

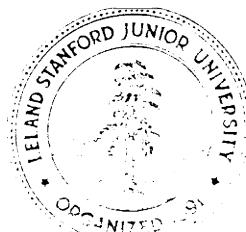
ON SUBGRAPH NUMBER INDEPENDENCE IN TREES

by

R. L. Graham
E. Szemerédi

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School of Humanities and Sciences
STANFORD UNIVERSITY



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R. L. Graham
Stanford University
and
Bell Laboratories, Murray Hill, New Jersey
and

E. Szemerédi
Stanford University
and
The Hungarian Academy of Sciences

Abstract

For finite graphs F and G , let $N_F(G)$ denote the number of occurrences of F in G , i.e., the number of subgraphs of G which are isomorphic to F . If \mathcal{F} and \mathcal{G} are families of graphs, it is natural to ask them whether or not the quantities $N_F(G)$, $F \in \mathcal{F}$, are linearly independent when G is restricted to \mathcal{G} . For example, if $\mathcal{F} = \{K_1, K_2\}$ (where K_n denotes the complete graph on n vertices) and \mathcal{G} is the family of all (finite) trees then of course $N_{K_1}(T) - N_{K_2}(T) = 1$ for all $T \in \mathcal{G}$. Slightly less trivially, if $\mathcal{F} = \{S_n : n=1,2,3,\dots\}$ (where S_n denotes the star on n edges) and \mathcal{G} again is the family of all trees then

$$\sum_{n=1}^{\infty} (-1)^{n+1} N_{S_n}(T) = 1 \quad \text{for all } T \in \mathcal{G}.$$

It will be proved that such a linear dependence can never occur if \mathcal{F} is finite, no $F \in \mathcal{F}$ has an isolated point and \mathcal{G} contains all trees. This result has important applications in recent work of L. Lovász and one of the authors [2].

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R. L. Graham
Stanford University
Stanford, California

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Bell Laboratories
Murray Hill, New Jersey

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E. Szemerédi
Stanford University
Stanford, California

and

The Hungarian Academy of Sciences
Budapest, Hungary

INTRODUCTION

It is a trivial observation (in fact, almost a definition) that in any finite tree T , the number of vertices of T always exceeds the number of edges of T by exactly 3. In [1], it was asked to what extent this can happen for graphs in general. That is, given a finite family \mathcal{F} of graphs G , when can there be a fixed linear dependence between the number of occurrences of the $G \in \mathcal{F}$ as subgraphs of a tree T which is valid for all finite* trees T . In this paper, we answer this question. In particular, this can never happen if none of the $G \in \mathcal{F}$ have isolated points;

* All graphs considered in this paper will be finite. For terminology see [3].

SOME NOTATION

For a graph G , we let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G , respectively. If H is a labelled graph (i.e., with distinguishable vertices) and G is an unlabelled graph, we define $N_G(H)$ to be the number of occurrences of G in H , i.e., the number of ways a subset of $|E(G)|$ edges can be selected from $E(H)$ together with i vertices from $V(H)$ if G has i isolated vertices, so that the resulting subgraph of H is isomorphic to G . Of course, the product of $N_G(H)$ and the order of the automorphism group of G is just $E_G(H)$, the number of ways of embedding G into H (considering G as labelled graph). For example, if G and H are as shown in Fig. 1 then $N_G(H) = 28$ and $E_G(H) = 112$.

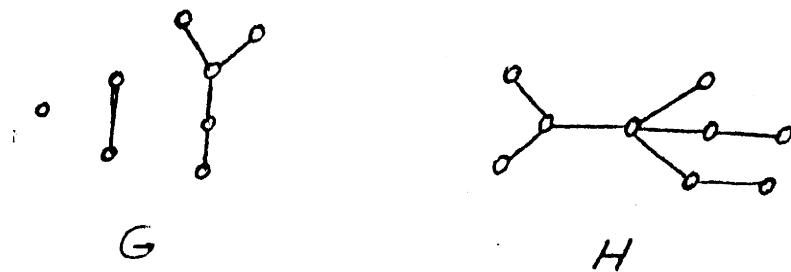


Fig. 1
Note that if the isolated point is removed from

G to form G' then $N_{G'}(H) = 14 = \frac{1}{2} N_G(H)$. Of course, in general, if G is formed from a graph G' having no isolated points by adjoining i isolated points then

$$(1) \quad N_G(H) = \binom{|V(H)| - |V(G')|}{i} N_{G'}(H)$$

THE MAIN RESULT

The primary result of this paper can be stated as follows.

Theorem. Let \mathcal{F} be a finite family of forests,* each having no isolated points, and suppose there exist real numbers A_F , $F \in \mathcal{F}$, and A_0 such that the equation

$$(2) \quad \sum_{F \in \mathcal{F}} A_F N_F(T) = A_0$$

is valid for all trees T . Then $A_F = 0$ for all $F \in \mathcal{F}$.

Remark. Since any subgraph of a tree is a forest then there is no loss of generality in assuming \mathcal{F} is a family of forests.

Proof: We may assume without loss of generality that among all families for which an equation of the form (2) is possible, \mathcal{F} has the least number of elements. The basic idea of the proof will be to construct a very large tree W^* for which one of the quantities $N_F(W^*)$ is much larger than all the others, thereby forcing its coefficient A_F to be 0. However, this contradicts the minimality of $|\mathcal{F}|$.

If T is a tree with a distinguished vertex v , we let $T^{(k)}$ denote the tree formed from T by adjoining k disjoint paths of length k to v . (See Fig. 2).

* i.e., acyclic graphs,

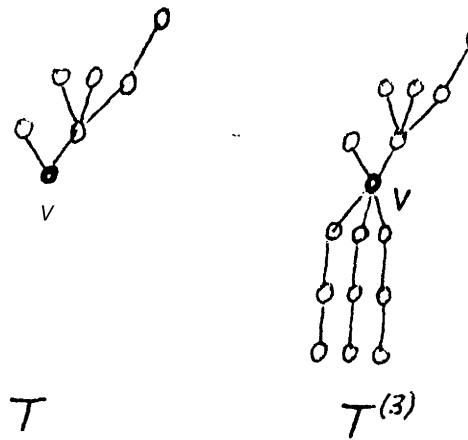


Fig. 2

Similarly, if F is a forest with components T_1, \dots, T_n having distinguished vertices v_1, \dots, v_n , respectively, then $F^{(k)}$ denotes the forest with components $T_1^{(k)}, \dots, T_n^{(k)}$.

We now define a (possibly empty) forest $W = W(\mathcal{F})$ with components W_i and distinguished vertices $w_i \in V(W_i)$, $1 \leq i \leq t$, as follows:

- (i) Some $F \in \mathcal{F}$ occurs as a subgraph of $W(k)$ for some k .
- (ii) $|E(W)|$ is minimal among all W satisfying (i).

Note that by (ii) no paths leave w_i in W_i . Define \mathcal{F}' to be the set $\{F \in \mathcal{F} : F \subseteq W(k) \text{ for some } k\}$.

Next, we choose s to be a large fixed integer, depending only on \mathcal{F} , to be determined later. For (large) integers n , define n_k by

$$n_k = [n^{1+s^{-k}}], \quad 1 \leq k \leq s(s+t).$$

We are finally ready to define the tree $W^* = W^*(n)$.

1. W^* will have a subset of $2s+t-1$ vertices, called special vertices, denoted by $X = \{x_1, \dots, x_s\}$, $Y = \{y_1, \dots, y_{s-1}\}$ and $\{w_1, \dots, w_t\}$.
2. For $1 \leq k \leq s$, x_k has n_k paths of length 1 attached to it.
3. For $1 \leq k \leq s-1$, y_k has n_{ks+j} paths of length j attached to it for $1 \leq j \leq s$.
4. For $1 \leq k \leq t$, w_k has $n_{s(s+k-1)+j}$ paths of length j attached to it for $1 \leq j \leq s$.
5. Also attached to w_k is a copy of W_k with w_k being the distinguished vertex of W_k .
6. The special vertices are joined sequentially by paths of length s , i.e., between adjacent vertices in the sequence $(x_1, \dots, x_s, y_1, \dots, y_{s-1}, w_1, \dots, w_t)$ are placed paths of length s .

This completes the construction of W^* . In Fig. 3 we illustrate the structure of W^* .

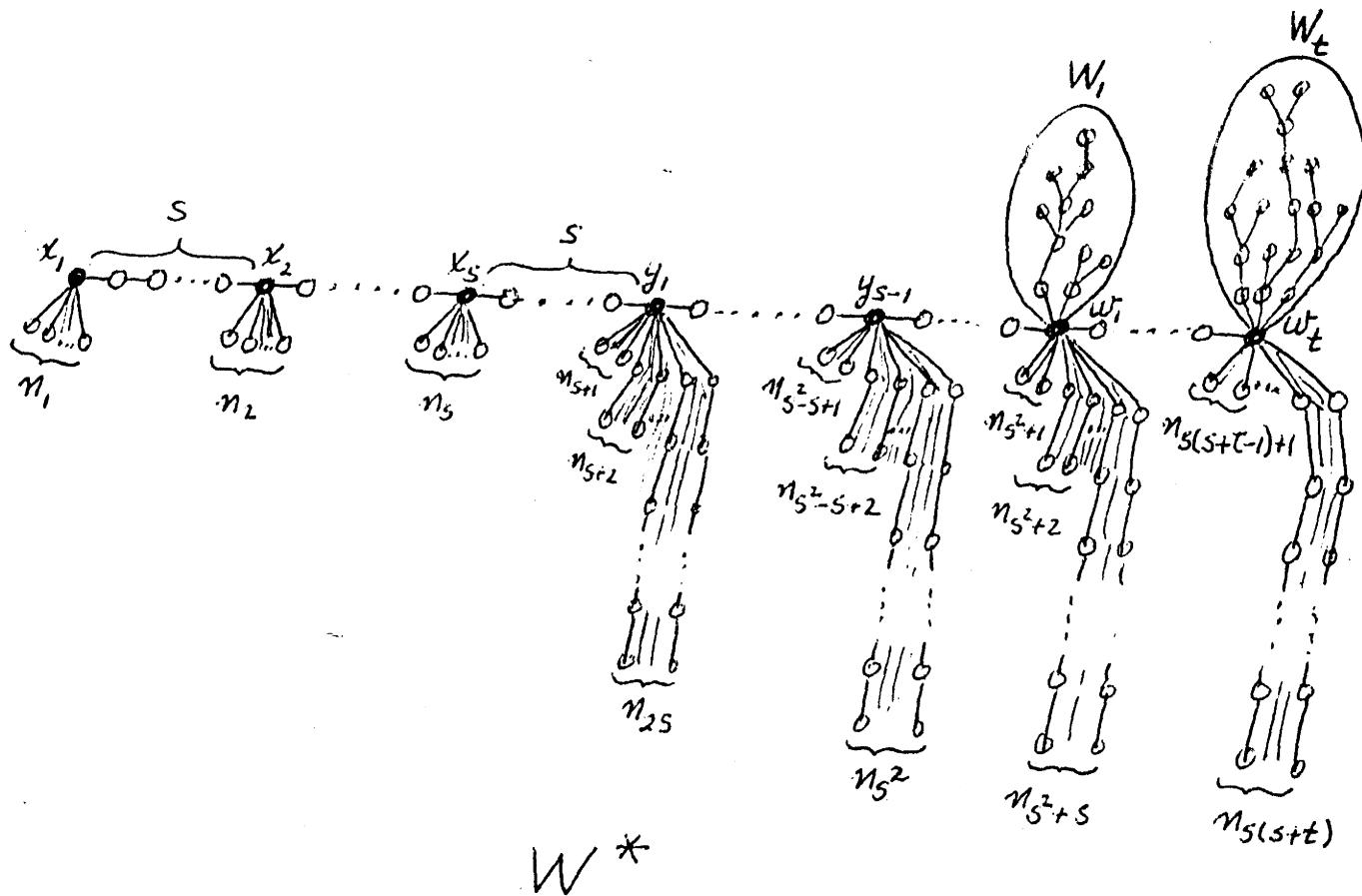


Fig. 3

By hypothesis, we have

$$\sum_{F \in \mathcal{F}} A_F N_F(W^*(n)) = A_0$$

for all n . However, since by the definition of \mathcal{F}' , no $F \in \mathcal{F} - \mathcal{F}'$ occurs as a subgraph of $W(k)$ for any k , then it is not difficult to see that $N_F(W^*(n)) = 0$ for these F , provided we have chosen s and n sufficiently large. Hence, we have

$$(3) \quad \sum_{F \in \mathcal{F}'} A_F N_F (W^*(n)) = A_0$$

for all sufficiently large n . It is important to note that by the minimality assumptions we have made, any embedding of any $F \in \mathcal{F}'$ into W^* must use all the edges of all the w_i , $1 < i < t$, in W^* , again, provided s and n are sufficiently large.

Fact. For any distinct $F, F' \in \mathcal{F}'$, either

$$N_F(W^*)/N_{F'}(W^*) > n^{s-s^3}$$

or

$$N_{F'}(W^*)/N_F(W^*) > n^{s-s^3}$$

for n sufficiently large.

For suppose the Fact holds. Since we must have $|\mathcal{F}'| > 1$, then there is some element $F^* \in \mathcal{F}'$ such that

$$N_{F^*}(W^*)/N_F(W^*) > n^{s-s^3}$$

for all $F \in \mathcal{F}' - \{F^*\}$. By (3) we have

$$(4) \quad A_{F^*} + \sum_{F \in \mathcal{F}' - \{F^*\}} A_F \left(\frac{N_F(W^*)}{N_{F^*}(W^*)} \right) = \frac{A_0}{N_{F^*}(W^*)} .$$

But as $n \rightarrow \infty$, all terms in (4) tend to zero except A_{F^*} which is nonzero 'by hypothesis. This contradiction would then prove the theorem.

Proof of Fact: Let F and F' be two distinct elements of \mathcal{F}' . Partition the components of F into three classes: F_1 , the set of stars, i.e., trees with at most one vertex of degree > 2 ; F_2 , the non-stars which are star-like, i.e., trees with at most one vertex of degree ≥ 3 ; and F_3 , the non-star-like trees, i.e., those having at least two vertices of degree > 3 .

Define F'_1 , F'_2 and F'_3 in an analogous way for F' . As we have noted earlier, F'_3 must consist of t trees T_1, \dots, T_t where T_k is formed from W_k by adjoining a (nonempty!) set of paths to w_k (with a similar remark applying to F'_3).

We need one more concept. A weak attachment α of F to W^* is formed as follows. A vertex u_i is selected from each component C_i of F . These u_i are mapped by an injection α into the set of special vertices of W^* with the restrictions that:

$$\alpha(u_i) = \begin{cases} x_j \text{ for some } j \text{ if } C_i \in F_1, \\ y_j \text{ for some } j \text{ if } C_i \in F_2, \\ w_j \text{ for some } j \text{ if } C_i \in F_3. \end{cases}$$

A weak attachment α of F to W^* is said to be proper if α can be extended to an embedding of F into W^* . We let $|\alpha|$ denote the number of ways α can be extended to an embedding of F into W^* . Note that in a proper weak attachment α of F to W^* , u_i must be a vertex of C_i of maximal degree if $C_i \in F_1 \cup F_2$. Define the sequence $\tau(\alpha) = (\tau_1, \tau_2, \dots, \tau_{s(s+t)})$ as follows:

$$\tau_k = \begin{cases} \text{number of paths of length 1 leaving } u_i \text{ for } \alpha(u_i) = x_k, 1 < k \leq s, \\ \text{number of paths of length } j \text{ leaving } u_i \text{ for } \alpha(u_i) = y_\ell \text{ where } k = \ell s + j \text{ for } 1 \leq j \leq s, 1 < \ell < s-1, \\ \text{number of paths of length } j \text{ leaving } u_i \text{ for } \alpha(u_i) = w_m \text{ where } k = s^2 + (m-1)s + j \text{ for } 1 < } \end{cases}$$

It is then clear that

$$|\alpha| < K_0 \prod_{k=1}^{s(s+t)} n_k^{\tau_k}$$

where K_0, K_1, \dots will denote constants depending on s and not on n . The sequences $\tau(\alpha)$ can be linearly ordered as follows.

For $\tau(\alpha) = (\tau_1, \tau_2, \dots, \tau_{s(s+t)})$ and

$\tau(\alpha') = (\tau'_1, \tau'_2, \dots, \tau'_{s(s+t)})$, we define

$\tau(\alpha') > \tau(\alpha)$ if either:

$$(i) \quad \sum_{k=1}^{s(s+t)} \tau'_k > \sum_{k=1}^{s(s+t)} \tau_k; \text{ or}$$

$$(ii) \quad \sum_{k=1}^{s(s+t)} \tau'_k = \sum_{k=1}^{s(s+t)} \tau_k \text{ and } \tau(\alpha') \text{ is lexicographically}$$

greater than $\tau(\alpha)$, i.e., for some m , $\tau'_k = \tau_k$ for $1 \leq k < m$ and $\tau'_m > \tau_m$.

We let $\tau^{(F)} = (\tau_1^{(F)}, \dots, \tau_{s(s+t)}^{(F)})$ denote a maximal sequence $\tau(\alpha)$ in this ordering as α ranges over all proper weak attachments of F to W^* . The proof of the Fact will depend on the following assertion.

Claim: If $\tau^{(F^I)} > \tau^{(F)}$ then $N_{F^I, (W^*)}/N_{F, (W^*)} > n^{s-s^3}$ for n sufficiently large,

Proof of Claim: Suppose $\tau^{(F^I)} > \tau^{(F)}$. It is easily seen that

$$(6) \quad N_{F^I, (W^*)} \geq \prod_{k=1}^{s(s+t)} \left(\frac{n_k}{\tau_k^{(F')}} \right) > K_1 \prod_{k=1}^{s(s+t)} n_k^{\tau_k^{(F')}}$$

On the other hand, it is not hard to show that

$$(7) \quad N_F(W^*) < K_2 \prod_{k=1}^{s(s+t)} n_k^{\tau_k(F)}$$

To see this, we consider F as a labelled forest and we show that

$$E_F(W^*) < K_3 \prod_{k=1}^{s(s+t)} n_k^{\tau_k(F)}$$

for a suitable constant $K_3 = K_3(s)$.

First, the non-star-like trees in F_3 can only be embedded into the W_i parts of W^* and, since the total number of proper weak attachments of F_3 to W^* is bounded by a function of s , then the embedding of the non-star-like trees of F contributes a factor of at most $K_4 \prod_{k=s^2+1}^{s(s+t)} n_k^{\tau'_k}$ where $\tau'(\beta) = (\tau'_{s^2+1}, \dots, \tau'_{s(s+t)})$ is a (maximal) sequence derived from some proper weak attachment β of F_3 to W^* .

Next, consider an embedding of a star-like tree $T \in F_2$ which is not a star. Suppose T is formed by adjoining m_k paths of length k , $1 < k < s$, to the "center" vertex u . Although it may be possible to embed T into W^* by mapping u onto some $x_i \in X$ (e.g., when at most two of the m_k , $k \geq 2$, are nonzero), when this is done we must use edges in one of the paths of length s connecting x_i to adjacent special vertices of W^* , and so, there are at most

$K_5 \frac{n^{-1}}{n_1} \sum_{k=1}^s m_k$ such embeddings, However, this factor is negligible compared to the corresponding factor of

$K_6 \frac{n^{-s}}{n_2} \sum_{k=1}^s m_k$ which we obtain if we embed T by mapping u onto some $y_i \in Y$ since

$$\frac{n^m}{s^2} \frac{n^{m-1}}{n_1} > K_7 \frac{n^{m(1+s^{-s^2})}}{n^{m(1+s^{-1})-1}} > K_8 n^{1/2}$$

provided s has been chosen sufficiently large for \mathfrak{F} and n is sufficiently large.

Finally, we consider a star $S \in \mathcal{F}_1$, say, consisting of m paths of length 1 adjoined to a vertex u . If $m \geq 3$ then in any embedding of F into W^* , u must be mapped onto some vertex in $X \cup Y$ since these are the only available vertices of degree ≥ 3 . However, since $n_k/n_{k+1} \rightarrow \infty$ as $n \rightarrow \infty$ then the dominant contribution will certainly come from the embeddings which map u onto some $x_i \in X$ (in fact, the smaller the index i , the better). If $m \leq 2$ then there are many ways of embedding S into W^* , for example, so that u does not map onto a special vertex of W^* . Again, however, the dominant term clearly comes from those embeddings which take u onto some special vertex $x_i \in X$.

Thus, all except a negligible fraction of the embeddings of F into W^* are extensions of proper weak

attachments α of F to W^* . Note that if α and α' are proper weak attachments of F to W^* and $\tau(\alpha') > \tau(\alpha)$ then by definition, either

$$\sum_{k=1}^{s(s+t)} \tau'_k > \sum_{k=1}^{s(s+t)} \tau_k$$

or

$$\sum_{k=1}^{s(s+t)} \tau'_k = \sum_{k=1}^{s(s+t)} \tau_k \text{ and for some } m \leq s(s+t),$$

$$\tau'_k = \tau_k \text{ for } 1 \leq m < k, \text{ and } \tau'_{m-1} > \tau_m.$$

In the first case,

$$\prod_{k=1}^{s(s+t)} n_k \tau'_k > K_9 \prod_{k=1}^{s(s+t)} n_k \tau_k^{(1+s^{-k})}$$

$$= K_9 \cdot n \sum_{k=1}^{s(s+t)} \tau'_k \cdot n \sum_{k=1}^{s(s+t)} \tau_k^{s^{-k}}$$

$$\geq K_9 n^{1+\sum_{k=1}^{s(s+t)} \tau_k}$$

$$> K_{10} n^{1/2s} \prod_{k=1}^{s(s+t)}$$

for s and n sufficiently large. In the second case

$$\prod_{k=1}^{s(s+t)} n_k^{\tau'_k} > K_9 \frac{\sum_{k=1}^{s(s+t)} \tau'_k}{n} \cdot \frac{\sum_{k=1}^{s(s+t)} \tau'_k s^{-k}}{n}$$

$$= K_9 \frac{\sum_{k=1}^{s(s+t)} \tau_k}{n} \cdot \frac{\sum_{k=1}^{m-1} \tau_k s^{-k}}{n} \cdot \frac{\sum_{k=m}^{s(s+t)} \tau'_k s^{-k}}{n}$$

But $\sum_{k=m}^{s(s+t)} \tau'_k s^{-k} \geq (\tau_{m+1}) s^{-m} = \tau_m s^{-m} + s^{-m}$

and $\sum_{k=m}^{s(s+t)} \tau_k s^{-k} \leq \tau_m s^{-m} + \sum_{k=m+1}^{s(s+t)} s^{1/2} s^{-k}$

$$\leq \tau_m s^{-m} + 2s^{-m-1/2}.$$

Hence, in either case,

$$(8) \quad \prod_{k=1}^{s(s+t)} n_k^{\tau'_k} \left/ \prod_{k=1}^{s(s+t)} n_k^{\tau_k} \right. > K_{11} n^{s^{-m} - 2s^{-m-1/2}} > K_{11} n^{s^{-2s^2}}$$

$$> K_{11} n^{1/s^3}$$

But since there are at most $K_{12} = K_{12}(s)$ proper weak attachments of F to W^* then by (5), (8), and the definition of $\tau^{(F)}$ we have

$$(9) \quad E_F(W^*) < K_{13} \prod_{k=1}^{s(s+t)} n_k^{\tau_k^{(F)}}$$

Hence, from (7) and (9), we have

$$N_{F'}(W^*)/N_F(W^*) \geq N_{F'}(W^*)/E_F(W^*)$$

$$> K_{14} \prod_{k=1}^{s(s+t)} n_k^{\tau_k^{(F')}} / \prod_{k=1}^{s(s+t)} n_k^{\tau_k^{(F)}} > n^{1/s} s^3$$

for n sufficiently large and the Claim is proved.

From the preceding discussion it is not difficult to see that if $\tau^{(F)} = \tau^{(F')}$ then F and F' are isomorphic which contradicts the hypothesis that they are distinct elements of \mathcal{F}' . Therefore, we must have $\tau^{(F)} \neq \tau^{(F')}$ and so the Fact always holds, provided s is sufficiently large. This completes the proof of the theorem. ■

CONCLUDING REMARKS

As we have seen in Eq. (1), when F has isolated points then $N_F(T)$ can be written as

$$(9) \quad N_F(T) = P(n)N_{F'}(T)$$

where $P(n)$ is a polynomial (depending on F) in $n = |V(T)|$ and F' has no isolated points. However, such an expression, valid for all trees T , can always be written in the form

$$(10) \quad P(n)N_{F'}(T) = \sum_{F \in \mathcal{F}_{F'}(d)} A_F N_F(T)$$

where $\mathcal{F}_{F'}(d)$ consists of all those forests which can be formed by adjoining exactly $d = \text{deg } P(n)$ additional edges to F' . This follows by the observation that

$$(11) \quad \binom{n-1-|E(F')|}{d} N_{F'}(T) = \sum_{F \in \mathcal{F}_{F'}(d)} N_F(F) N_F(T)$$

since the left-hand side of (11) can be interpreted as counting the number of ways of selecting a copy of F' in T together with d additional edges of T . For example, if F' is the forest shown in Fig. 4(a) then

$$(12) \quad (n-4)N_{F'}(T) = 2N_{F_1}(T) + 4N_{F_2}(T) + 2N_{F_3}(T) + 3N_{F_4}(T)$$

where the F_i are given in Fig. 4(b).

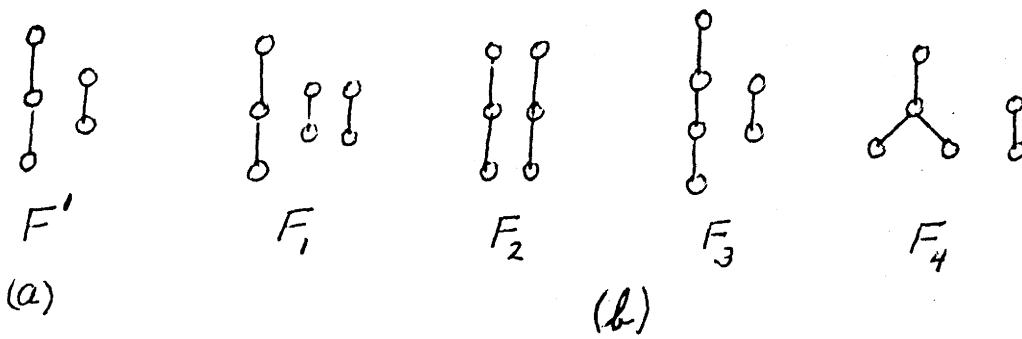


Fig. 4

We remark that if \mathcal{F} is allowed to be infinite then nontrivial linear dependences among the $N_F(T)$, $F \in \mathcal{F}$, can exist. For example, if S_k denotes the star with k edges,

then for $\mathcal{F} = \{S_k : k=1, 2, \dots\}$ we have

(13)

$$\sum_{k=1}^{\infty} (-1)^{k+1} S_k (T) = 1$$

for all trees T .

REFERENCES

- [1] M. Edelberg, M. R. Garey, and R. L. Graham, On the distance matrix of a tree (to appear).
- [2] Distance matrices of trees, R. L. Graham and L. Lovász (to appear).
- [3] F. Harary, Graph Theory, Addison-Wesley, New York (1969).