now take two of the four end vertices of T and identify them with the vertex x in one of the copies of G_0 , then identify the remaining two end vertices of T with the vertex x in the other copy of G_0 .

REMARK. Corollary 2 shows that in general, it is very hard to decide whether an undirected graph G admits an orientation of diameter two. However, if G has too few edges then the answer is always negative. More precisely, Katona and Szemerédi[9] proved that no undirected graph with n vertices and fewer than $\frac{n}{2} \log_2 \frac{n}{2}$ edges admits an orientation of diameter two.

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