THE BOSE-NELSON SORTING PROBLEM

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Dedicated to R. C. Bose on his Seventieth birthday

A typical "sorting network" for four numbers is illustrated in Figure 1; the network involves five "comparators", shown as directed wires connecting two lines. Four numbers are input at the left, and as they move towards the right each comparator causes an interchange of two numbers if necessary so that the larger number appears at the point of the arrow. At the right of the network the numbers have been sorted into nondecreasing order from top to bottom; it is easy to verify that this will be the case no matter what numbers are input, since the first four comparators select the smallest and the largest elements and the final comparator ranks the middle two.

Sorting networks were originally constructed prior to 1957 by R. J. Nelson, who developed special networks for eight or less elements. Nelson also showed that n more comparators always suffices to go from n elements to $n+1$ (see O'Connor and Nelson [1962]).

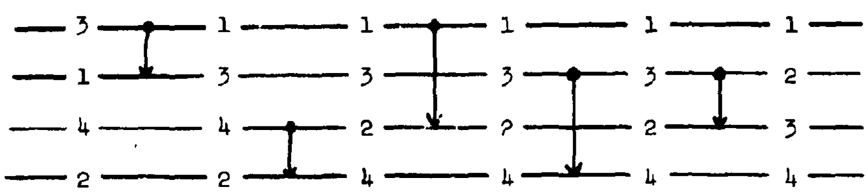

Figure 1. A Sorting Network.

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In 1960-1961 he and R. C. Bose constructed n-element sorting networks which were considerably more economical as $n \to \infty$ (see Bose and Nelson [1962]). The Bose-Nelson sorting problem is the problem of determining $S(n)$, the minimum number of comparators needed in an n-element sorting network.

Bose and Nelson gave an upper bound for $S(n)$, and conjectured that their method actually gave $S(n)$ exactly; but subsequent constructions have shown that their upper bound can be improved for all $n > 8$ (see Floyd and Knuth [1967], Batcher [1968]). In this paper we develop a few aspects of the theory, and prove that Bose and Nelson's conjecture was correct for $n \leq 8$.

Table 1 outlines some of the early work on the Bose-Nelson sorting problem, and summarizes its current status; see Knuth [1971] for further details of recent constructions, due to M. W. Green, A. Waksman, and G. Shapiro. The upper bounds listed for $n \leq 12$ are probably exact.

In order to study the problem in detail, it is convenient to introduce a few notational conventions. Let $x = \langle x_1,\ldots,x_n\rangle$ and $y = \langle y_1,\ldots,y_n\rangle$ be sequences of n real numbers; x is said to be sorted if $x_1 \leq x_2 \leq \cdots \leq x_n$. We define two operators on such sequences, the exchange operation (ij) and the comparator operation $[ij]$, for $1 \leq i,j \leq n$, $i \neq j$, as follows:

$$x(ij) = y \quad \text{iff} \quad y_i = x_j\,,\ y_j = x_i\,,\ y_k = x_k \ \text{ for } i \neq k \neq j\,; \qquad (1)$$

$$x[ij] = x \quad \text{if} \quad x_i \leq x_j\,,\ x[ij] = x(ij) \ \text{ if } x_i > x_j\,. \qquad (2)$$

Thus $x[ij] = y$ iff $y_i = \min(x_i,x_j)$, $y_j = \max(x_i,x_j)$, and $y_k = x_k$ for $i \neq k \neq j$. It is clear that, when i,j,k,ℓ are distinct, we have (see Figure 2)

Approx. date	n =	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Asymptotic
1956	Nelson	0	1	3	5	9	12	18	19	27	36	46	57	69	72	86	101	$\frac{1}{2}n^2$
1960	Bose and Nelson	0	1	3	5	9	12	16	19	27	32	33	42	50	55	61	65	$n^{1.6}$
1964	Floyd and Knuth	0	1	3	5	9	12	16	19	25	31	37	41	49	54	60	64	$n^{1+c/\sqrt{\log n}}$
1964	Batcher	0	1	3	5	9	12	16	19	26	31	37	41	48	53	59	63	$\frac{1}{4}\,n(\log_2 n)^2$
1969	Best upper bounds known	0	1	3	5	9	12	16	19	25	29	35	39	46	51	56	60	$\frac{1}{4}\,n(\log_2 n)^2$
1966	Best lower bound known	0	1	3	5	9	12	16	19	22	25	28	31	34	39	42	46	$n \log_2 n$

Table 1. Bounds for $S(n)$.

$$(ij)[ij] = [ij] \ , \ [ij](ij) = [ji] \ ; \tag{3}$$

$$(ij)[jk] = [ik](ij) \ , \ (ij)[kj] = [ki](ij) \ , \ (ij)[k\ell] = [k\ell](ij) . \tag{4}$$

A comparator network α is a sequence of zero or more exchange and/or comparator operations; a sorting network α is a comparator network such that $x\alpha$ is sorted for all x . We write $\alpha\beta$ for the network consisting of α followed by β ; and we say that

$$\alpha \subseteq \beta \ \text{ iff } \ U\alpha \subseteq U\beta \ , \ \alpha \equiv \beta \ \text{ if } \ U\alpha = U\beta \ , \tag{5}$$

where U is the set of all sequences $\langle x_1,\ldots,x_n\rangle$. Figure 1 illustrates the sorting network $[12][34][13][24][23]$. Clearly $\alpha \subseteq \beta$ implies that $\alpha\gamma \subseteq \beta\gamma$. Furthermore, if β is a sorting network and $\alpha \subseteq \beta$ we must have $\alpha \equiv \beta$; in fact, $x\alpha = x\beta$ for all x in this case, since $x\alpha$ must be sorted.

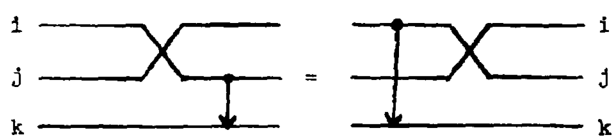

Figure 2

Sorting networks can also be interpreted in a more general way, if we allow m numbers to be contained in each line for some fixed $m \geq 1$. If $x_1,\ldots,x_n$ are multisets (i.e., sets with the possibility of repeated elements), containing m elements each, we can redefine the comparator $[ij]$ to be the operation of replacing x_i and x_j by the smallest and largest m elements, respectively, of the original $2m$ elements in x_i and x_j . See Figure 3, which illustrates the case $m = 2$. Our first result gives a basic property of this general interpretation.

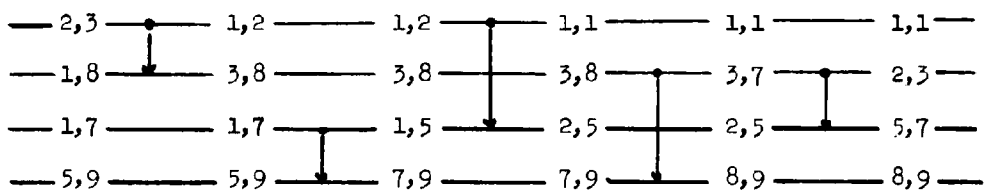

Figure 3. Another interpretation of the network in Figure 1.

Theorem 1. *Let α be a comparator network for n elements, and let i and j be indices such that $(x\alpha)_i \not\leq (x\alpha)_j$ for some x and for some $m \geq 1$; in other words, the m elements of the multiset $(x\alpha)_i$ are not all less than or equal to the m elements of $(x\alpha)_j$. Then there is a sequence $y = \langle y_1,\ldots,y_n \rangle$ of zeroes and ones such that $(y\alpha)_i = 1$ and $(y\alpha)_j = 0$.*

Proof: Let $\alpha = f_1 \ldots f_t$ where each f_s is an exchange or a comparator. Let u be the smallest element of $(x\alpha)_j$; we shall use the name A to stand for any number $\leq u$, and B for any number $> u$. By hypothesis, at least one element of $(x\alpha)_i$ is a B . We shall define a sequence $y^{(s)}$ of zeroes and ones, for $0 \leq s \leq t$, such that

$$y_p^{(s)} = 0 \text{ implies that } (xf_1 \ldots f_s)_p \text{ contains an } A \text{ ,} \qquad (6)$$

$$y_p^{(s)} = 1 \text{ implies that } (xf_1 \ldots f_s)_p \text{ contains a } B \text{ ,} \qquad (7)$$

for $1 \leq p \leq n$. First for $s = t$ we define $y_i^{(t)} = 1$, $y_j^{(t)} = 0$, and other elements $y_k^{(t)}$ are defined in any manner consistent with the above conventions (6), (7).

Assuming that $y^{(s)}$ has been defined for $s \geq 1$, we define $y^{(s-1)}$ as follows:

Case 1, $f_s = (pq)$. Then $y^{(s-1)} = y^{(s)}(pq)$.

Case 2, $f_s = [pq]$ and $y_p^{(s)} = y_q^{(s)}$. Then $y^{(s-1)} = y^{(s)}$. This fulfills the above conditions, since $y_q^{(s)} = 0$ implies that $(xf_1 \ldots f_s)_q$ contains at least one A , hence $(xf_1 \ldots f_s)_p$ contains <u>all</u> A's , hence there are more than m A's in all; some A's must be present in both $(xf_1 \ldots f_{s-1})_p$ and $(xf_1 \ldots f_{s-1})_q$. Similarly $y_p^{(s)} = 1$ implies that $(xf_1 \ldots f_{s-1})_p$ and $(xf_1 \ldots f_{s-1})_q$ both contain at least one B .

Case 3, $f_s = [pq]$ and $y_p^{(s)} \neq y_q^{(s)}$. Then $(y_p^{(s-1)}, y_q^{(s-1)})$ are defined to be either (0,1) or (1,0) , in any manner consistent with the above conventions; and $y_r^{(s-1)} = y_r^{(s)}$ for $p \neq r \neq q$. This definition of $y^{(s-1)}$ is justified because $(xf_1 \ldots f_{s-1})_p$ and $(xf_1 \ldots f_{s-1})_q$ are not both all A's or both all B's . Note that $y_q^{(s)} = 0$ is impossible, since it implies as in Case 2 that $(xf_1 \ldots f_s)_p$ is all A's , contradicting our convention; thus $y_p^{(s)} = 0$, $y_q^{(s)} = 1$.

According to this definition, $y^{(s-1)} f_s = y^{(s)}$, hence $y^{(0)}\alpha = y^{(t)}$; therefore $y = y^{(0)}$ satisfies the conditions of the theorem. ∎

When $m = 1$, Theorem 1 implies that a network will necessarily sort all possible inputs if we can prove that it sorts the 2^n sequences of zeroes and ones:

Corollary 1. A comparator network is a sorting network if and only if it sorts all sequences of zeroes and ones.

Corollary 2. Let $M(m)$ be the minimum number of comparators needed to merge two sets of m elements, i.e., to sort all sequences $\langle x_1,\ldots,x_{2m}\rangle$ such that $x_1 \le \cdots \le x_m$ and $x_{m+1} \le \cdots \le x_{2m}$. Then

$$S(mn) \le nS(m) + M(m)S(n) \quad , \tag{8}$$

$$M(mn) \le M(m)M(n) \quad . \tag{9}$$

Proof: Replace each line in an n-element sorting network by m parallel lines, and replace each comparator by $M(m)$ comparators which merge the $2m$ lines corresponding to the original 2 lines. Append n m-element sorting networks at the left, in order to sort each of the groups; this yields a sorting network for mn elements having $M(m)S(n) + nS(m)$ comparators. If we start with a sorting network that was constructed in this way for $n = 2$, the righthand part of the network has $M(m)$ comparators; expanding each line to m' lines makes the $M(m)M(m')$ comparators of the righthand part capable of merging two ordered groups of mm' elements. ∎

An example of the construction in Corollary 2 appears in Figure 4. Bose and Nelson proved Corollary 2 in the special case of binary merging, $n = 2$; this shows that $M(2^n) \le 3^n$, and $S(2^n) \le 3^n-2^n$. When $S(n) \approx n^\beta - n$ and $M(n) \approx n^\beta$, the inequalities in Corollary 2 do not allow us to lower the exponent β ; and in fact these inequalities do not lead to an especially efficient way to construct sorting networks, compared to other known methods. Yet the special case $m = 7$, $n = 3$ shows that 21 elements can be sorted with one less comparator than predicted by the

Bose-Nelson conjecture, and this is what first showed us that the conjecture was false in general (see Floyd [1964]). We went on to find that the conjecture is false for all $n > 8$. (This is, perhaps, poetic justice, since Bose made the conjecture shortly after he had helped to disprove Euler's famous Latin-squares conjecture, for all $n > 6$, after having first disproved it for $n = 50$! And our own sorting networks have by now been shown to be nonoptimal for all $n > 9$.)

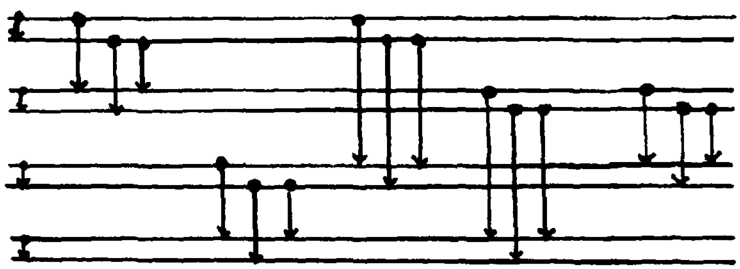

Figure 4. An 8-element sorting network, constructed from Figure 1 and Corollary 2 $(m = 2)$.

Let us now examine the properties of comparator networks a little more closely. In the first place, we can use identities (3), (4) to transform any comparator network so that all comparators precede all exchanges, and so that all comparators $[ij]$ have $i < j$. (Working from left to right, we replace $[ji]$ by $[ij](ij)$ when $i < j$, and we permute exchanges with comparators. This process clearly converges in a finite number of steps.) In this way a network α is transformed into $\alpha'\alpha''$ where α' has only "downward" comparators and α'' has only exchanges. If α is a sorting network, we can see by considering the effect of α on $\langle 1,2,\ldots,n \rangle$ that α'' must be the identity transformation, so $\alpha \equiv \alpha'$.

Let us say that a sorting network is in <u>standard form</u> if it consists entirely of comparators $[ij]$ with $i < j$, and no exchanges. We have proved

Lemma 1. <u>Every sorting network is equivalent to a network in standard form, having the same number of comparators.</u> ∎

When the network is in standard form and when x_n exceeds $x_1,\ldots,x_{n-1}$, all comparators $[in]$ are essentially inoperative; hence we can construct a sorting network for $n-1$ elements by simply removing all such comparators $[in]$ from an n-element sorting network in standard form. T. Hibbard [1963] observed that sorting networks having the same number of comparators as those originally constructed by Bose and Nelson can be obtained in this way by starting with a Bose-Nelson network for $2^k > n$ elements and deleting all comparators involving $x_{n+1},\ldots,x_{2^k}$.

We can now obtain a lower bound for the merging problem:

Theorem 2. $M(2n) \geq 2M(n) + n$.

<u>Proof</u>: Consider a network with $M(2n)$ comparators in standard form, which sorts $\langle x_1,\ldots,x_{4n} \rangle$ whenever $x_1 \leq x_3 \leq \cdots \leq x_{4n-1}$ and $x_2 \leq x_4 \leq \cdots \leq x_{4n}$. We separate the comparators $[ij]$ into three types,

A: $i \leq 2n$, $j \leq 2n$

B: $i \leq 2n$, $j > 2n$

C: $i > 2n$, $j > 2n$.

Since $x_{2n+1},\ldots,x_{4n}$ may be very large, there must be at least $M(n)$

comparators of type A; similarly there must be at least $M(n)$ of type C. And since we might have $x_i = 1$ when i is odd, 0 when i is even, there must be at least n comparators of type B in order to let n 0's rise to the top half of the diagram. ∎

A similar proof shows that $M(2n+1) \geq M(n) + M(n+1) + n$; and the same relations also hold with S in place of M. It follows that $M(n) \geq \frac{1}{2} n \log_2 n + n$ for all n ; this is why sorting networks based recursively on binary merging involve the order of $n(\log n)^2$ comparators, at least.

The best sorting networks known for $n > 8$ do not use binary merging, so Theorem 2 does not give us useful information about lower bounds for $S(n)$. When n is comparatively small, exact lower bounds can be found, as we shall now see. First we shall examine a general commutativity condition:

Lemma 2. If $\alpha = [i_1 j_1][i_2 j_2]\ldots[i_t j_t]$, where $\{i_1, i_2, \ldots, i_t\} \cap \{j_1, j_2, \ldots, j_t\} = \emptyset$, and if β is any rearrangement of the comparators of α , then $\alpha \equiv \beta$.

Proof: We shall show that $U\alpha$ is the set $S(\alpha)$ of all vectors $\langle x_1, \ldots, x_n \rangle$ such that $x_{i_s} \leq x_{j_s}$ for $1 \leq s \leq t$. All $x \in S(\alpha)$ satisfy $x\alpha = x$, hence $S(\alpha) \subseteq U\alpha$.

Conversely, suppose that $x\alpha \notin S(\alpha)$, i.e., $(x\alpha)_{i_s} > (x\alpha)_{j_s}$ for some s . Clearly s must be less than t ; let $\alpha' = [i_1 j_1]\ldots[i_{t-1} j_{t-1}]$. By induction on t , we have $(x\alpha')_{i_s} \leq (x\alpha')_{j_s}$, hence $[i_t j_t]$ has either increased the i_s component or decreased the j_s component of $x\alpha'$. This means that $i_s = j_t$ or $j_s = i_t$, contradicting the hypothesis. ∎

Lemma 3. *Let α be a sorting network for 3 or more elements. Then there is a sorting network, with no more comparators than α, in which the first three operations are either*

Case A, [12][13][23] ;

or Case B, [12][13][45] ;

or Case C, [12][34][13] ;

or Case D, [12][34][14] .

Proof: Clearly α includes at least three comparators, or it couldn't sort. Since $(ij)\alpha \equiv \alpha$, we may use (3) and (4) to transform α into a sorting network α' in which the first operation is [12] . For example, if $\alpha = [47]\beta$, we may take $\alpha' = (14)(27)\alpha = [12](14)(27)\beta$; and if $\alpha = [21]\beta$ we may take $\alpha' = \alpha = [12](12)\beta$. Similarly we may assume that the second operation is either [12] or [13] or [23] or [34] ; and since [12][12] = [12] , we may rule out the case [12][12] . If the second operation is [24] , we may observe that whenever $\alpha = [i_1 j_1]\ldots[i_t j_t]$ is a sorting network, so is the "dual" network $\alpha' = [j_1 i_1]\ldots[j_t i_t]\tau$ where τ is the sequence of exchanges which transforms $\langle x_1, x_2, \ldots, x_n\rangle$ into $\langle x_n, \ldots, x_2, x_1\rangle$. Hence when $\alpha = [12][23]\beta$ we may consider the sorting network $\alpha' = [21][32]\beta'\tau = [12][13](13)(12)\beta'\tau$. Therefore we may assume that the first two operations are [12][13] or [12][34] .

Proceeding in this way we can analyze the possibilities for the third comparator, as follows.

$[12][13][12] = [12][13]$.

$[12][13][23] = A$.

$[12][13][14] \supseteq [12][34][13]$.

$[12][13][24] \rightarrow [13][12][34](23) \equiv [12][13][34](23)$.

$[12][13][34] \rightarrow [12][14][34] = [12][34][14]$.

$[12][13][45] = B$.

$[12][34][13] = C$.

$[12][34][14] = D$.

$[12][34][23] \rightarrow [34][12][41](13)(24) = [12][34][14](14)(13)(24)$.

$[12][34][24] \sim [21][43][42] = [12][34][13](13)(34)(12)$.

$[12][34][15] = [12][15][34] \rightarrow [12][13][45](45)$.

$$[12][34][25] = [12][25][34] \sim [21][52][43] \rightarrow [21][32][45](35)$$
$$= [12][13][45](13)(12)(35)$$.

$[12][34][35] = [34][35][12] \rightarrow [12][13][45](45)(35)(24)(13)$.

$[12][34][45] \rightarrow [34][12][25](24)(13) = [12][34][25](24)(13)$.

Here "$\rightarrow$" denotes an appropriate left-multiplication by one or more exchanges, "$\equiv$" denotes an application of Lemma 2, and "$\sim$" denotes dualization as above.

Finally, if the first three operations are $[12][34][56]$, we may consider the first comparator which has an index in common with a previous one; this will reduce to a case already considered. ∎

The exhaustive method in this proof can be extended to show that there are essentially eleven ways to choose the first four comparators, when $n \geq 4$, namely

A1. [12][13][23][24] .

A2. [12][13][23][45] .

B1. [12][13][45][14] .

B2. [12][13][45][46] .

B3. [12][13][45][56] .

C1. [12][34][13][24] .

C2. [12][34][13][35] .

C3. [12][34][13][45] .

C4. [12][34][13][56] .

D1. [12][34][14][35] .

D2. [12][34][14][56] .

Details are omitted here, since we shall not need this fact.

Theorem 3. $S(n) \geq S(n-1) + 3$, for $n \geq 5$.

Proof: There is a sorting network with n comparators, in standard form, having one of the four forms stated in Lemma 3. If we suppress all comparators $[ij]$ with $i = 1$ we have a sorting network for $x_2, \ldots, x_n$, so we must show that at least three comparators have $i = 1$. This is obvious, since in each case we already know two of the comparators, and at least one more is required to bring the smallest element to the required position. ∎

Theorem 3 probably possesses the unique property that it has exactly two applications, no more and no less! Once $S(5)$ has been shown to equal 9 , we can use Theorem 3 to show that $S(6) = 12$; and $S(7) = 16$ will imply that $S(8) = 19$. Besides these results, the theorem appears to be quite useless.

We always have $S(n) \geq \log_2 n!$ by an elementary information-theoretic argument, hence the values of $S(1), S(2), S(3), S(4)$ are immediately established. But information theory tells us only that $S(5) \geq 7$, and Theorem 3 shows that $S(5) \geq 8$; the following theorem shows how to strengthen Theorem 3 when $n = 5$.

Theorem 4. $S(5) = 9$.

Proof: We need only show that $S(5) \geq 9$, in view of Bose and Nelson's construction. Proceeding as in Theorem 3, if the sorting network begins as in Case D we may permute the lines so that the first three comparators are [14][25][15] . Then we must have at least $S(3)$ more comparators [ij] with $i < j \leq 3$, and at $S(3)$ more with $3 \leq i < j$, to complete the sort. This makes 9 comparators.

For Cases A, B, and C we may permute the lines to obtain a sorting network in standard form in which the first three comparators are respectively

[12][15][25] in Case A,

[13][14][25] in Case B,

[14][25][12] in Case C.

Applying these to all 32 combinations $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ of zeroes and ones (cf. Corollary 1), then replacing all zeroes at the left and all ones at the right by asterisks, discarding all duplicates and all sequences which are nothing but asterisks, we obtain the 5-tuples

```
* * * 1 0
* * 1 0 0
* * 1 1 0
* * 1 0 *          (10)
* 1 0 0 *
* 1 1 0 *
* 1 0 * *
```

plus the "special" 5-tuples

```
1 1 0 0 * ,  1 1 0 * * ,  1 1 1 0 *   in Case A,
1 0 1 1 0 ,  1 0 * * *                in Case B,     (11)
1 1 0 * * ,  * 1 0 1 0 ,  * 1 1 1 0   in Case C.
```

In order to sort (10), we need at least $S(3)$ comparators with $2 \leq i < j \leq 4$ and $S(3)$ with $3 \leq i < j \leq 5$; and there must also be another with $i = 1$. The only way to do this with five more comparators is to use the sequence [34][23][45][34] or [34][45][23][34] , with an additional [1j] inserted somewhere. But then it is not difficult to verify that the special 5-tuples in (11) cannot all be sorted. ∎

Theorem 5. $S(7) = 16$.

Proof: This theorem was proved by exhaustive enumeration on a CDC G-21 computer at Carnegie Institute of Technology in 1966. The program was written by Mr. Richard Grove, and its running time was approximately 20 hours. The algorithm consisted of constructing a set S_t of sequences such that, for all α of the form $[i_1 j_1] \ldots [i_t j_t]$, there exist permutations π and ρ with $\pi\alpha\rho \supseteq \beta$ for some $\beta \in S_t$. The sets S_t were generated successively for $t = 1, 2, \ldots, 16$, taking care to keep each set rather small; for this purpose a 128-bit vector was maintained for each element of S_t , characterizing those 7-tuples of zeroes and ones which are output by the network. Most of the computation (about 13 hours) was spent in the cases $t = 8$ and 9 , since S_9 had 729 elements. None of the six elements in S_{15} was a sorting network. ∎

The methods of proof used to establish these lower bounds on $S(n)$ are of course quite unsatisfactory for larger values of n . We have no idea how to prove that $S(n)$ grows as $c\,n(\log n)^2$, although the best upper bounds known to date have this asymptotic behavior.

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13. ABSTRACT

Comparison/exchange modules can be combined to construct networks capable of sorting elements into nondecreasing order. In this paper we consider combinatorial properties of such networks, and we determine the minimum number of modules required to sort eight or less elements.

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Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
sorting networks						
merging networks						
comparators						

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Unclassified
Security Classification