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Large [g,d] Sorting Networks

by

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DIGITAL SYSTEMS LABORATORY

STANFORD ELECTRONICS LABORATORIES

STANFORD UNIVERSITY . STANFORD, CALIFORNIA

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ABSTRACT

With only a few exceptions the minimum-comparator N-sorter networks employ the generalized "divide-sort-merge" strategy. That is, the N inputs are divided among g ≥ 2 smaller sorting networks -- of size $N_1, N_2, \ldots, N_g$, where $N = \Sigma_{k=1}^{g} N_k$ -- that comprise the initial portion of the N-sorter network. The remainder of the N-sorter is a comparator network that merges the outputs of the Nl-, N_2-, ... and N_g-sorter networks into a single sorted sequence. The most economical merge networks yet designed, known as the "[g,d]" merge networks, consist of d smaller merge networks -- where d is a common divisor of $N_1, N_2, \ldots, N_g$ -- followed by a special comparator network labeled a "[g,d] f-network." In this paper we describe special constructions for $[2^r, 2^r]$ f-networks, $r > 1$, which enable us to reduce the number of comparators required by a large N-sorter network from $.25N(\log_2 N)^2 - .25N(\log_2 N) + O(N)$ to $.25N(\log_2 N)^2 - .37N(\log_2 N) + O(N)$.

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I. Introduction

A comparator network with 4 inputs is illustrated in **Fig.** 1(a). Each of the 5 comparators, labeled A, B, C, D, and E, compares its two inputs and emits the smaller on its higher output lead and the larger on its lower output lead. An abbreviated diagram for this comparator Let-work is given in Fig. 1(b), where each comparator is replaced by a vertical line connecting the two comparands.

A comparator network with N input and output leads is called an N-sorter network, or simply an N-sorter, if for any set* of inputs I = $\{i_1,i_2,\ldots,i_N\}$, the resulting outputs 0 = $\{o_1,o_2,\ldots,o_N\}$ satisfy: 1) 0 is a permutation of I; and 2) $o_j \le o_k$ if j ≤ k. The net-work depicted in Fig. 1 is a b-sorter, since comparators A through D move the smallest input to 01 and the largest input to o_4, and then comparator E orders the remaining two inputs.

The most economical general strategy known for constructing N-sorter networks, the [g,d] strategy, is introduced in [2]. Although it rep-resents an extension of the earlier paper, the present paper is self-contained. The earlier paper describes the [g,d] strategy for arbitrary g,d, and gives specific results for N-sorter networks with N ≤ 36. The present paper describes the especially efficient networks that have been discovered for the case g = d = 2^r.

---

* Since we wish to allow for the possibility that two or more inputs have the same value, we probably should refer to I and 0 as "**multisets,**" rather than as "sets." And we should then refer to particular members of I **as a "submultiset"** of I, rather than as a "subset." (See D. E. Knuth [1].) However, we prefer to use the more familiar terms "set" and "subset," and will do so consistently, even when all members of I are required to have one of the values 0 or 1.

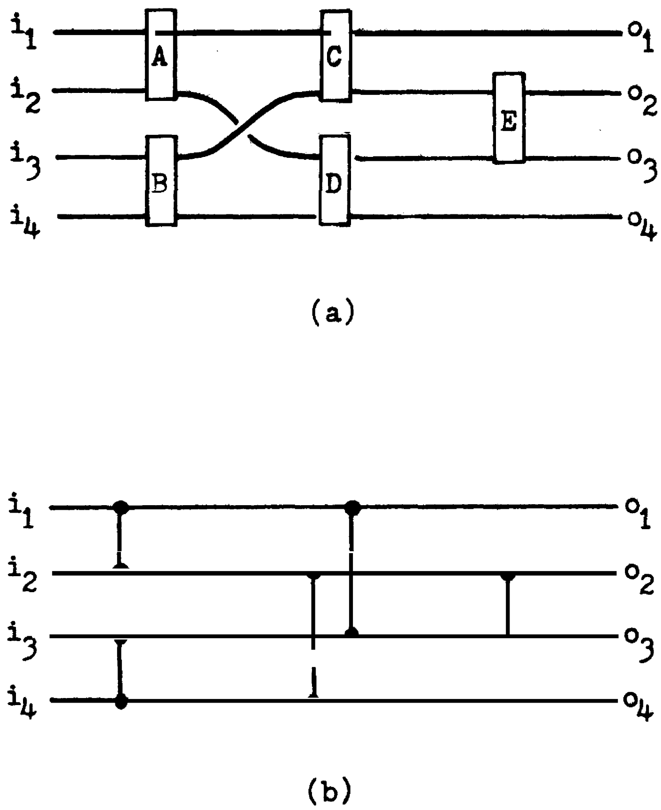


Fig. 1. b-sorter network.

II. [g,d] Sorting Networks

One way to determine whether a comparator network with N inputs and N outputs is an N-sorter network is to verify that it will sort all N! permutations of the numbers 1, 2, ...N as inputs. However, the following theorem reduces to 2^N the number of input patterns required to test a comparator network.

Theorem 1: (Zero-One Principle)

A comparator network with N inputs and N outputs is an N-sorter network if and only if it will sort all 2^N combinations of N inputs where each input is either 0 or 1.

Proof: See references 2, 3, and 4.

Although 2^N grows much more slowly than N!, it is not feasible to test large networks for 2^N different combinations of inputs. Therefore, if we desire large sorting networks, we must build them in such a way that we can guarantee "by construction" that they will sort all combinations of inputs. The most economical strategy known for designing large sorting networks, the <u>[g,d] strategy</u>, is introduced in [2]. In the remainder of this section we present a brief description of the [g,d] networks.

The purpose of the [g,d] N-sorter network, where N = gd, is to accept as input the unordered set $I = \{i_1, i_2, \ldots, i_N\}$ and to produce as output the set $O = \{o_1, o_2, \ldots o_N\}$, where O is a permutation of I, and $o_1 \le o_2 \le \ldots \le o_N$. In order to specify the internal structure of

the $[g,d]$ (gd)-sorter network precisely, we find it convenient to consider I to be a g X d array, with $I_{(\alpha,\beta)} = i_{(\alpha-1)d+\beta}$. The g rows and d columns of I are given by

$$I_{(\alpha,*)} = \bigcup_{1\le\beta\le d} \{I_{(\alpha,\beta)}\}, \quad 1 \le \alpha \le g; \tag{1}$$

$$I_{(*,\beta)} = \bigcup_{1\le\alpha\le g} \{I_{(\alpha,\beta)}\}, \quad 1 \le \beta \le d. \tag{2}$$

Using this notation, we define the $[g,d]$ (gd)**-sorter** network as

i) g d-sorters for $I_{(\alpha,*)}$, $1 \le \alpha \le g$; followed by

ii) d g-sorters for $I_{(*,\beta)}$, $1 \le \beta \le d$; followed by

iii) a special comparator network called a $[g,d]$ f-network, which is defined below.

It has been shown [2 , 5] that the g+d small sorting networks in i) and ii) leave the rows and columns of I sorted. In order to distinguish the original unordered set I from the set with sorted rows and columns, we relabel the latter $V = \{v_1, v_2, \ldots, v_N\}$. **The** $[g,d]$ f-network is defined informally to be a network that contains whatever comparators are sufficient to transform the partially ordered set V into the completely ordered set 0.

The Zero-One Principle guarantees that a comparator network which begins with g d-sorters and d g-sorters is a (gd)**-sorter** network if it sorts I when **each member** of I is either 0 or 1. Therefore, when designing a [g,d] f-network that will complete the ordering of V, we may assume -- without loss of generality -- that all members of V are either 0 or 1. We make this assumption throughout the remainder of this paper.

If the number of 0's in $V_{(*,j)}$ is denoted $Z(V_{(*,j)})$, then it can be shown that since the rows and columns of V are sorted,

$$Z(V_{(g,*)}) \le Z(V_{(g-1,*)}) \le \ . \ . \ . \le Z(V_{(1,*)}) \le Z(V_{(g,*)}) \le d; \qquad (3)$$

$$Z(V_{(*,d)}) \le Z(V_{(*,d-1)}) \le \ . \ . \ . \le Z(V_{(*,1)}) \le Z(V_{(*,d)}) \le g. \qquad (4)$$

We are now in a position to make the following definition.

Definition 1:

A sequence of comparators is called a <u>[g,d] f-network</u> for N = td items if and only if it will complete the ordering of the partially ordered set $V = \{v_1, v_2, \ldots, v_N\}$, where a) the columns $V_{(*,j)}$, $1 \le j \le d$, are ordered and b) the number of 0's in $V_{(*,j)}$ satisfies (4).

The best f-networks known for g,d = 2,4 are given in Table 1. Each of the tabulated f-networks is described by a sequence of <u>templates</u> of the form $V_{(i,\alpha)}:V_{(i+j,\beta)}$ -- where $1 \le \alpha, \beta \le d$, $j \ge 0$, and $\alpha < jd + \beta$ -- followed by a <u>range</u> for i, which is specified in terms of t = N/d.

Let $f_{[g,d]}(N)$ represent the minimum number of comparators required by a [g,d] f-network for N items. (This function is only defined when N is a multiple of d.) Since we have not proved that the f-networks in Table 1 are minimal, we have labeled the number of comparators they require $a_{[g,d]}(N)$. Note that $\hat{f}_{[g,d]}(N)$ is linear in N, i.e. that

$$a_{[g,d]}(N) = a_{[g,d]} N - b_{[g,d]}, \qquad (5)$$

where $a_{[g,d]}$ is (1/d) times the number of templates required by the [g,d] f-network and $b_{[g,d]}$ is a positive constant.

[g,d]	f-network for N-sorter, $N = td$		$\hat{f}_{[g,d]}(N)$
[2,2]	$V_{(i,2)}:V_{(i+1,1)}$,	$1 \le i \le t-1.$	$t-1 = \frac{1}{2}N-1$
[2,4]	$V_{(i,3)}:V_{(i+1,1)}$,	$1 \le i \le t-1;$	$4t-3 = N-3$
	$V_{(i,4)}:V_{(i+1,2)}$,	$1 \le i \le t-1;$	
	$V_{(i,2)}:V_{(i,3)}$,	$1 \le i \le t;$	
	$V_{(i,4)}:V_{(i+1,1)}$,	$1 \le i \le t-1.$	
[4,2]	$V_{(i,2)}:V_{(i+2,1)}$,	$1 \le i \le t-2;$	$2t-3 = N-3$
	$V_{(i,2)}:V_{(i+1,1)}$,	$1 \le i \le t-1,$	
[4,4]	$V_{(i,3)}:V_{(i+2,1)}$,	$1 \le i \le t-2;$	$8t-11 = 2N-11$
	$V_{(i,4)}:V_{(i+2,2)}$,	$1 \le i \le t-2;$	
	$V_{(i,2)}:V_{(i+1,1)}$,	$1 \le i \le t-1;$	
	$V_{(i,4)}:V_{(i+1,3)}$,	$1 \le i \le t-1;$	
	$V_{(i,3)}:V_{(i+1,1)}$,	$1 \le i \le t-1;$	
	$V_{(i,4)}:V_{(i+1,2)}$,	$1 \le i \le t-1;$	
	$V_{(i,2)}:V_{(i,3)}$,	$2 \le i \le t-1;$	
	$V_{(i,4)}:V_{(i+1,1)}$,	$1 \le i \le t-1.$	

Table 1. Small f-networks.

We may use the [g,d] strategy recursively to obtain N-sorters for arbitrarily large N, provided we can construct [g,d] f-networks for large N. Theorems 2 and 3 below, which are proved in [2], describe two methods for constructing large f-networks using several copies of smaller f-networks.

Theorem 2:

Let the set $V = \{v_1, v_2, \ldots, v_N\}$, where N = tsd, be considered a t x s x d array, with $V_{(i,j,k)} = v_{(i-1)sd+(j-1)d+k}$. Then we can construct a [g,sd] f-network for V using:

i) d [g,s] f-networks for $V_{(*,*,k)}$, $1 \le k \le d$;

followed by

ii) one [g,d] f-network for V.

Theorem 3:

Let V be as in Theorem 2. Then we can construct an [sg,d] f-network for V using:

i) s [g,d] f-networks for $V_{(*,j,*)}$, $1 \le j \le s$;

followed by

ii) one [s,d] f-network for V.

As an example of the constructions described by Theorems 2 and 3, suppose that we desire to construct a $[2,2^r]$ f-network for the set $V = \{v_1, v_2, \ldots v_N\}$, where $N = t \cdot 2^r$. According to Theorem 3, we should consider V to be a $t \times 2 \times 2^{r-1}$ array, and use i) 2^{r-1} [2,2] f-networks for $V_{(*,*,k)}$, $1 \le k \le 2^{r-1}$, followed by ii) a $[2,2^{r-1}]$ f-network for V. From Table 1 we find that the [2,2] f-network for $V_{(*,*,k)}$ requires the comparators $V_{(i,2,k)}:V_{(i+1,1,k)}$, $1 \le i \le t-1$, so that all of the comparators required by i) are described by $V_{(i,2,k)}:V_{(i+1,1,k)}$, $1 \le i \le t-1$, $1 \le k \le 2^{r-1}$.

It is not really necessary to consider V to be a t X s X d array in order to apply Theorems 2 and 3, although this assumption does simplify the description of the [g,sd] and [sg,d] f-networks. In the next section we find it necessary to describe a $[2,2^r]$ and a $[2^r,2]$ f-network for a t $\times$ 2^r array. It is readily verified that the comparators prescribed by Theorems 2 and 3 for these two f-networks are those given in Corollaries 1 and 2 below.

Corollary 1:

Let the set $V = \{v_1,v_2,\ldots,v_N\}$, $N = t \cdot 2^r$, 'be considered a t X 2^r array. Then we can construct a $[2,2^r]$ f-network for V using:

i) the $\frac{1}{2}(N-2^r)$ comparators $V_{(i,s+2^{r-1})}:V_{(i+1,s)}$, $1 \le i \le t-1$, $1 \le s \le 2^{r-1}$; followed by

ii) one $[2,2^{r-1}]$ f-network for V.

Corollary 2:

Let V be as in Corollary 1. Then we can construct a $[2^r,2]$ f-network for V using:

i) the $\frac{1}{2}(N-2^r)$ comparators $V_{(i,2s)}:V_{(i+1,2s-1)}$, $1 \le i \le t-1$, $1 \le s \le 2^{r-1}$; followed by

ii) one $[2^{r-1},2]$ f-network for V.

The number of comparators required by the best f-network that can be constructed out of smaller f-networks using the construction of Theorem 2 and/or Theorem 3 is given by

$$\hat{f}_{[g,d]}(N) = \min_{\substack{1 \le q < g \\ g \bmod q = 0}} \; \min_{\substack{1 \le p < d \\ 2 < q + p \\ d \bmod p = 0}} F(g,d,N,q,p) \;, \tag{6}$$

where

$$F(g,d,N,q,p) = q \cdot p \cdot \hat{f}_{[g/q,d/p]}(N/(q \cdot p)) + \hat{f}_{[q,p]}(N)$$

$$+ \; q \cdot a_{[g/q,p]}(N/q) + p \cdot \hat{f}_{[q,d/p]}(N/p). \tag{7}$$

Note that $a_{[g,1]}(N) = \hat{f}_{[1,d]}(N) = 0$, so that: a) if $q = 1$, then (7) describes a construction that uses only Theorem 2; b) if $p = 1$, then (7) describes the use of Theorem 3 alone; and c) if $p,q > 1$, then (7) describes a network built using both theorems. The case $p = q = 1$ is not allowed, since it would reduce (6) to an identity.

We may use Equations (5)-(7) to show that the number of comparators required by the best $[2^i,2^j]$ f-network that can be constructed according to Theorems 2 and 3 is given by

$$\hat{f}_{[2^i,2^j]} = a_{[2^i,2^j]}N - b_{[2^i,2^j]}, \tag{8}$$

where

$$a_{[2^i,2^j]} = \min_{0 \le r < i} \; \min_{\substack{0 \le s < j \\ 0 < r + s}} \left\{ a_{[2^{i-r},2^{j-s}]} + a_{[2^r,2^s]} ,\; a_{[2^{i-r},2^s]} + a_{[2^r,2^{j-s}]} \right\} ; \tag{9}$$

$$b_{[2^i,2^j]} = \max_{0 \le r < i} \; \max_{\substack{0 \le s < j \\ 0 < r + s}} \left\{ 2^{r+s} b_{[2^{i-r},2^{j-s}]} + b_{[2^r,2^s]} 2^r b_{[2^{i-r},2^s]} + 2^s b_{[2^r,2^{j-s}]} \right\} . \tag{10'}$$

Starting with $a_{[2,2]} = \frac{1}{2}$, $b_{[2,2]} = 1$, which we obtain from Table 1, we may use (9) and (10) to show that the most economical [2,4],[4,2], and [4,4] f-networks that can be constructed using Theorems 2 and 3 are described by

$$a_{[2,4]} = a_{[4,2]} = 1;$$

$$b_{[2,4]} = b_{[4,2]} = 3; \tag{11}$$

$$a_{[4,4]} = 2;$$

$$b_{[4,4]} = 9.$$

The [2,4] and [4,2] f-networks listed in Table 1 are, in fact, constructed according to Theorem 2 and Theorem 3, respectively. However, the [4,4] f-network given in Table 1, which achieves $b_{[4,4]} = 11$, is the smallest example of ,a more economical construction that has been discovered for $[2^r,2^r]$ f-networks, $r > 1$. This construction is described in the next section.

III. Constructing $[2^r,2^r]$ f-networks

In this section we describe a particularly efficient method for constructing $[2^r,2^r]$ f-networks, $r > 1$. The construction depends upon the concept of a "redundant" comparator. Now the purpose of the comparator $v_\alpha : v_\beta$ is to compare v_a and v_β and to interchange the two if $v_\alpha > v_\beta$, which is to say, if $v_\alpha = 1$ and $v_\beta = 0$. The comparator $v_\alpha : v_\beta$ is said to be "redundant" if it can be shown that, as a result of previous comparators, $(v_a = 1) \Rightarrow (v_\beta = 1)$. A redundant comparator never makes any interchanges; therefore, the network performance is not altered by removing any redundant comparators.

The method used in this section for constructing a $[2^r,2^r]$ f-network is: a) to determine the templates required by the $[2^r,2^r]$ f-network derived using Theorems 2 and 3; b) to reorder the templates in such a way that, although the resulting network still orders V, some of the comparators become redundant; and c) to remove the redundant comparators. The number of comparators required by the efficient $[2^r,2^r]$ f-network is just the number determined by Equations (8)-(10), minus the number that become redundant when the templates are reordered. Since the economical construction does not reduce the number of templates, the linear coefficient $a_{[2^r,2^r]}$ is not changed from (9). We shall see that the improvement is reflected by an increase in $b_{[2^r,2^r]}$ over (10).

Suppose that we desire to construct a $[2^r,2^r]$ f-network for the set $V = \{v_1,v_2,\ldots,v_N\}$, where $N = t \cdot 2^r$. According to Theorem 2, the $[2^r,2^r]$ f-network can be constructed using: $a\,[2^r,2^{r-1}]$

f-network for the odd members of **V**, labeled V_o; a $[2^r,2^{r-1}]$ f-network for the even members V_e; and a $[2^r,2]$ f-network. Furthermore, according to Theorem 3, each of the $[2^r,2^{r-1}]$ f-networks can itself be built out of two $[2^{r-1},2^{r-1}]$ and one $[2,2^{r-1}]$ f-networks. The successive levels of detail for the resulting $[2^r,2^r]$ f-network are displayed in Fig. 2. Considering V to be a t X 2^r array, we define the six subsets of V appearing in Fig. 2 as follows.

$$V_{01} = \bigcup_{i\ \mathrm{odd}} \bigcup_{j\ \mathrm{odd}} \{V_{(i,j)}\} \qquad (12)$$

$$V_{02} = \bigcup_{i\ \mathrm{even}} \bigcup_{j\ \mathrm{odd}} \{V_{(i,j)}\} \qquad (13)$$

$$V_{e1} = \bigcup_{i\ \mathrm{odd}} \bigcup_{j\ \mathrm{even}} \{V_{(i,j)}\} \qquad (14)$$

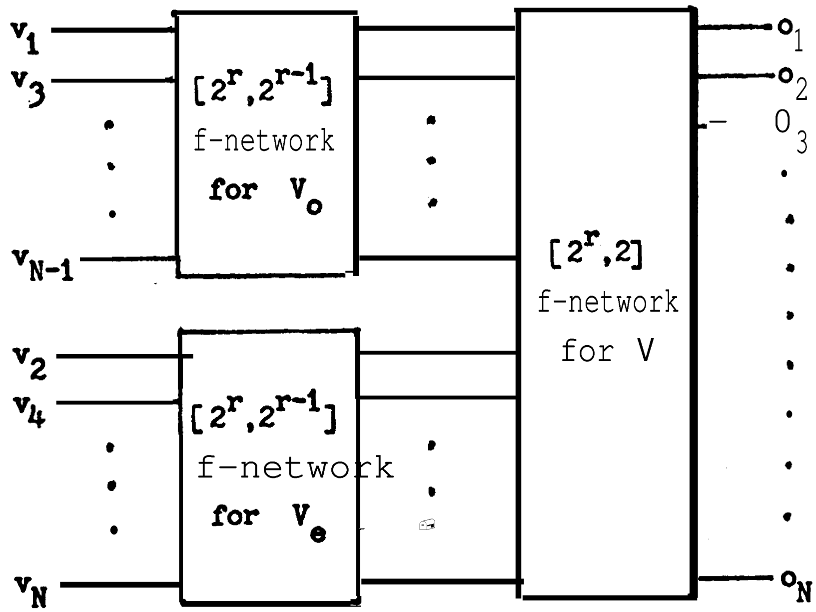
$$V_{e2} = \bigcup_{i\ \mathrm{even}} \bigcup_{j\ \mathrm{even}} \{V_{(i,j)}\} \qquad (15)$$

$$V_o = V_{o1} \cup V_{o2} \qquad (16)$$

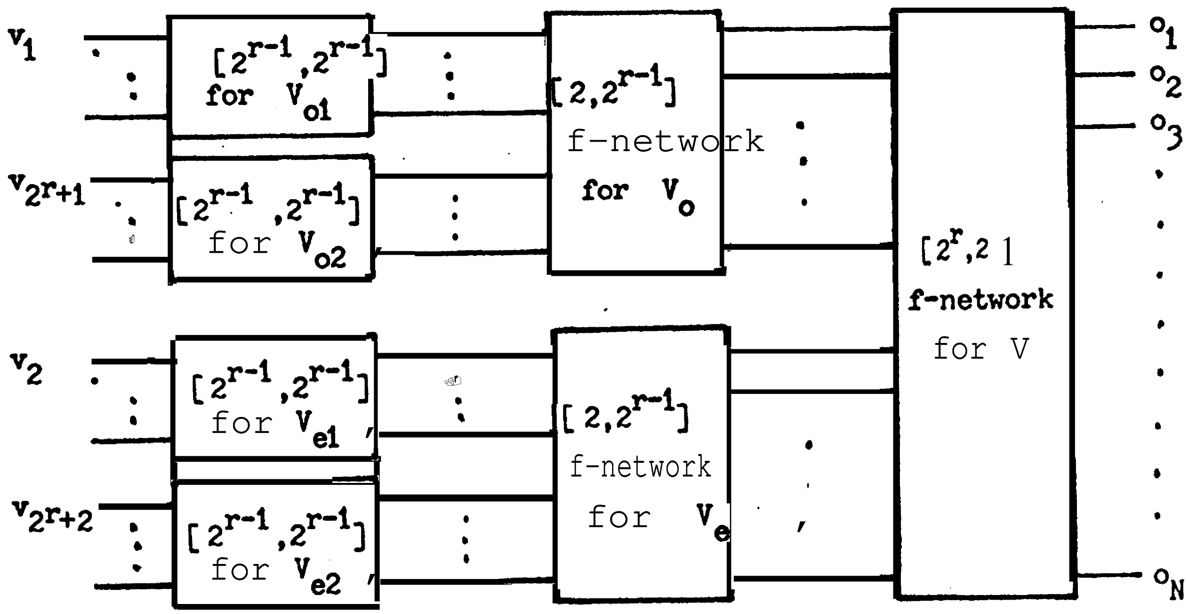
$$V_e = V_{e1} \cup V_{e2} \qquad (17)$$

These subsets are illustrated for the case $t = 2^r = 4$ in Fig. 3, and for the case t + 10, $2^r = 8$ in Fig. 6.

We may use Corollary 2 to express the $[2^r,2]$ f-network in Fig. 2(b) as $\frac{1}{2}(N-2^r)$ comparators followed by a $[2^{r-1},2]$ f-network



(a)



(b)

Fig. 2, $[2^r,2^r]$ f-network constructed using
(a) Theorem 2 and (b) Theorem 3 twice.

for V. The resulting $[2^r,2^r]$ f-network contains the following four groups of comparators, which appear sequentially.

1) Four $[2^{r-1},2^{r-1}]$ f-networks for $V_{01}, V_{o2}, V_{e1}, V_{e2}$;

2) Two $[2,2^{r-1}]$ f-networks for V_o and V_e;

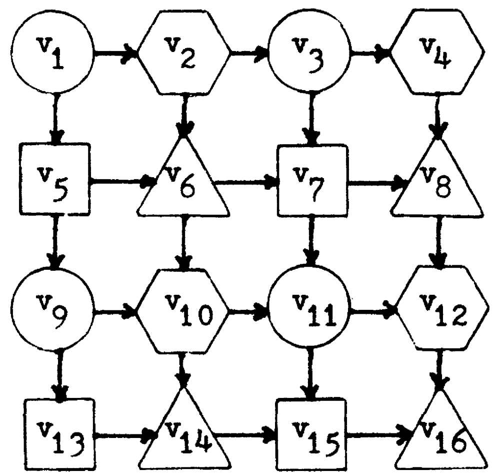
3) The $\frac{1}{2}(N-2^r)$ comparators $V_{(i,2s)}:V_{(i+1,2s-1)}$, $1 \le i \le t-1$, $1 \le s \le 2^{r-1}$; (18)

4) A $[2^{r-1},2]$ f-network for V.

The economical $[2^r,2^r]$ f-networks take advantage of the following observation (which is proved below): If we interchange the order of 2) and 3), then not only does the resulting network still order V, but also 2^{r-1} of the comparators in the $[2^{r-1},2]$ f-network become redundant.

Before proving this observation, we shall illustrate the construction, using the [4,4] 16-sorter as an example. The partial ordering in the intermediate set V is illustrated in Fig. 3(a), with an arrow from v_α to v_β representing the relation $v_\alpha \le v_\beta$. The dashed lines in Fig. 3(b) represent the four [2,2] f-networks required by 1) for the four sets $V_{o1}, V_{o2}, V_{e1}, V_{e2}$. The dashed lines in 3(c) through 3(e) represent, respectively: the 6 comparators called for in 3); the [2,2] f-networks for V_o and V_e required by 2); and the [2,2] f-network for V given in 4).

The comparators illustrated in 3(b) through 3(e) are exactly those described for the [4,4] f-network in Table 1 -- plus two extra comparators in 3(e), namely $v_2:v_3$ and $v_{14}:v_{15}$. These two comparators are redundant. The partial ordering in V depicted in Fig. 3(a) requires that $v1 = 0$ if $Z(V) \ge 1$, and that

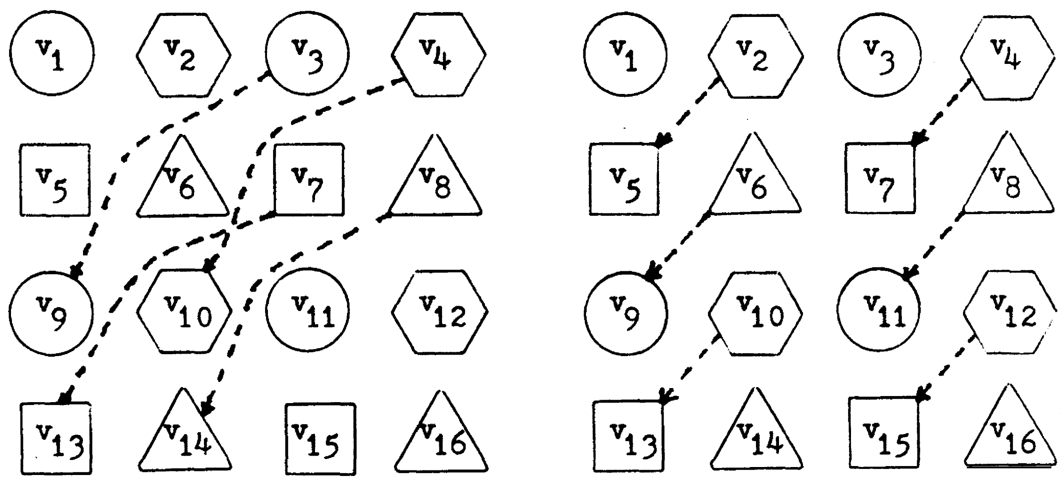


(a)

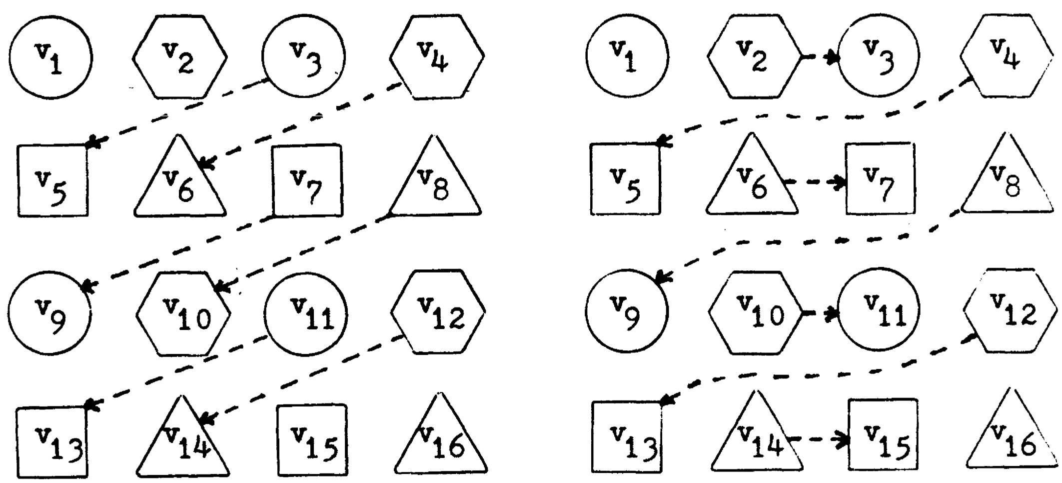
Fig. 3. f-network for

[4,4] **16-sorter.**

Key: V_{o1} = ○ ;
$V_{\theta 2}$ = ⎣⎦ ;
V_{e1} = ⬡ ;
V_{e2} = △.



(b) (c)



(d) (e)

$v_2 = 0$ or $v_5 = 0$ if $Z(V) \geq 2$. The comparator $v_2:v_5$ in 3(c) guarantees that $v_2 = 0$ if $Z(V) \geq 2$. Therefore,

$$(v_2 = 1) \Rightarrow (z(v) < 2) \Rightarrow (v_3 = 1), \qquad (19)$$

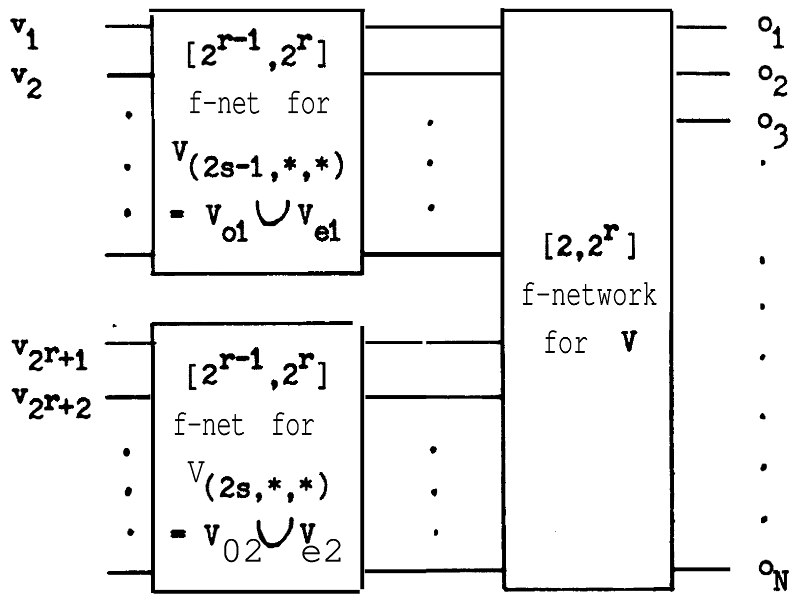
so that the comparator $v_2:v_3$ is redundant. By symmetry, the comparator $v_{14}:v_{15}$ is also redundant.

Although the economical $[2^r,2^r]$ f-network is a modification of the network depicted in Fig. 2 and described by (18), it still begins with four $[2^{r-1},2^{r-1}]$ f-networks for V_{01}, V_{02}, V_{e1}, and V_{e2}. In Fig. 4 we display successive levels of detail for a third possible construction for a $[2^r,2^r]$ f-network. Note that it, too, begins with four $[2^{r-1},2^{r-1}]$ f-networks for V_{01}, V_{02}, V_{e1}, and V_{e2}, although the remainder of the network differs from that in Fig. 2. All three $[2^r,2^r]$ f-networks share the construction depicted in Fig. 5, namely four $[2^{r-1},2^{r-1}]$ f-networks followed by a special comparator network that we shall call an <u>[r] h-network.</u>

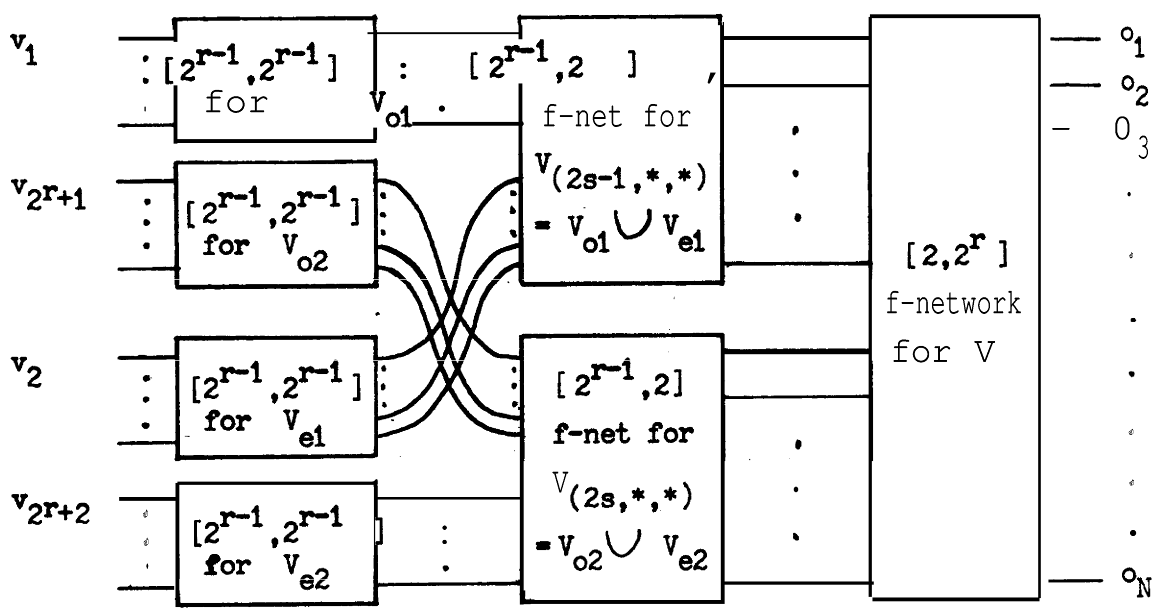
An [r] h-network is defined informally to be a network that contains whatever comparators are sufficient to complete the ordering in V. Fig. 2 and -Fig. 4 illustrate two different [r] h-networks.

In order to define an h-network precisely, let us consider the partial ordering in V achieved by the four $[2^{r-1},2^{r-1}]$ f-networks. Clearly they order the sets V_{o1}, V_{o2}, V_{e1}, V_{e2}. Since the construction of Fig. 2 and (18) guarantees that a $[2,2^{r-1}]$ f-network will complete the ordering of V_o, once V_{o1} and V_{02} are ordered,

$$z(V_{o2}) \leq z(V_{o1}) \leq z(V_{o2}) + 2^{r-1}. \qquad (20)$$



(a)



(b)

Fig. 4. $[2^r,2^r]$ f-network for V constructed using (a) Theorem 3 and (b) Theorem 2 twice.

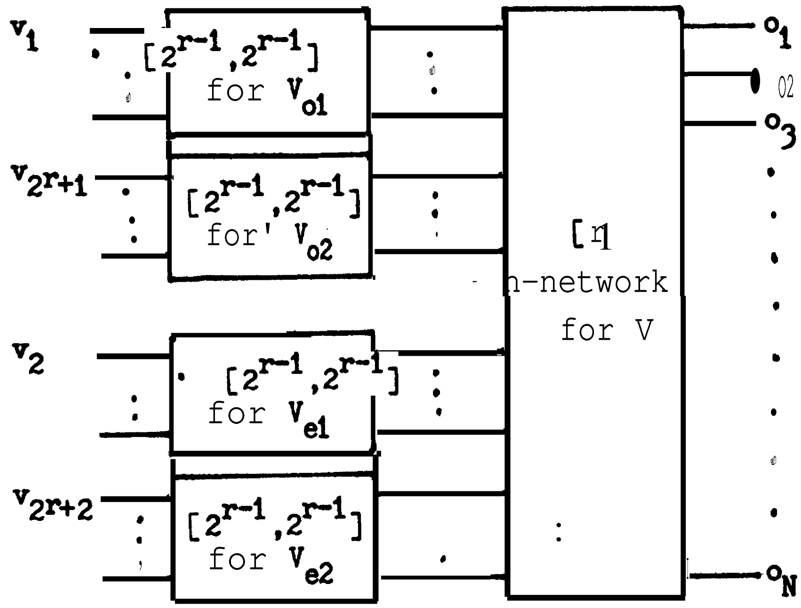


Fig. 5. $[2^r,2^r]$ f-network.

Similarly, since another $[2,2^{r-1}]$ f-network will order V_{e}, once V_{e1} and V_{e2} are ordered,

$$z(V_{e2}) \leq z(V_{e1}) \leq z(V_{e2}) + 2^{r-1} \tag{21}$$

According to the construction depicted in Fig. 4, one $[2^{r-1}, 2]$ f-network will order $V_{o1} \cup V_{e1}$ and another will order $V_{o2} \cup V_{e2}$, once V_{o1}, V_{o2}, V_{e1}, and V_{e2} are all ordered. Therefore,

$$z(V_{e1}) \leq z(V_{o1}) \leq z(V_{e1}) + 2^{r-1}; \tag{22}$$

$$z(V_{e2}) \leq z(V_{o2}) \leq z(V_{e2}) + 2^{r-1}. \tag{23}$$

We are now ready for the following formal definition.

Definition 2:

A sequence of comparators is called an <u>[r] h-network</u> for $N = t \cdot 2^{r}$ items if and only if it will complete the ordering of the partially ordered set $V = \{v_1, v_2, \ldots, v_N\}$, where a) the four subsets of V defined by (12)-(15) are each ordered and b) the number of 0's in these subsets satisfies (20)-(23).

From out discussion of Fig. 2 we conclude that one possible construction of an [r] h-network consists of items 2), 3), and 4) from (18). Lemma 1 shows that we may interchange the order of 2) and 3) in (18).

Lemma 1:

Let the set $V = \{v_1, v_2, \ldots, v_N\}$, where $N = t \cdot 2^r$ and t is even, be considered a $t \times 2^r$ array. Then we can construct an [r] h-network for V using:

i) the $\frac{1}{2}(N-2^r)$ comparators $V_{(i,2s)}:V_{(i+1,2s-1)}$, $1 \le i \le t-1$, $1 \le s \le 2^{r-1}$, that produce the intermediate set v; followed by

ii) one $[2,2^{r-1}]$ f-network for $\hat{V}_o$ and another $[2,2^{r-1}]$ f-network for v_e; followed by

iii) one $[2^{r-1},2]$ f-network for V.

Proof:

The complete proof of Lemma 1 is given in Appendix A. Essentially we show that the comparators in i) tend to move 0's from V_o to V_e, while maintaining the partial ordering in the four subsets of V. Specifically, we prove that $\hat{V}_{o1}$, $\hat{V}_{o2}$, $\hat{V}_{e1}$, and $\hat{V}_{e2}$ are **all** ordered and that

$$z(\hat{V}_{o2}) \le z(\hat{V}_{o1}) \le z(\hat{V}_{o2}) + 2^{r-1}; \tag{24}$$

$$z(\hat{V}_{e2}) \le z(\hat{V}_{e1}) \le z(\hat{V}_{e2}) + 2^{r-1}; \tag{25}$$

$$z(\hat{V}_{e}) \le z(\hat{V}_{o}) \le z(\hat{V}_{e}) + 2^{r-1}. \tag{26}$$

Therefore, the $[2,2^{r-1}]$ f-networks in ii) will complete the ordering of V_o and $\hat{V}_e$, so that the $[2^{r-1},2]$ f-network in iii) will then order V.

Q.E.D.

We can use Corollary 1 to express the $[2,2^{r-1}]$ f-network for $\hat{V}_o$ in ii) of Lemma 1 as the $\frac{1}{2}(\frac{1}{2}N-2^{r-1})$ comparators $\hat{V}_{(i,s+2^{r-1})}:\hat{V}_{(i+1,s)}$, $1 \le i \le t-1$, $1 \le s \le 2^{r-1}$, s odd, followed by a $[2,2^{r-2}]$ f-network. Similarly, we can use Corollary 1 to express the $[2,2^{r-1}]$ f-network for $\hat{V}_e$ as $\frac{1}{2}(\frac{1}{2}N-2^{r-1})$ comparators followed by a $[2,2^{r-2}]$ f-network. This leads to the following recursive construction for an $[r]$ h-network.

Theorem 4:

Let V be as in Lemma 1. Then we can construct an [r] h-network for V using:

i) the $\frac{1}{2}(N-2^{r})$ comparators $V_{(i,2s)}:V_{(i+1,2s-1)}$, $1 \le i \le t-1$, $1 \le s \le 2^{r-1}$, that produce the intermediate set $\hat{V}$; followed by

ii) the $\frac{1}{2}(N-2^{r})$ comparators $\hat{V}_{(i,s+2^{r-1})}:\hat{V}_{(i+1,s)}$, $1 \le i \le t-1$, $1 \le s \le 2^{r-1}$, $1 \le k \le 2$, that produce the intermediate set $\tilde{V}$; followed by

iii) an [r-1] h-network for $\tilde{V}$.

Proof:

Lemma 1 and Corollary 1 imply that the intermediate set $\tilde{V}$ can be ordered by: a $[2,2^{r-2}]$ f-network for $\tilde{V}_o$ and another $[2,2^{r-2}]$ f-network for $\tilde{V}_e$, followed by a $[2^{r-1},2]$ f-network for $\tilde{V}$. As noted above, these three f-networks constitute one example of an [r-1] h-network. (Simply replace r in Fig. 2 by r-1.) A complete proof of Theorem 4, which shows that the number of 0's in the four subsets of $\tilde{V}$ satisfies (20)-(23) with r replaced by r-1, is given in Appendix B.

Q.E.D.

Consider the [r] h-network illustrated in Fig. 2, namely a $[2,2^{r-1}]$ f-network for V_o and a $[2,2^{r-1}]$ f-network for V_e followed by a $[2^r,2]$ f-network for V. We may use Corollary 1 to express the $[2,2^{r-1}]$ f-network for V_o as the sequence of templates $\alpha_{r-1},\alpha_{r-2},\ldots,\alpha_1$, where a) α_{r-1} is the template $V_{(i,s+2^{r-1})}:V_{(i+1,s)}$, $1 \le i \le t-1$, $1 \le s \le 2^{r-1}$, s odd; and b) the sequence $\alpha_p,\alpha_{p-1},\ldots,\alpha_1$ represents the templates for the $[2,2^p]$ f-network for V_o. We may use Corollary 1 to express the $[2,2^{r-1}]$ f-network for V_e as a similar sequence of templates $\beta_{r-1},\beta_{r-2},\ldots,\beta_1$. Since the templates α_p and β_p are identical except that α_p requires s odd and β_p requires s even, we can combine the two templates α_p and β_p into a single template τ_p.

In a similar manner we may use Corollary 2 to express the $[2^r,2]$ f-network for V as the sequence of templates $\pi_r,\pi_{r-1},\ldots,\pi_1$, where a) π_r is the template $V_{(i,2s)}:V_{(i+1,2s-1)}$, $1 \le i \le t-1$, $1 \le s \le 2^{r-1}$; and b) the sequence $\pi_p,\pi_{p-1},\ldots,\pi_1$ represents the templates for the $[2^p,2]$ f-network for V. The [r] h-network illustrated in Fig. 2 may then be represented as the sequence $\tau_{r-1},\tau_{r-2},\ldots,\tau_1,\pi_r,\pi_{r-1},\ldots,\pi_1$. However, Theorem 4 embodies the following corollary.

Corollary 3:

Let V be as in Lemma 1; let the two sequences of templates $\tau_{r-1},\tau_{r-2},\ldots,\tau_1$ and $\pi_r,\pi_{r-1},\ldots,\pi_1$ be as defined above. Then the following sequence of templates constitutes an [r] h-network for V.

$$\pi_r,\tau_{r-1},\pi_{r-1},\ldots,\tau_1,\pi_1. \qquad (27)$$

The partial ordering in the intermediate set V achieved by the $[2^r,2^r](4^r)$**-sorter** network is not completely specified by (20)-(23). Since the odd rows of V contain members of V_{o1} alternated with members of V_{e1}, while odd columns of V consist of members of V_{o1} separated by members of V_{o2}, and since the rows and columns of V are ordered,

$$Z(V_{o1}) \le Z(V_{e1}) + Z(V_{o2}) + 1. \qquad (28)$$

If the number of 1's in V_{e2} is represented by $|V_{e2}|$, then we can show by symmetry that when t is even,

$$|V_{e2}| \le |V_{e1}| + |V_{o2}| + 1. \qquad (29)$$

We shall see that the additional ordering in V specified by (28) and (29) guarantees that $2^r - 2$ of the comparators in the [r] h-network for V described by Corollary 3 are redundant. To show this, it is convenient to use two lemmas.

Lemma 2:

Let $V = \{v_1, v_2, \ldots, v_N\}$, where $N = t\cdot 2^r > 2^r$ and t is even. Suppose that the four subsets V_{o1}, V_{o2}, V_{e1}, and V_{e2} of V are each ordered and that they satisfy (20)-(23). Suppose also that

$$Z(V_{e1}) < 2^{r-1} \Rightarrow Z(V_{o2}) \le Z(V_{e1}) ; \qquad (30)$$

$$|V_{o2}| < 2^{r-1} \Rightarrow |V_{e1}| \le |V_{o2}|. \qquad (31)$$

Then if we apply the [r] h-network described by Theorem 4 to V, the 2^r comparators $V_{(i,2s)}:V_{(i+1,2s-1)}$, $i \in \{1,t-1\}$, $1 \le s \le 2^{r-1}$, are redundant.

Proof:

The $\frac{1}{2}(N-2^r)$ comparators $V_{(i,2s)}:V_{(i+1,2s-1)}$, $1 \le i \le t-1$, $1 \le s \le 2^{r-1}$, are illustrated in Fig. 6 for the case t = 10, $2^r = 8$. The comparator $V_{(1,2s)}:V_{(2,2s-1)}$ compares the s^{th} member of V_{e1}, written $(V_{e1})_s$, with the s^{th} member of V_{02}, $(V_{o2})_s$. Suppose that $(V_{e1})_s = 1$, where $1 \le s \le 2^{r-1}$. Then, since V_{e1} and V_{o2} are ordered we may use (30) to show that

$$\begin{aligned}(V_{e1})_s = 1 &\Rightarrow Z(V_{e1}) < s \le 2^{r-1} \\ &\Rightarrow Z(V_{o2}) \le Z(V_{e1}) < s \\ &\Rightarrow (V_{02})_s = 1 \ . \end{aligned} \tag{32}$$

Therefore, the comparators $(V_{e1})_s:(V_{o2})_s$ or $V_{(1,2s)}:V_{(2,2s-1)}$, $1 \le s \le 2^{r-1}$, are redundant.

If t is even, then the comparator $V_{(t-1,2s)}:V_{(t,2s-1)}$, $1 \le s \le 2^{r-1}$, may be rewritten as $(V_{e1})_{a+s}:(V_{o2})_{a+s}$, where $a = \frac{1}{4}N-2^{r-1}$. Suppose that $(V_{o2})_{a+s} = 0$, where $1 \le s \le 2^{r-1}$. Then since V_{o2} and V_{e1} are ordered, we may use (31) to show that

$$\begin{aligned}(V_{o2})_{a+s} = 0 &\Rightarrow |V_{o2}| < 2^{r-1}-s+1 \le 2^{r-1} \\ &\Rightarrow |V_{e1}| \le |V_{o2}| < 2^{r-1}-s+1 \\ &\Rightarrow (V_{e1})_{a+s} = 0. \end{aligned} \tag{33}$$

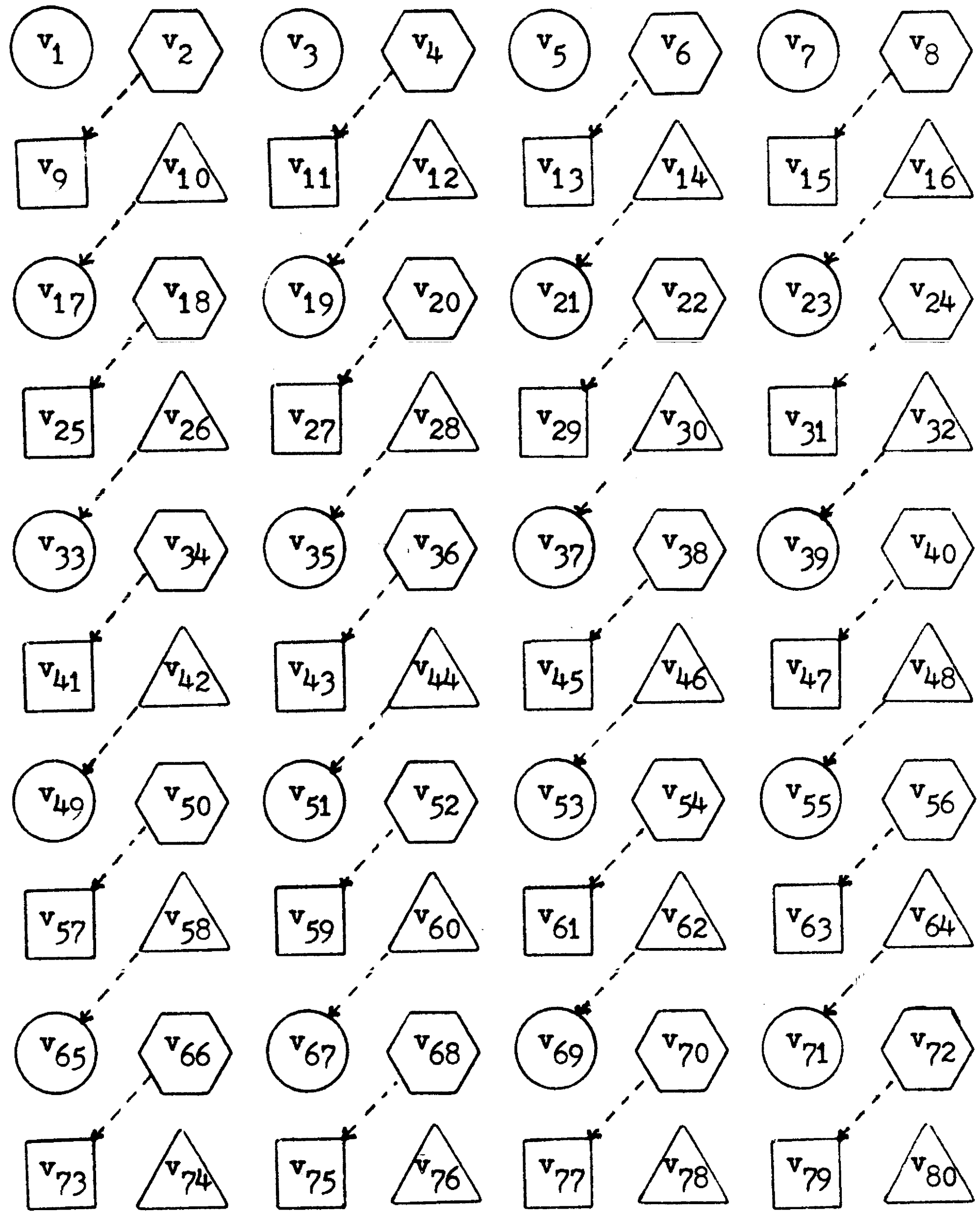


Fig. 6. The comparators $V_{(i,2s)}:V_{(i+1,2s-1)}$.

Key: V_{o1} = 0 ; V_{o2} = □ ;
V_{e1} = ⬡ ; V_{e2} = △ .

The contrapositive of (33) is $(V_{e1})_{a+s} = 1 \Rightarrow (V_{o2})_{a+s} = 1$, so that the comparators $V_{(t-1,2s)}:V_{(t,2s-1)}$, $1 \le s \le 2^{r-1}$, are also redundant.

Q.E.D.

Lemma 3:

Let $V = \{v_1, v_2, \ldots, v_N\}$, where $N = t \cdot 2^r > 2^r$ and t is even. Suppose that the four subsets V_{o1}, V_{o2}, V_{e1}, and V_{e2} of V are each ordered and that they satisfy (20)-(23), (28) and (29). Then if we apply the [r] h-network described by Theorem 4 to V, the subsets of the intermediate set $\tilde{V}$ satisfy (20)-(23) and (28)-(31), with r replaced by r-1.

Proof:

The proof of Lemma 3 is given in Appendix C.

Consider the [r] h-network described by the sequence of templates given by (27), where π_r operates on a set V, τ_r 1 operates on $\hat{V}$, and π_{r-1} operates on $\tilde{V}$. If the subsets of the original set V satisfy the hypotheses of Lemma 3, then Lemma 3 shows that $\tilde{V}$ satisfies the hypotheses of both Lemma 2 and Lemma 3. Lemma 2 shows then that 2^{r-1} of the comparators in π_{r-1} are redundant; repeated use of Lemmas 2 and 3 shows that 2^{p} comparators in π_p are redundant, $1 \le p \le r-1$. This inspires the following definition.

Definition 3:

Let $V = \{v_1, v_2, \ldots, v_N\}$, where $N = t \cdot 2^r$, be considered a $t \times 2^r$ array. Then a reduced [r] h-network or an <u>[r] &network</u> consists of:

i) the $\frac{1}{2}(N-2^r)-2^r$ comparators $V_{(i,2s)}:V_{(i+1,2s-1)}$, $2 \le i \le t-2$, $1 \le s \le 2^{r-1}$, that produce the intermediate set $\hat{V}$; followed by

a) nothing, if $r = 1$; or

b) the $\frac{1}{2}(N-2^r)$ comparators $\hat{V}_{(i,s+2^{r-1})}:\hat{V}_{(i+1,s)}$, $1 \le i \le t-1$, $1 \le s \le 2^{r-1}$, that produce the intermediate set $\tilde{V}$; followed by an [r-1] ℓ-network for $\tilde{V}$.

It is readily verified that the [r] ℓ-network requires $\Sigma_{p=1}^{r} 2^p = 2^{r+1} - 2$ fewer comparators than the [r] h-network. Theorem 5 shows that if V is the intermediate set for the $[2^r, 2^r]$ (4^r)-**sorter** network, then the [r-1] h-network in iii) of Theorem 4 may be replaced by an [r-1] ℓ-network, thereby saving 2^r-2 comparators.

Theorem 5:

We may complete the ordering of the intermediate set V achieved in the $[2^r, 2^r]$ (4^r)-**sorter** network using:

i) four $[2^{r-1}, 2^{r-1}]$ f-networks for the subsets $V_{01}, V_{02}, V_{e1}, V_{e2}$; followed by

ii) the [r] h-network described by Theorem 4 with the [r-1] h-network for the intermediate set $\tilde{V}$ replaced by an [r-1] &network.

This theorem summarizes the results of Theorem 4, Lemma 2, and Lemma 3.

The number of comparators required by the economical $[2^r,2^r]$ f-network described by Theorem 5 is given by

$$\hat{f}_{[2^r,2^r]}(N) = 4\,\hat{f}_{[2^{r-1},2^{r-1}]}(\tfrac{1}{4}N) + N - 2^r + \ell_{[r-1]}(N) \qquad (34)$$

where $\ell_{[r-1]}(N)$ is the number of comparators required by the [r-1] &network. We may use Definition 3 to show that $\ell_{[r-1]}(N)$ satisfies the recurrence relation

$$\ell_{[r]}(N) = \ell_{[r-1]}(N) + N - 2^{r+1}, \qquad (35)$$

with the boundary condition

$$\ell_{[1]}(N) = \tfrac{1}{2}N - 3. \qquad (36)$$

The solution to (35) and (36) is

$$\ell_{[r]}(N) = (r-\tfrac{1}{2})\,N - (2^{r+2} - 5).$$

We may use (37) along with the boundary condition $\hat{f}_{[2,2]}(N) = \tfrac{1}{2}N - 1$ to solve (34). In the notation of the last section,

$$\hat{f}_{[2^r,2^r]}(N) = a_{[2^r,2^r]}N - b_{[2^r,2^r]}, \qquad (38)$$

where

$$a_{[2^r,2^r]} = \tfrac{1}{2} r^2 , \tag{39}$$

$$b_{[2^r,2^r]} = \frac{4}{3} 4^r - 3\cdot 2^r + \frac{2}{3} \tag{40}$$

When i ≠ j and i,j>1, the most economical $[2^i,2^j]$ f-networks known use the economical $[2^r,2^r]$ f-networks as building blocks for the construction of Theorems 2 and 3. The number of comparators required by these networks is given by Equations (8)-(10). It is readily verified (by induction) that

$$a_{[2^i,2^j]} = \tfrac{1}{2}(i\cdot j), \tag{41}$$

which reduces to (39) when i = j = r. No closed form solution is known for $b_{[2^i,2^j]}$ with arbitrary i,j, and i ≠ j, although the following special result can be proved.

$$b_{[2^i,2^{ki}]} = \frac{(2^{ki}-1)}{(2^i\ \ 1)} b_{[2^i,2^i]}, \tag{42}$$

where $b_{[2^i,2^i]}$ is given by (40).

We have calculated $b_{[2^i,2^j]}$ for i,j ≤ 32, and give the results for i,j ≤ 8 in Table 2. The symmetry of (10) implies that

$$b_{[2^i,2^j]} = b_{[2^j,2^i]}, \tag{43}$$

which is observed in Table 2. For $i < j \le 32$ we find that the right-hand side of (10) is minimized if and only if $r = 0$ and $s \equiv j \pmod{i}$. Therefore, for $i \le j \le 32$, we may express $b_{[2^i, 2^j]}$ in the following recurrence relation.

$$b_{[2^i,2^j]} = 2^{j-i} b_{[2^i,2^i]} \; \square \; b_{[2^i,2^{j-i}]} \, . \tag{44}$$

We hypothesize that (44) holds for all $i \le j$; however, no closed form solution is known for (44).

i \ j	1	2	3	4	5	6	7	8
1	1	3	7	15	31	63	127	255
2	3	11	25	55	113	231	465	935
3	7	25	63	133	277	567	1141	2293
4	15	55	133	295	605	1235	2493	5015
5	31	113	277	605	1271	2573	5197	10445
6	63	231	567	1235	2573	5271	10605	21315
7	127	465	1141	2493	5197	10605	21463	43053
8	255	935	2293	5015	10445	21315	43053	86615

Table 2. **Small values of** $b_{[2^i,2^j]}$.

IV. (2^m)-sorter Networks

The minimum number of comparators required by a network that sorts N inputs is denoted S(N). Let G(N) represent the minimum number of comparators required by an N-sorter network that makes repeated use of the [g,d] strategy. In this section we examine the asymptotic growth of G(N), restricting our attention to the special case that N is a power of 2. (Results with [g,d] networks for $N \le 36$ are given in [2].) If $N = 2^m$, then since $N = gd$, g and d are also powers of 2. Clearly $G(2^m)$ satisfies the following recurrence relation.

$$G(2^m) = \min_{0<r<m} \left\{ 2^r\, G(2^{m-r}) + 2^{m-r}\, G(2^r) + \hat{f}_{[2^r,2^{m-r}]}(2^m) \right\}. \qquad (45)$$

We have calculated $G(2^m)$ for $m \le 64$ and give the results for $m \le 16$ in Table 3. Note that since $\hat{f}_{[2^i,2^j]}(N) = \hat{f}_{[2^j,2^i]}(N)$, we may restrict r to the range $\lceil \frac{1}{2}m \rceil \le r < m$. The column entitled r_G gives those values of $r \in [\lceil \frac{1}{2}m \rceil, m-1]$ that minimize the right-hand-side of (45). For example, when $m = 4$ the minimum is achieved only for $r = 2$, whereas when $m = 5$ the minimum occurs for both $r = 3$ and $r = 4$.

When m is even, our results in the last section indicate that $\hat{f}_{[2^r\ 2^{m-r}]}(2^m)$ is minimized by $r = \frac{1}{2}m$. We might expect, therefore, that the right-hand-side of (45) should be minimized by $r = \lceil \frac{1}{2}m \rceil$, so that when m is even the optimal (2^m)-sorter network should be "square." However, we observe from Table 3 that the minimum almost always occurs when r is a power of 2. This is explained as follows.

m	$N=2^m$	r_G	G(N)	r_S	$\hat{S}(N)$
1	2		1		1
2	4	1	5		5
3	8	2	19		19
4	16	2	61		60
5	32	3,4	187	4	185
6	64	4	525	4	521
7	128	4	1427	4	1419
8	256	4	3705	4	3673
9	512	6	9457	5,8	9395
10	1024	6,8	23357	6,8	23229
11	2048	8	56787	8	56531
12	4096	8	135417	8	134649
13	8192	8	319827	8	318291
14	16384	8	743421	8	740349
15	32768	8	1714003	8	1707859
16	65536	8	3907497	8	3891113

Table 3. $G(2^m)$ and $\hat{S}(2^m)$ for $m \leq 16$.

A $[g,d]$ (2^m)**-sorter** network begins with 2^m **2-sorters.** The remainder of the (2^m)-sorter is a succession of f-networks interspersed with **2-sorters.** When $m = 2^k$, each of the f-networks in the (2^m)**-sorter** can be one of the efficient square f-networks described in the last section; therefore, the (2^m)**-sorter** networks are particularly efficient whem $m = 2^k$. Now we show below that $G(2^m) \sim \frac{1}{4}m^2 2^m$, whereas from Equations (8) and (41) we know that $\hat{f}_{[2^r,2^{m-r}]}(2^m) \sim \frac{1}{2}r(m-r)2^m$. Since $r \geq \lceil \frac{1}{2}m \rceil$, this means that the dominant term in (45) is $2^{m-r}G(2^r)$. By choosing r to be a power of 2, we maximize the efficiency of the largest component of the (2^m)-sorter.

As noted above, when $m = 2^k$ the (2^m)**-sorter** network can restrict itself to the efficient square f-networks. This construction leads to the following recurrence relation.

$$G(2^{2m}) = 2^{m+1} G(2^m) + \hat{f}_{[2^m,2^m]}(2^{2m}), \qquad m = 2^k; \; k \geq 0. \tag{46}$$

Using (38)- (40) and the boundary condition $G(2) = 1$, we find that the solution to (46) is

$$G(2^m) = \left[m^2 - (\tfrac{1}{4} + \sigma_k)m + \tfrac{4}{3} \right] 2^m - \tfrac{5}{3}, \qquad m = 2^k; \; k \geq 0. \tag{47}$$

Where

$$\sigma_k = \frac{1}{6} \sum_{0 \leq r < k} 2^{-(2^r + r)} \tag{48}$$

Since σ_k converges rapidly to **.107,** the asymptotic growth of $G(N)$ may be expressed as

$$G(N) = .250\, N\,(\log_2 N)^2 - .357\, N\,(\log_2 N) + O(N). \tag{49}$$

Let $\hat{S}(N)$ represent the number of comparators required by the most economical N-sorter network known. For $m \le 3$, $\hat{S}(2^m) = G(2^m)$. However, M. W. Green [3] has designed a 16-sorter network which requires only 60 comparators, whereas $G(16) = 61$. For $m > 4$, the most economical (2^m)-sorter network uses the $[g,d]$ strategy, encorporating many copies of Green's economical 16-sorter. Therefore, for $m > 4$, $\hat{S}(2^m)$ satisfies

$$\hat{S}(2^m) = \min_{\lceil \frac{1}{2}m \rceil \le r < m} \left| 2^r\, \hat{S}(2^{m-4}) + 2^{m-r}\, \hat{S}(2^r) + \hat{f}_{[2^r, 2^{m-r}]}(2^m) \right|, \quad m > 4. \tag{50}$$

We have included $\hat{S}(2^m)$ in Table 3, along with the values of r, labeled r_S, that minimize the right-hand-side of (50). Again we observe that the minimum normally occurs when r is a power of 2, which leads to the same recurrence relation obtained above for G.

$$\hat{S}(2^{2m}) = 2^{m+1}\, \hat{S}(2^m) + \hat{f}_{[2^m, 2^m]}(2^{2m}), \quad m = 2^k;\ k \ge 2. \tag{51}$$

Using (38)-(40) and the boundary condition $\hat{S}(16) = 60$ we find that

$$\hat{S}(2^m) = \left[\frac{1}{4} m^2 - \left(\frac{17}{64} + \sigma_k\right) m + \frac{4}{3}\right] 2^m - \frac{5}{3}, \quad m = 2^k;\ k \ge 2. \tag{52}$$

The asymptotic growth of $\hat{S}(N)$ is given by

$$\hat{S}(N) = .250\, N\, (\log_2 N)^2 - .372\, N\, (\log_2 N) + O(N). \qquad (53)$$

V. Conclusion

Prior to the [g,d] strategy, the most economical N-sorter network known (for most values of N), used: i) a $\lceil \frac{1}{2}N \rceil$-sorter; ii) a $\lfloor \frac{1}{2}N \rfloor$-sorter; and iii) a $(\lceil \frac{1}{2}N \rceil, \lfloor \frac{1}{2}N \rfloor)$ merge network designed by K. E. Batcher [6]. A close examination of Batcher's N-sorter network for the set $I = \{i_1, i_2, \ldots, i_N\}$, where $N = 2d$, reveals the following. If I is considered to be a $2 \times d$ array, then Batcher's (2d)-sorter network begins with 2 d-sorters, one for $I_{(1,*)}$ and one for $I_{(2,*)}$, followed by d 2-sorters for $I_{(*,j)}$, $1 \le j \le d$. Therefore, Batcher's (2d)-sorter network uses what we would call the [2,d] strategy. The [g,d] strategy is simply an extension of Batcher's strategy to include values of g > 2.

The number of comparators required by Batcher's N-sorter network is denoted B(N). With the boundary condition B(1) = 0, Batcher shows that

$$B(2^m) = (\tfrac{1}{4}m^2 - \tfrac{1}{4}m + 1)\, 2^m - 1, \quad m \ge 0; \tag{54}$$

Using the Green's 16-sorter as a boundary condition, i.e. using B(16) = 60 leads to

$$B(2^m) = (\tfrac{1}{4}m^2 - \tfrac{1}{4}m + \tfrac{13}{16})\, 2^m - 1, \quad m \ge 4; \tag{55}$$

Given the [2,2] f-network in Table 1, Theorems 2 and 3 guarantee the existence of $[2^i, 2^j]$ f-networks for arbitrary i, j. Let $\tilde{G}(2^m)$ represent the number of comparators required by a [g,d] (2^m)-sorter that uses only the f-networks constructed according to Theorems 2 and 3 from the [2,2] f-network. Then the boundary condition $\tilde{G}(1) = 0$ leads to

$$\tilde{G}(2^m) = (\tfrac{1}{4}m^2 - \tfrac{1}{4}m + 1)\ 2^m - 1, \quad m \geq 0; \tag{56}$$

which is exactly the same as (54). However, using the Green's 16-sorter as a boundary condition leads to

$$\tilde{G}(2^m) = (\tfrac{1}{4}\ m^2 - \tfrac{19}{64}m + 1)\ 2^m - 1, \quad \begin{matrix} m = 4k, \\ k \geq 1; \end{matrix} \tag{57}$$

The savings of (57) over (55) is possible because the [g,d] (2^m)-sorter can take better advantage of Green's 16-sorter. For example, the $[2^4,2^{4+k}]$ (2^{8+k})-sorter can use 2^{5+k} copies of the efficient 16-sorter, whereas Batcher's $[2,2^{8+k}]$ (2^{8+k}) sorter can only use 2^{4+k} copies.

We have seen that the existence theorems for $[2^i,2^j]$ f-networks (i.e. Theorems 2 and 3) lead to N-sorter networks that require $\sim \frac{3}{64} N(\log_2 N)$ fewer comparators than the best networks previously known. In addition, we found that reordering the comparators in the $[2^r,2^r]$ f-networks prescribed by the existence theorems leads to the more substantial savings of $\sim (\sigma_k + \frac{1}{64})N(\log_2 N)$ comparators. (Compare $\hat{S}(2^m)$ given by (52) with $B(2^m)$ given by (55).)

Appendix A: Proof of Lemma 1

Lemma 1:

Let the set $V = \{v_1, v_2, \ldots, v_N\}$, where $N = t \cdot 2^r$ and t is even be considered a $t \times 2^r$ array. Then we can construct an $[r]$ h-network for V using:

i) the $\frac{1}{2}(N-2^r)$ comparators $V_{(i,2s)}:V_{(i+1,2s-1)}$, $1 \le i \le t-1$, $1 \le s \le 2^{r-1}$, that produce the intermediate set $\hat{V}$; followed by

ii) one $[2,2^{r-1}]$ f-network for $\hat{V}_o$ and another $[2,2^{r-1}]$ f-network for $\hat{V}_e$; followed by

iii) one $[2^{r-1},2]$ f-network for $\hat{V}$.

Proof:

According to Definition 2, the comparators described by i) through iii) constitute an $[r]$ h-network if and only if they will complete the ordering of V given that a) the four subsets of V defined by (12)-(15) are each ordered and b) the number of 0's in these subsets satisfies (20)-(23).

Let us assume that the partial ordering in V satisfies a) and b). Then, as noted in the text, to prove the lemma we need to show that $\hat{V}_{o1}$, $\hat{V}_{o2}$, $\hat{V}_{e1}$, and $\hat{V}_{e2}$ are all ordered and that the number of 0's in these subsets of $\hat{V}$ satisfies (24)-(26). If we let $(V_{01})_j$ represent the j^{th} member of V_{o1}, then the comparators in i) may be expressed as

$$(V_{e1})_j : (V_{o2})_j, \qquad 1 \le j \le \tfrac{1}{4}N; \tag{58}$$

$$(V_{e2})_j : (V_{o1})_{j+2^{r-1}} , \qquad 1 \le j \le \tfrac{1}{4}N-2^{r-1}. \tag{59}$$

(See Fig. 6.) Therefore,

$$(\hat{V}_{e1})_3 = (V_{e1})_j \wedge (V_{o2})_j, \qquad 1 \le j \le \tfrac{1}{4}N; \tag{60}$$

$$(\hat{V}_{02})_j = (V_{e1})_j \vee (V_{o2})_j, \qquad 1 \le j \le \tfrac{1}{4}N; \tag{61}$$

$$(\hat{V}_{e2})_j = \begin{cases} (V_{e2})_j \wedge (V_{o1})_{j+2^{r-1}} , & 1 \le j \le \tfrac{1}{4}N-2^{r-1}; \\ (V_{e2})_j , & \tfrac{1}{4}N-2^{r-1} \le j \le \tfrac{1}{4}N; \end{cases} \tag{62}$$

$$(\hat{V}_{o1})_j = \begin{cases} (V_{o1})_j , & 1 \le j \le 2^{r-1}; \\ (V_{o1})_j \vee (V_{e2})_{j-2^{r-1}} , & 2^{r-1} \le j \le \tfrac{1}{4}N. \end{cases} \tag{63}$$

Here "A" and 'V" represent the boolean "and" and "or" functions, so that, for instance, $(\hat{V}_{e1})_3 = 1$ iff $(V_{e1})_j = (V_{o2})_j = 1$.

It is easy to verify that, since V_{o1}, V_{o2}, V_{e1}, and V_{e2} are all ordered, Equations (60)-(63) imply that $\hat{V}_{o1}$, $\hat{V}_{o2}$, $\hat{V}_{e1}$, and $\hat{V}_{e2}$ are all ordered as well. Furthermore,

$$z(\hat{V}_{o1}) = 2^{r-1} + \min[z(V_{e2}) , z(V_{o1}) - 2^{r-1}] , \tag{64}$$

$$z(\hat{V}_{o2}) = \min [z(V_{e1}) , z(V_{o2})] , \tag{65}$$

$$z(\hat{V}_{e1}) = \max [z(V_{e2}) , z(V_{o2})] , \tag{66}$$

$$z(\hat{V}_{e2}) = \max [z(V_{e2}) , z(V_{o1}) - 2^{r-1}] . \tag{67}$$

From (65) and (66) we see that $Z(\hat{V}_{o2}) \leq Z(\hat{V}_{e1})$; also, (64) and (67) imply that $Z(\hat{V}_{o1}) \leq Z(\hat{V}_{e2}) + 2^{r-1}$. We may use (20)-(23) and (64)-(67) to show that $Z(\hat{V}_{e2}) \leq Z(\hat{V}_{o2})$ and that $Z(\hat{V}_{e1}) \leq Z(\hat{V}_{o1})$. These relations are all summarized by

$$Z(\hat{V}_{e2}) \leq Z(\hat{V}_{o2}) \leq Z(\hat{V}_{e1}) \leq Z(\hat{V}_{o1}) \leq Z(\hat{V}_{e2}) + 2^{r-1}. \tag{68}$$

Since Relation (68) embodies Relations (24)-(26), the lemma is proved.

Q.E.D.

Appendix B: Proof of Theorem 4

Theorem 4:

Let V be as in Lemma 1. Then we can construct an [r] h-network for V using:

i) the $\frac{1}{2}(N-2^r)$ comparators $V_{(i,2s)}:V_{(i+1,2s-1)}$, $1 \le i \le t-1$, $1 \le s \le 2^{r-1}$, that produce the intermediate set $\hat{V}$; followed by

ii) the $\frac{1}{2}(N-2^{r-2})$ comparators $\hat{V}_{(i,s+2^{r-1})}:\hat{V}_{(i+1,s)}$, $1 \le i \le t-1$, $1 \le s \le 2^{r-1}$, that produce the intermediate set $\tilde{V}$; followed by

iii) an [r-1] h-network for $\tilde{V}$.

Proof:

We may use Lemma 1 and Corollary 1 to show that a $[2,2^{r-2}]$ f-network will order $\tilde{V}_o$ and that another $[2,2^{r-2}]$ f-network will order $\tilde{V}_e$. Therefore, each of the four subsets $\tilde{V}_{o1}$, $\tilde{V}_{o2}$, $\tilde{V}_{e1}$, and $\tilde{V}_{e2}$ is ordered. Furthermore, the number of 0's in these subsets satisfies

$$z(\tilde{V}_{o2}) \le z(\tilde{V}_{o1}) \le z(\tilde{V}_{o2}) + 2^{r-2}; \qquad (69)$$

$$z(\tilde{V}_{e2}) \le z(\tilde{V}_{e1}) \le z(\tilde{V}_{e2}) + 2^{r-2}. \qquad (70)$$

In order to prove that an [r-1] h-network will complete the ordering of $\tilde{V}$, we must show that the number of 0's in the four subsets also satisfies

$$z(\tilde{V}_{e1}) \le z(\tilde{V}_{o1}) \le z(\tilde{V}_{e1}) + 2^{r-2}; \tag{71}$$

$$z(\tilde{V}_{e2}) \le z(\tilde{V}_{o2}) \le z(\tilde{V}_{e2}) + 2^{r-2}; \tag{72}$$

Now the four subsets $\tilde{V}_{o1}$, $\tilde{V}_{o2}$, $\tilde{V}_{e1}$, and $\tilde{V}_{e2}$ of $\tilde{V}$ are defined by (12)-(15), with t replaced by 2t and with r replaced by r-1, so that $\tilde{V}$ is considered to be a $2t \times 2^{r-1}$ array. However, since the comparators listed in i) and ii) assume that V and $\hat{V}$ are $t \times 2^r$ arrays, it is convenient to consider $\tilde{V}$ to be $t \times 2^r$ as well. In this case the four subsets of $\tilde{V}$ are given by

$$\tilde{V}_{o1} = \bigcup_{1 \le i \le t} \bigcup_{\substack{j \text{ odd} \\ j \le 2^{r-1}}} \{\tilde{V}_{(i,j)}\}; \tag{73}$$

$$\tilde{V}_{o2} = \bigcup_{1 \le i \le t} \bigcup_{\substack{j \text{ odd} \\ j > 2^{r-1}}} \{\tilde{V}_{(i,j)}\}; \tag{74}$$

$$\tilde{V}_{e1} = \bigcup_{1 \le i \le t} \bigcup_{\substack{j \text{ even} \\ j \le 2^{r-1}}} \{\tilde{V}_{(i,j)}\}; \tag{75}$$

$$\tilde{V}_{e2} = \bigcup_{1 \le i \le t} \bigcup_{\substack{j \text{ even} \\ j > 2^{r-1}}} \{\tilde{V}_{(i,j)}\}. \tag{76}$$

(See Fig. 7.)

We may use the right-hand-sides of (73)-(76), with $\tilde{V}$ replaced by $\hat{V}$, to define four similar subsets X_{01}, X_{o2}, X_{e1}, and X_{e2} of $\hat{V}$. For example, we define

$$X_{01} = \bigcup_{1 \le i \le t} \bigcup_{\substack{j \text{ odd} \\ j \le 2^{r-1}}} \{\hat{V}_{(i,j)}\}, \tag{77}$$

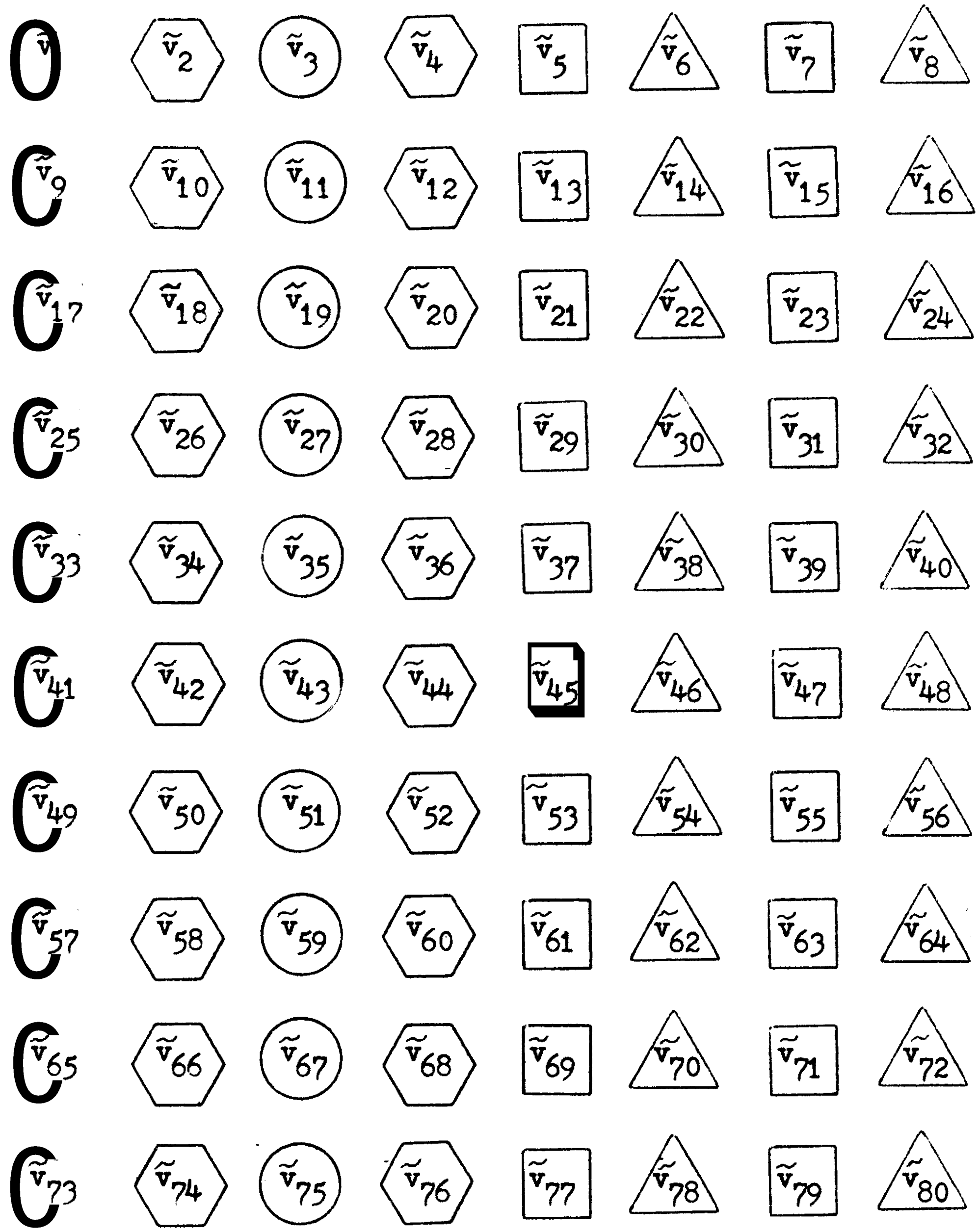


Fig. 7. The subsets of $\tilde{V}$.

Key: $\tilde{V}_{o1}$ = ○ ; $\tilde{V}_{o2}$ = □ ;
$\tilde{V}_{e1}$ = ⬡ ; $\tilde{V}_{e2}$ = △ .

Using this notation we can express the comparators in ii) as

$$(X_{o2})_j : (X_{o1})_{j+2^{r-2}}, \qquad 1 \le j \le \tfrac{1}{4}N - 2^{r-2}; \tag{78}$$

$$(X_{e2})_j : (X_{e1})_{j+2^{r-2}}, \qquad 1 \le j \le \tfrac{1}{4}N - 2^{r-2}. \tag{79}$$

These comparators are illustrated in Fig. 8.

From the proof of Lemma 1 we know that the four subsets $\hat{V}_{o1}$, $\hat{V}_{o2}$, $\hat{V}_{e1}$, and $\hat{V}_{e2}$ of $\hat{V}$ are each ordered and that the number of 0's in these subsets satisfies (68). Let us represent the number of 0's in $\hat{V}_{o1}$ as

$$Z(\hat{V}_{o1}) = \alpha_{o1} 2^{r-1} + \beta_{o1} 2^{r-2} + \gamma_{o1}, \tag{80}$$

where α_{o1}, β_{o1}, and γ_{o1} are integers satisfying

$$\begin{aligned} 0 &\le \alpha_{o1} \le \tfrac{1}{2}t; \\ 0 &\le \beta_{o1} \le 1; \\ 0 &\le \gamma_{o1} < 2^{r-2}. \end{aligned} \tag{81}$$

We can represent $\hat{V}_{o2}$ in terms of similar coefficients α_{o2}, β_{o2}, and γ_{o2}. Note that (68) implies that $\alpha_{o2} \le \alpha_{o1} \le \alpha_{o2} + 1$.

The subset X_{o1} of $\hat{V}$ includes the first 2^{r-2} members of each row of $\hat{V}_{o1}$ and the first 2^{r-2} members of each row of $\hat{V}_{o2}$. (See Fig. 8). Since $\hat{V}_{o1}$ and $\hat{V}_{o2}$ are each ordered,

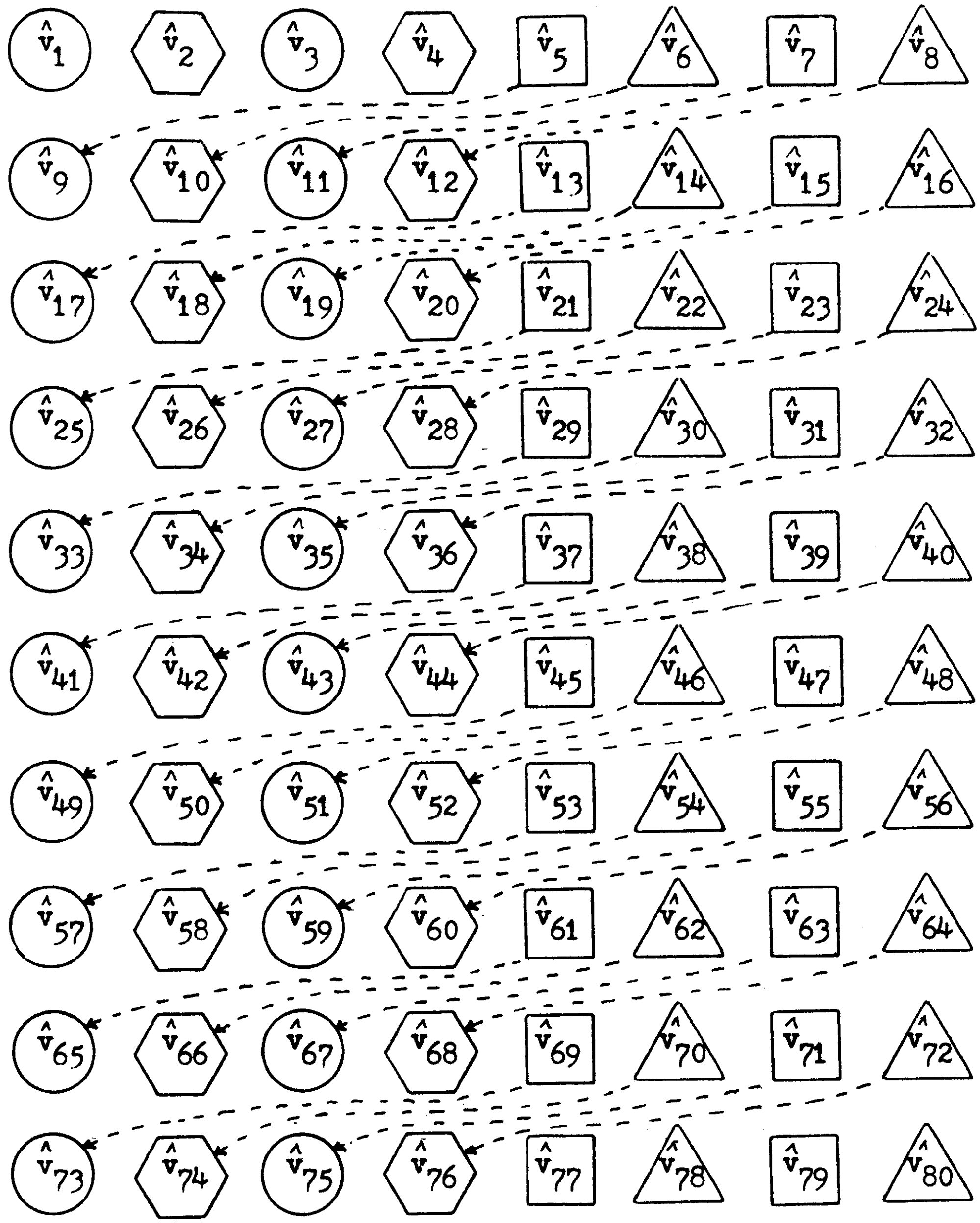


Fig. 8, The comparators $\hat{V}_{(i,s+2^{r-1})} : \hat{V}_{(i+1,s)}$.

Key: X_{o1} = ○ ; X_{o2} = □ ;

X_{e1} = 0 ; X_{e2} = △ .

$$z(X_{o1}) = (\alpha_{o1} + \alpha_{o2})2^{r-2} + \lambda_1 + \lambda_2, \tag{82}$$

where

$$(\alpha_{01} = \alpha_{o2}) \Rightarrow \begin{cases} \lambda_1 = \beta_{o1}2^{r-2} + (1-\beta_{o1})\gamma_{o1}, \\ \lambda_2 = \beta_{o2}2^{r-2} + (1-\beta_{o2})\gamma_{o2}; \end{cases} \tag{83}$$

$$(\alpha_{01} = \alpha_{o2}+1) \Rightarrow \begin{cases} \lambda_1 = \beta_{o2}2^{r-2} + (1-\beta_{o2})\gamma_{o2}, \\ \lambda_2 = \beta_{o1}2^{r-2} + (1-\beta_{o1})\gamma_{o1}. \end{cases} \tag{84}$$

The first $\alpha_{o1} + \alpha_{o2}$ "rows" of X_{01}, given by

$$\bigcup_{\substack{j \text{ odd} \\ j \le 2^{r-1}}} \{ \hat{V}_{(i,j,1)} \}, \; 1 \le i \le \alpha_{o1} + \alpha_{02}, \tag{85}$$

each contain 2^{r-2} 0's. The next two "rows" contain λ_1 and λ_2 0's, respectively. Note that if $0 < \lambda_2 \le X1 < 2^{r-2}$, then the subset X_{o1} is not ordered.

The s&set X_{o2} of $\hat{V}$ includes the last 2^{r-2} members of each row of $\hat{V}_{01}$ and $\hat{V}_{o2}$. (See Fig. 8.) Therefore,

$$z(X_{o2}) = (\alpha_{o1} + \alpha_{o2})2^{r-2} + \mu_1 + \mu_2, \tag{86}$$

where

$$(\alpha_{01} = \alpha_{o2}) \Rightarrow \begin{cases} \mu_1 = \beta_{o1}\gamma_{o1}, \\ \mu_2 = \beta_{o2}\gamma_{o2}; \end{cases} \tag{87}$$

$$(a_{01} = \alpha_{o2}+1) \Rightarrow \begin{cases} \mu_1 = \beta_{o2}\gamma_{o2}, \\ \mu_2 = \beta_{o1}\gamma_{o1}. \end{cases} \tag{88}$$

The first $\alpha_{o1} + \alpha_{o2}$ "rows" of X_{02} are all O's; the next two "rows" contain μ_1 and μ_2 O's, respectively. Note that X_{o2} is not ordered if $\mu_2 > 0$, since $\mu_1 < 2^{r-2}$.

The comparators given by (78)-(79) transform the intermediate set $\hat{V}$ into the set $\tilde{V}$. As indicated in Fig. 8, the i^{th} "row" of X_{02} is compared, item by item, with the $(i+1)^{st}$ "row" of X_{o1}; the "row" containing more O's becomes the i^{th} "row" of $\tilde{V}_{o2}$ while the "row" with fewer O's becomes the $(i+1)^{st}$ "row" of $\tilde{V}_{o1}$. Therefore,

$$Z(\tilde{V}_{o1}) = (\alpha_{o1}+\alpha_{o2})2^{r-2} + \lambda_1 + \min[\mu_1,\lambda_2]; \tag{89}$$

$$Z(\tilde{V}_{o2}) = (\alpha_{o1}+\alpha_{o2})\alpha^{r-2} + \max[\mu_1,\lambda_2] + \mu_2. \tag{90}$$

We may refer back to the definitions of λ_i and μ_i to verify that (89)-(90) imply that

$$Z(\tilde{V}_{o1}) = 2^{r-2} + \min[Z(\hat{V}_{o2}) , Z(V_{o1}) - 2^{r-2}]; \tag{91}$$

$$Z(\tilde{V}_{o2}) = \max[Z(\hat{V}_{o2}) , Z(\hat{V}_{o1}) - 2^{r-2}]. \tag{92}$$

For example, if $\alpha_{o1} = \alpha_{o2}$, then (89) reduces to

$$\begin{aligned} Z(\tilde{V}_{o1}) &= (\alpha_{o1}+\alpha_{o1})2^{r-2} + \beta_{o1}2^{r-2} + (1-\beta_{o1})\gamma_{o1} \\ &\quad + \min[\beta_{o1}\gamma_{o1} , \beta_{o2}2^{r-2} + (1-\beta_{o2})\gamma_{o2}] \\ &= \min[Z(\hat{V}_{o1}) , Z(\hat{V}_{o2}) - \beta_{o2}\gamma_{o2} + \beta_{o1}2^{r-2} \\ &\quad - (1-\beta_{01})\gamma_{01}] . \end{aligned} \tag{93}$$

Now $z(\hat{v}_{o1}) < z(\hat{v}_{o2}) + 2^{r-2}$ implies that $\beta_{o1} = \beta_{o2}$ and that $0 \le z(\hat{v}_{o1}) - z(\hat{v}_{o2}) = \gamma_{o1} - \gamma_{o2} < 2^{r-2}$, so that (93) becomes

$$z(\tilde{v}_{o1}) = \min[z(\hat{v}_{o1}), z(\hat{v}_{o2}) + 2^{r-2} - \gamma_{o1}]$$

$$= z(\hat{v}_{o1})$$

$$= \min[z(\hat{v}_{o1}), z(\hat{v}_{o2}) + 2^{r-2}]; \qquad (94)$$

whereas $z(\hat{v}_{o1}) \ge z(\hat{v}_{o2}) + 2^{r-2}$ implies that $\beta_{o1} = 1, \beta_{o2} = 0,$ $0 \le \gamma_{o1} - \gamma_{o2} < 2^{r-2}$, so that (93) becomes

$$z(\tilde{v}_{o1}) = \min[z(\hat{v}_{o1}), z(\hat{v}_{o2}) + 2^{r-2}]. \qquad (95)$$

Equations (94) and (95) are equivalent to (91).

In a similar manner we can show that

$$z(\tilde{v}_{e1}) = 2^{r-2} + \min[z(\hat{v}_{e2}), z(\hat{v}_{e1}) - 2^{r-2}]; \qquad (96)$$

$$z(\tilde{v}_{e2}) = \max[z(\hat{v}_{e2}), z(\hat{v}_{e1}) - 2^{r-2}]. \qquad (97)$$

Equations (91) - (92), (96) - (97), and (68) together imply (71)-(72); as noted above, this is sufficient to prove the theorem.

Q.E.D.

Appendix C: Proof of Lemma 3

Lemma 3:

Let $V = \{v_1, v_2, \ldots, v_N\}$, where $N = t \cdot 2^r > 2^r$ and t is even. Suppose that the four subsets V_{o1}, V_{o2}, V_{e1}, and V_{e2} of V are each ordered and that they satisfy (20)-(23),(28), and (29). Then if we apply the [r] h-network described by Theorem 4 to V, the subsets of the intermediate set $\tilde{V}$ satisfy (20)-(23) and (28)-(31), with r replaced by r-1.

Proof:

The proof of Theorem 4 indicates that if the subsets of V are ordered and satisfy (20)-(23), then the subsets of $\tilde{V}$ are also ordered and satisfy (20)-(23), with r replaced by r-1. We shall prove that if the subsets of V satisfy (28) as well, then

$$z(\tilde{V}_{o1}) \le z(\tilde{V}_{e1}) + z(\tilde{V}_{o2}) + 1; \tag{98}$$

$$z(\tilde{V}_{e1}) < 2^{r-2} \Rightarrow z(\tilde{V}_{o2}) \le z(\tilde{V}_{e1}). \tag{99}$$

The proof that (29) implies that the subsets of $\tilde{V}$ satisfy (29) and (31), with r replaced by r-1, follows from symmetry.

Suppose that $z(\tilde{V}_{e1}) \ge 2^{r-2}$. Then (99) does not apply. And since the proof of Theorem 4 demonstrates that

$$z(\tilde{V}_{o1}) \le z(\tilde{V}_{o2}) + 2^{r-2}, \tag{100}$$

Equation (98) holds when $Z(\tilde{V}_{e1}) \geq 2^{r-2}$.

Now suppose that (28) holds and that $Z(\tilde{V}_{e1}) < 2^{r-2}$. Then from (96) we see that

$$Z(\tilde{V}_{e1}) = Z(\hat{V}_{e1}) < 2^{r-2}. \tag{101}$$

Equations (101) and (66) imply that

$$Z(V_{e1}) < 2^{r-2}; \tag{102}$$

$$Z(V_{o2}) < 2^{r-2}. \tag{103}$$

And we may use (21), (28), and (102)-(103) to conclude that

$$Z(V_{e2}) < 2^{r-2}; \tag{104}$$

$$Z(V_{o1}) < 2^{r-1}. \tag{105}$$

Equations (102)-(105) categorize the distribution of 0's in V; we may use these values in (64)-(67) to show that

$$Z(\hat{V}_{o1}) = Z(V_{o1}) \leq Z(V_{o2}) + Z(V_{e1}) + 1$$
$$\leq Z(\hat{V}_{o2}) + 2^{r-2}, \tag{106}$$

$$Z(\hat{V}_{o2}) = \min[Z(V_{o2}), Z(V_{e1})] < 2^{r-2}; \tag{107}$$

$$z(\hat{v}_{e1}) = \max[z(v_{o2}), z(v_{e1})] < 2^{r-2}; \tag{108}$$

$$z(\hat{v}_{e2}) = z(v_{e1}) < 2^{r-2}. \tag{109}$$

And finally, (91) - (92) and (96) - (97) then imply that

$$z(\tilde{v}_{o1}) = z(\hat{v}_{o1}) = z(v_{o1}); \tag{110}$$

$$z(\tilde{v}_{o2}) = z(\hat{v}_{o2}) -= \min[z(v_{o2}), z(v_{e1})]; \tag{111}$$

$$z(\tilde{v}_{e1}) = z(\hat{v}_{e1}) = \max[z(v_{o2}), z(v_{e1})]; \tag{112}$$

$$z(\tilde{v}_{e2}) = z(\hat{v}_{e2}) = z(v_{e2}). \tag{113}$$

From (111)-(112) we see that $z(\tilde{v}_{o2}) \leq z(\tilde{v}_{e1})$ so that (99) holds, and that $z(\tilde{v}_{o2}) + z(\tilde{v}_{e1}) = z(v_{o1}) + z(v_{e1})$, so that (98) holds as well.

Q.E.D.

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