CHROMATIC AUTOMORPHISMS OF GRAPHS

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STAN-CS-72-273

MARCH 1972

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY

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Abstract

The coloring group and the full automorphism group of an n-chromatic graph are independent if and only if n is an integer ≥ 3 .

AMS (MOS subject classification numbers (1970):

Primary: 05C25, 05C15

Key words: Graph, coloring, automorphism group

This research was supported by the National Science Foundation under grant number GJ-992.

1. Introduction.

When coloring highly symmetric graphs, one often finds that the symmetries of a given graph determine to a certain extent the symmetries of its minimal colorings. We will say that an automorphism a and a coloring

$$c : V \longrightarrow R \qquad (1)$$

of a graph $H = (V,E)$ are <u>compatible</u> if there is a bijection $p : R \longrightarrow R$ with $c(a(v)) = p(c(v))$ for all $v \in V$. One might expect that a graph H having at least one non-identical automorphism always admits a non-identical automorphism a compatible with some minimal coloring of H (a minimal coloring of H is a coloring (1) with $|R|$ equal to the chromatic number $\chi(H)$ of H). However, this is not always the case. The 3-chromatic graph H in Fig.1 admits 30 distinct 3-colorings and four distinct non-identical automorphisms but none of the 120 pairs are compatible.

(Fig.1)

In discussions with Dr. Jarik Nešetřil of Charles University, we were led to the concept of a <u>chromatic automorphism</u> of H : this is an automorphism compatible with every minimal coloring of H . Obviously, the chromatic automorphisms form a subgroup C(H) of the full automorphism group A(H) of H . Besides, C(H) is always a normal subgroup of A(H) . To see this, let f be an arbitrary automorphism of H and let $a \in C(H)$. If c is a minimal coloring of H , then $c \circ f^{-1}$ is another such coloring and there is a $p : R \longrightarrow R$ such that $c \circ (\circ f^{-1} \circ a \circ f) = (c \circ f^{-1}) \circ a \circ f =$ $= p \circ (c \circ f^{-1}) \circ f = p \circ c$, that is, $f^{-1} \circ a \circ f \in C(H)$.

It is well-known that any group G is isomorphic to the full automorphism group of some graph H (Frucht [1] has been the first to prove this). Now, it is natural to ask which pairs (G,N) - where G is a group and N a normal subgroup of G - are representable as (A(H),C(H)) of some graph H . The answer is given in the next section.

2. The main result.

THEOREM. Let G be a group and let N be a normal subgroup of G . Let $n \geqslant 3$ be an integer. Then there exists an n-chromatic graph H with $A(H) \cong G$ and $C(H) \cong N$.

Proof. If G is the one-element group, then the statement follows immediately from the main result of [2] . From now on we shall assume that $|G| > 1$.

A graph H with the required properties will be constructed. To help the reader, we give first an informal description of the construction with $n = 3$ and then proceed in a more precise manner. Let e be the unit element of G and let $<$ be an arbitrary well-ordering of the set $G - \{e\}$. For each pair $(x,y) \in (G - \{e\})^2$ with $x < y$ we take a copy of the graph in Fig.3, for each pair $(x,y) \in G^2$ we take a copy of the graph in Fig.2. Identifying all the vertices with equal labels we obtain the desired 3-chromatic graph H.

(Fig.2)

(Fig.3)

More generally and more precisely, we set

$$\mathrm{ess}G^2 = \{(x,y) : x,y \in G \ , \ x \neq j\},$$

$$R = \{(x,y) : x,y \in G - \{e\} \ , \ x < y\} .$$

The vertex-set of H will be $V = V_1 \cup V_2 \cup \ldots \cup V_6$, where

$$V_1 = G \times \{1\} ,$$

$$V_2 = (G - \{e\}) \times \{2\} ,$$

$$V_3 = (G - \{e\}) \times \{3\} ,$$

$$V_4 = \mathrm{ess}G^2 \times \{1,2,\ldots,2n-1\} ,$$

$$V_5 = G/N \times (G - \{e\}) \times \{1,2,\ldots,n-1\} ,$$

$$V_6 = \{1,2,3\} \times R .$$

The edges of H will be all the two-point sets

$$\{((x,y),j),((x,y),k)\} \quad (0 < |j-k| < n),$$

$$\{(x,1),((x,y),j)\} \quad (0 < j < n) ,$$

$$\{(y,1),((x,y),j)\} \quad (n < j < 2n) ,$$

$$\{(x,1),(xN,z,j)\} ,$$

$$\{((x,y),j),(xN,x^{-1}y,k)\} \quad (0 < j < n, \ j \neq k) ,$$

$$\{(z,2),(z,3)\} ,$$

$$\{(z,2),(xN,z,j)\} ,$$

$\{(x,3),(j,(x,y))\}$ $(j \in \{1,2,3\})$,

$\{(y,3),(j,(x,y))\}$ $(j \in \{2,3\})$,

$\{(2,(x,y)),(j,(x,y))\}$ $(j \in \{1,3\})$

and no other ones. Now, we will show that the graph described above has all the desired properties.

Let a be an arbitrary automorphism of G . First of all, we note that the elements of V_2 are the only vertices of H not contained in any triangle of H . Therefore $a(V_2) = V_2$. When V_2 is removed, the resulting graph has just two components: the component induced by $V_3 \cup V_6$, which contains vertices of degree two in H , while the other component , induced by $V_1 \cup V_4 \cup V_5$, contains no such vertices. Thus $a(V_3 \cup V_6) = V_3 \cup V_6$ and $a(V_1 \cup V_4 \cup V_5) = V_1 \cup V_4 \cup V_5$. The elements of V_3 are the only vertices of the first component which are adjacent to the elements of V_2 , so that $a(V_3) = V_3$ and $a(V_6) = V_6$. A similar argument applied to $V_1 \cup V_4 \cup V_5$ yields $a(V_5) = V_5$ and $a(V_1 \cup V_4) = V_1 \cup V_4$. Since the group G is non-trivial, the degrees of the elements of V_1 are not smaller than 3n-3 , while V_4 contains only vertices whose degrees do not exceed 3n-4 . Thus $a(V_1) = V_1$ and $a(V_4) = V_4$. Altogether, $a(V_i) = V_i$ for i = 1,2,...,6 .

So we are in position enabling us to define bijections $a': G - \{e\} \longrightarrow G - \{e\}$ and $a^*: G \longrightarrow G$ by

$$a(x,2) = (a'(x),2) , \quad a(x,1) = (a^*(x),1) .$$

Since $(x,3)$ is the only element of V_3 adjacent to $(x,2)$, we have $a(x,3) = (a'(x),3)$. Moreover, it is easy to see that $x < y$ if and only if H has a vertex v of degree two whose distance from $(y,3)$ is two and which is adjacent to $(x,3)$. Consequently,

$$x < y \text{ if and only if } a'(x) < a'(y) . \qquad (2).$$

A well-ordered set, however, is a rigid structure: the only bijective transformation a' satisfying (2) is the identity mapping. Hence $a'(x) = x$ for all $x \in G - \{e\}$; we conclude that $a(u) = u$ for all $u \in V_2 \cup V_3$, which yields $a(u) = u$ for all $u \in V_6$ as an easy consequence.

The vertex $((x,y),n-1)$ is the only vertex in V_4 of degree $3n-4$ which is adjacent to $(x,1)$ and has distance two from $(y,1)$. Hence $a((x,y),n-1) = (a^*(x),a^*(y),n-1)$. Now, by a series of similar easy arguments, there it follows that $a((x,y),j) = (a^*(x),a^*(y),j)$ for all $j = 1,2,\ldots,2n-1$. Since $(xN,x^{-1}y,k)$ is the only vertex in V_5 adjacent to all $((x,y),j)$ with $0 < j < n$, $j \neq k$, the equality $a(xN,x^{-1}y, j) = (a^*(x)N,a^*(x)^{-1}a^*(y),j)$ must hold. Finally, $(x^{-1}y,2)$ is the only vertex in V_2 adjacent to each $(xN,x^{-1}y,j)$; hence $a(x^{-1}y,2) = (x^{-1}y,2)$ must also be adjacent to $(a^*(x)N,a^*(x)^{-1}a^*(y),j)$ for all j . Consequently,

$$a^*(x)^{-1}a^*(y) = x^{-1}y \qquad (3)$$

whenever $(x,y) \in \mathrm{ess}G^2$. Setting $x = e$ in (3) and writing $z = a^*(e)$ we obtain

$$a^*(y) = zy \tag{4}$$

for all $y \neq e$; $a^*(e) = z$ by definition. Our findings can be summarized as follows. Given any $a \in A(H)$ there is a $z_a = z \in G$ such that

$$\left.\begin{aligned} &a(x,1) = (zx,1) ,\\ &a((x,y),j) = ((zx,zy),j) ,\\ &a(xN,w,j) = (zxN,w,j) ,\\ &a(u) = u \quad \text{for all } u \in V_2 \cup V_3 \cup V_6 . \end{aligned}\right\} \tag{5}$$

Conversely, it is easy to verify that the formulas (5) define an automorphism of H for an arbitrary $z \in G$. It is clear that the assignment $a \longmapsto z_a$ is a group isomorphism of $A(H)$ onto G .

It is quite obvious that H is n-chromatic. Given any two vertices u, v of H , $u \sim v$ will mean that $c(u) = c(v)$ for each n-coloring c of H . It is not difficult to see that

$$(x,1) \sim ((x,y),n) \sim (y,1) \sim (x^{-1}y,2) ,$$
$$((x,y),j) \sim ((x,y),j+n) \sim (xN,x^{-1}y,j) \qquad (0 < j < n) .$$

If $z = z_a \in N$, then $zxN = xN$ for all $x \in G$; the corresponding automorphism a (defined by (5)) satisfies

$$a(u) = u \qquad \text{for all } u \in V_2 \cup V_3 \cup V_5 \cup V_6 ,$$
$$a(u) \sim u \qquad \text{whenever } u \in V_1 \text{ or } u = ((x,y),n) ,$$
$$a((x,y),j) = ((zx,zy),j) \sim (zxN,(zx)^{-1}(zy),j) = (xN,x^{-1}y,j) \sim ((x,y),j) ,$$

$$a((x,y),j+n) = ((zx,zy),j+n) \sim ((zx,zy),j) \sim$$
$$\sim ((x,y),j) \sim ((x,y),j+n) ,$$

whenever $0 < j < n$. Altogether, we have $a(u) \sim u$ for all $u \in V$; a is compatible with every minimal coloring, i.e., $a \in C(H)$.

Conversely, let $z = z_a \in G - N$. Set $p(1) = 2$, $p(2) = 1$, $p(j) = j$ for $j = 3,\dots,n$ and define a mapping $c : V \longrightarrow \{1,\dots,n\}$ by

$$c(u) = n \quad (u \in V_1 \cup V_2) ,$$
$$c((x,y),n) = n ,$$
$$c(u) = 1 \quad (u \in V_3) ,$$
$$c(2,(x,y)) = 2 ,$$
$$c(1,(x,y)) = c(3,(x,y)) = 3 ,$$
$$c(N,w,j) = c((x,y),j) = c((x,y),j+n) = j \ (0 < j < n) \quad \text{if } x \in N ,$$
$$c(xN,w,j) = c((x,y),j) = c((x,y),j+n) = p(j) \quad (0 < j < n) \text{ if } x \notin N .$$

It is easy to verify that c is a coloring of H . Let us note that $c(N,w,1) = 1 = c(x,3)$; however, $c(a(N,w,1)) = c(zN,w,1) = p(1) = 2$, while $c(a(x,3)) = c(x,3) = 1$. Hence a is not compatible with c , $a \notin C(H)$. We have shown that an automorphism a is chromatic if and only if $z_a \in N$. Thus $C(H) \cong N$ - which finishes the proof.

3. Concluding remarks.

Our theorem is best possible in the sense that the range of the chromatic number n of the representing graph cannot be extended without imposing additional restriction on the choice of N and G . The case n = 1 is trivial: every graph H =(V,E) with $\chi(H) = 1$ has $A(H) = C(H) \cong Sym_{|V|}$. The smallest pair (G,N) which is not realizable as (A(H),C(H)) of a 2-colorable H is $(C_3,\{e\})$. Indeed, if H = (V,E) is a 2-chromatic graph with $A(H) \cong C_3$ and $C(H) \cong \{e\}$, then H must be disconnected (otherwise H is uniquely colorable and every automorphism is chromatic). No two different components of H are isomorphic - if there were isomorphic components, A(H) would have an element of order two. Exactly one component has a non-trivial automorphism (otherwise $|A(H)| \geqslant 4$); denote this component by H_0 and the rest of the graph by H_1 . Let a be one of the two non-trivial automorphisms of H ; a is not chromatic. Let c be a 2-coloring of H which is not compatible with a . Since H_0 is uniquely colorable, c(u) = = c(v) is equivalent to c(a(u)) = c(a(v)) for all $u,v \in H_0$. As a is not compatible with c , c(a(u)) = 2 if c(u) = 1 and c(a(v)) = 1 if c(u) = 2 for all $u \in H_0$. But then $c(a^3(u)) \neq$ $\neq c(u)$, which is a contradiction as a^3 is the identity mapping.

Finally, we will show that (C_3,C_3) is not realizable as

(A(H),C(H)) of a graph H with infinite chromatic number n . Assume that there is such a graph H . It contains at most one vertex adjacent to all other vertices (if there were two such vertices u, v , then the mapping $a : V \longrightarrow V$ defined by a(u) = v , a(v) = u , a(w) = w for all the other vertices, would be an automorphism of H). V contains three distinct vertices u,v,w with a(u) = v , a(v) = w , a(w) = u ; at least one of them - say u - is not related to some other vertex u* . But then $\{a(u),a(u^*)\} \neq \{u,u^*\}$; since n+1 = n , there is a minimal coloring c of H with c(u) = c(u*) and $c(a(u)) \neq c(a(u^*))$. a is not chromatic, $C(H) \neq A(H)$, which is a contradiction.

References.

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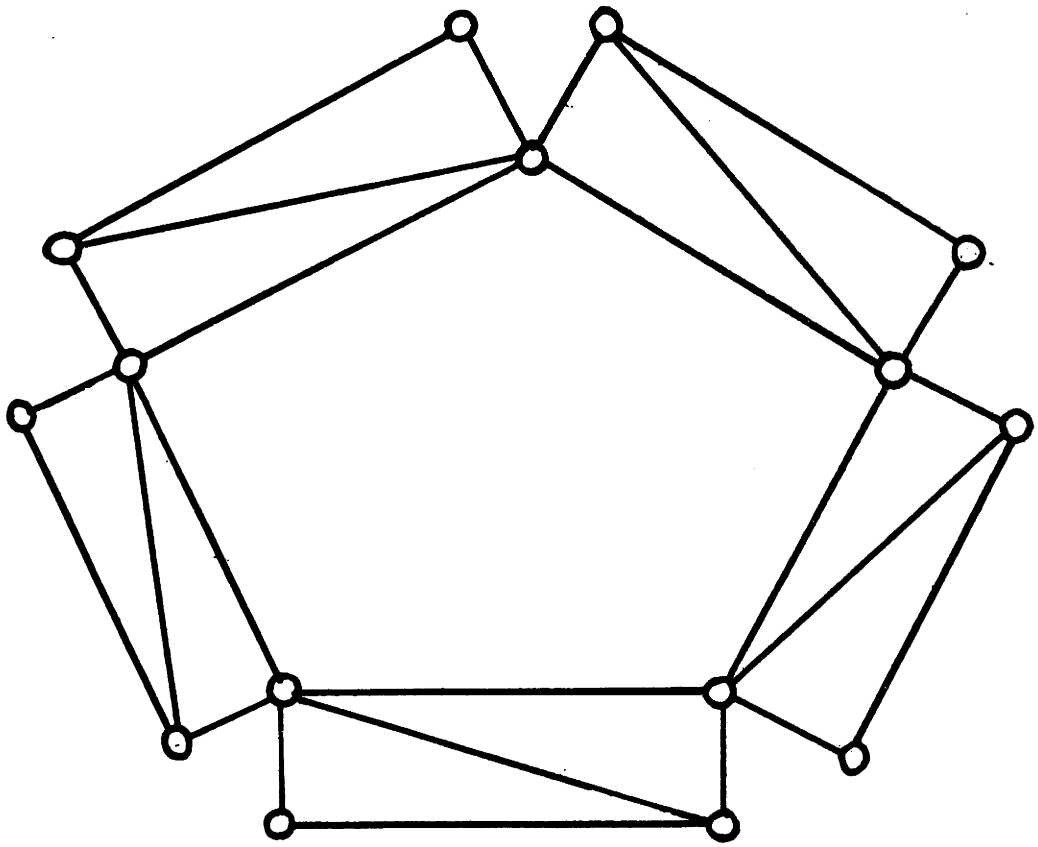
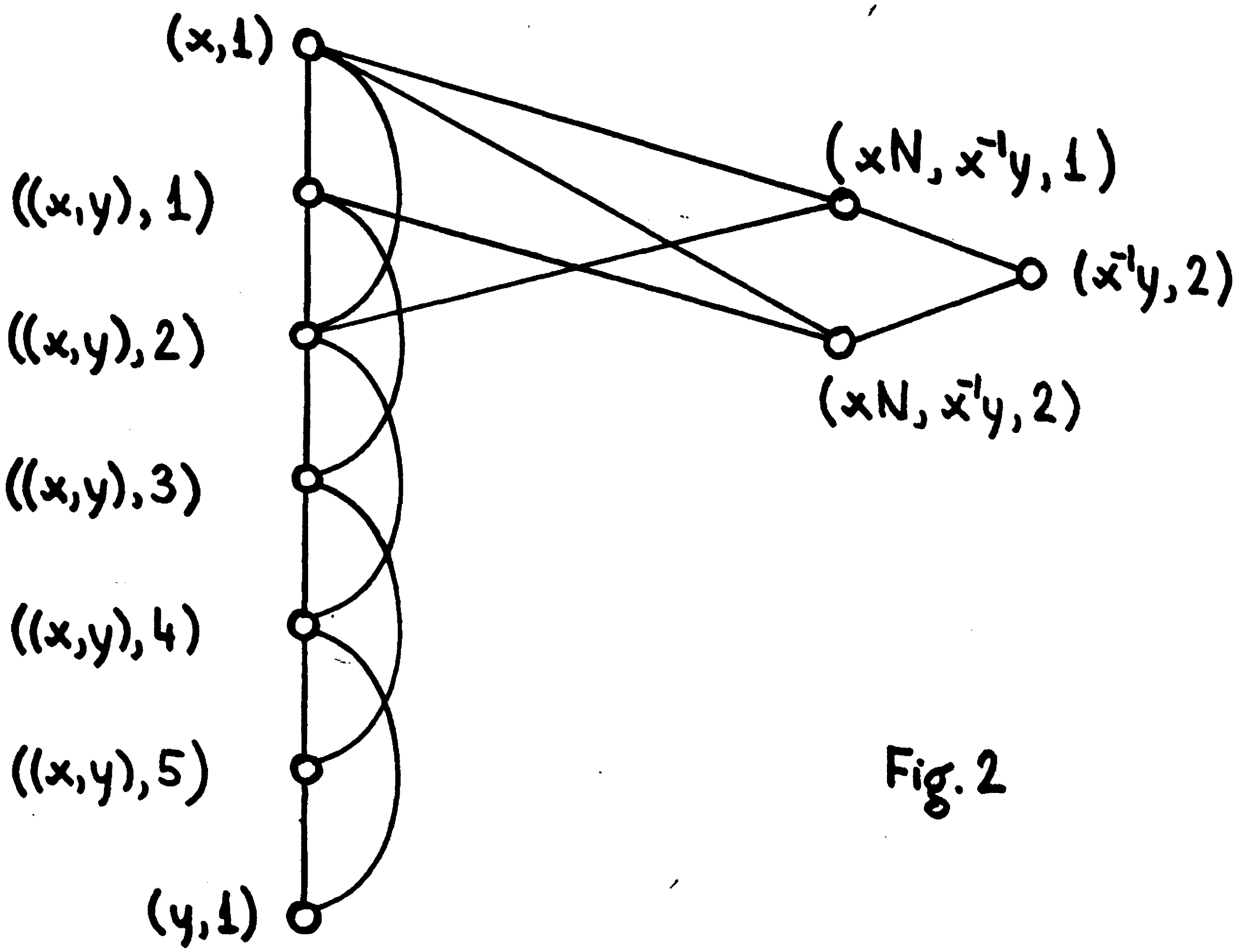
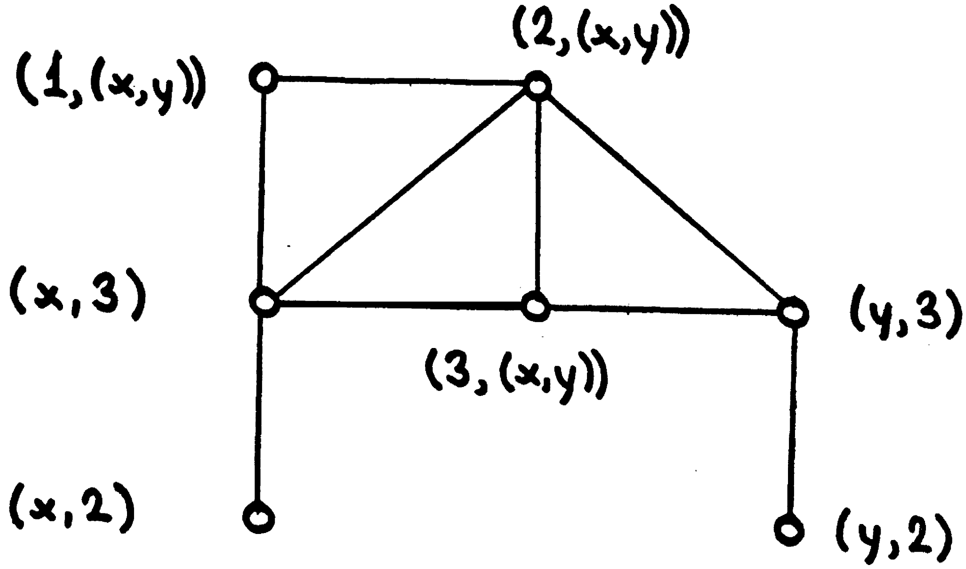


Fig. 1



(x,1)
((x,y),1)
((x,y),2)
((x,y),3)
((x,y),4)
((x,y),5)
(y,1)
(xN, x⁻¹y, 1)
(x⁻¹y, 2)
(xN, x⁻¹y, 2)

Fig. 2



(1,(x,y))
(2,(x,y))
(x,3)
(y,3)
(3,(x,y))
(x,2)
(y,2)

Fig. 3