"ON THE PROPERTIES OF THE DERIVATIVES OF THE SOLUTIONS OF LAPLACE'S EQUATION AND THE ERRORS OF THE METHOD OF FINITE DIFFERENCES FOR BOUNDARY VALUES IN C_2 AND $C_{1,1}$"

BY E. A. VOLKOV

TRANSLATED BY
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ON THE PROPERTIES OF THE DERIVATIVES OF THE SOLUTIONS OF LAPLACE'S EQUATION AND THE ERRORS OF THE METHOD OF FINITE DIFFERENCES FOR BOUNDARY VALUES IN C_2 AND $C_{1,1}$

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In § 1 of the present work it is shown that if in a circular disk a harmonic function u is given whose boundary values are twice continuously differentiable ($u \in C_2(\gamma)$, where γ is the boundary of the disk), then the same function u need not have bounded second derivatives in the open disk nor on any fixed line. In § 2 is investigated the error of the ordinary finite difference methods of solving the Dirichlet problem for Laplace's equation, when at the interior nodes of the net the solution is the arithmetic mean of the values at the four neighboring nodes, and at the nodes near the boundary there is applied Collatz's method of linear interpolation. In the case where the solution has second derivatives in the closed disk which satisfy a Hölder condition with exponent $\lambda > 0$, it is established in [1] under very weak conditions on the boundary of the region that this method gives uniform convergence on the net with a speed h^2 (h is the mesh constant), and that the order of convergence cannot be improved by any power of h (see [2]). In the present work it is assumed that the boundary everywhere has a tangent line whose angle of turning satisfies a Lipschitz condition ($\gamma \in C_{1,1}$) and that the boundary value has a first derivative satisfying a Lipschitz condition (belongs to $C_{1,1}(\gamma)$), and there is derived a uniform estimate for the error in the finite difference method which has the order $h^2 \ln h^{-1}$. In § 3 it is proved that this estimate cannot be improved under the stated conditions. Moreover, it is established that the speed of convergence of the scheme being considered can be worse than h^2 in a region with an arbitrarily smooth boundary, for example in a circular disk, and with more stringent conditions on the boundary values. Indeed, for any function $g(x)$ satisfying the properties that the ratio $g(x)/\ln x$ is positive for $x = 2$, is strictly monotonically decreasing as x increases, and takes values from infinity to zero, there exists a function harmonic in the circle with boundary values in $C_2(\gamma)$, for which the difference scheme considered above gives convergence not better than $h^2 g(h^{-1})$. In § 4 is presented a special scheme for a square net which ensures uniform convergence with speed h^2 in a region with boundary $\gamma \in C_{1,1}$ and with boundary values in $C_{1,1}(\gamma)$. In §5 it is proved that the given requirements on the boundary and boundary values, generally speaking, cannot be weakened in terms of the classes $C_{k,\lambda}$ and still obtain methods with order of convergence $h^2 \ln h^{-1}$, considered in § 2, or order of convergence h^2, considered in § 4.

The unimprovable error estimate for finite difference methods of order $h^2 \ln h^{-1}$, as derived in § 2, is stronger for the class of regions with boundaries -in $C_{1,1}$ than the corresponding result in [1], since the present result is established under weaker conditions on the solution of the Dirichlet problem for Laplace's equation than in [1]. Moreover, these conditions are -imposed in a natural manner only on the boundary of the region and the boundary values, and in a definite sense cannot be weakened. In [1] the error estimate of order $h^2 \ln x^{-1}$ was derived under essentially weaker conditions on the boundary than in the present work, but under the assumption of boundedness in the region of the second derivatives of the unknown solution.

§1. On the unboundedness of the second derivatives of harmonic functions with boundary values in C_2

Let $\Omega = \Omega\{(x-1)^2 + y^2 < 1\}$ be a disk with boundary γ, and let

$$v(x,y,\epsilon) = -\operatorname{Im}\{z_\epsilon^2 \ln z_\epsilon\},$$

where $z_\epsilon = x + \epsilon + iy$ and ϵ is a parameter. Obviously, for arbitrary $\epsilon > 0$ the function v is twice continuously differentiable along the arcs of the boundary γ. Indeed, as is shown by an elementary calculation,

$$\sup_{0<\epsilon<1} \max_{\gamma} |v_{s^k}^{(k)}| = V^k < \infty \qquad k = 0,1,2; \tag{1.1}$$

$$v_{xy}^{(2)}(x,0,\epsilon) = -2 \ln (x+\epsilon) - 4, \qquad x + \epsilon > 0. \tag{1.2}$$

We consider the function

$$w(x,y) = \sum_{n=1}^{\infty} \frac{1}{n^2} v(x,y,e^{-n^2}) \tag{1.3}$$

In view of (1.1) the value of w is twice continuously differentiable along γ, and also, because of Weierstrass's Theorem, the function w is continuous on $\bar{\Omega}$ and harmonic in Ω. Define

$$Q(x,y,m) = \sum_{n=1}^{2m} \frac{1}{n^2} v(x,y,e^{-n^2}) .$$

By virtue of (1.2),

$$Q^{(2)}_{xy}(0,0,m) > 4m - 8 \quad . \tag{1.4}$$

Let N be an arbitrarily large but fixed natural number, $N > 8$. In view of (1.4) and the continuity on $\bar{\Omega}$ the mixed derivative $Q^{(2)}_{xy}(x,y,N)$ at the point $(x_N,0) \in \Omega$, where $0 < x_N < e^{-3}/2$, satisfies the relation

$$Q^{(2)}_{xy}(x_N,0,N) > 2N.$$

Hence, by (1.2),

$$Q^{(2)}_{xy}(x_N,0,N^*) \geq Q^{(2)}_{xy}(x_N,0,N) > 2N, \tag{1.5}$$

where
$$N^* = \max \left\{ N, \left[\frac{4V^o}{\pi\, x_N^2 N} \right] + 1 \right\}.$$

We have (cf. [3], § 3) :

$$|w^{(2)}_{xy}(x_N,0) - Q^{(2)}_{xy}(x_N,0,N^*)| \leq \frac{8}{\pi\, x_N^2} \max_{\gamma} \; |w(x,y) - Q(x,y,N^*)|$$

$$< \frac{4V^o}{\pi\, x_N^2 N^*} CN.$$

Hence it follows from (1.5) that

$$|w^{(2)}_{xy}(x_N,0)| > N.$$

Because N was arbitrary,

$$\sup_{0 < x < 1} |w^{(2)}_{xy}(x,0)| = \infty \quad . \tag{1.6}$$

It will be proved in § 2 below that

$$\sup_{0<x<1} \left(\left| w^{(2)}_{x^2}(x,0) \right| + \left| w^{(2)}_{y^2}(x,0) \right| \right) < \infty \, . \tag{1.7}$$

Hence from (1.6) it follows that

$$\sup_{\Omega} \left| \frac{\partial^2 w}{\partial \ell^2} \right| = \infty, \tag{1.8}$$

where $\partial/\partial\ell$ is the operator of **differentiation** along an arbitrary fixed direction coinciding with neither the x nor y axes.

As is not difficult to show, the boundary values of the function w are thrice continuously differentiable everywhere on γ except at the origin of the coordinates. Hence [4] the function w has all its second derivatives bounded in an arbitrary subregion of the disk Ω whose closure does not contain (0,0). Hence from (1.8) it follows that for the function

$$w^*(x,y) = w(x,y) + w\left(\frac{x+y-1}{\sqrt{2}} + 1, \frac{x-y-1}{\sqrt{2}} \right) , \tag{1.9}$$

which is harmonic in Ω and has boundary values in $C_2(\gamma)$, the second derivatives computed in **an arbitrary** direction, including those parallel to the coordinate axes, are not bounded in the open disk Ω .

§2. An upper bound for the error of the finite difference method

We consider the Dirichlet problem

$$\Delta u = 0 \quad \text{in } \Omega , \qquad u = \varphi \quad \text{on } \gamma , \tag{2.1}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, γ is the boundary of the finite region Ω , and φ is a given function. We assume that $\gamma \in C_{1,1}$, that is, each point of γ has a tangent whose angle with the x-axis satisfies a Lipschitz condition with respect to arc length s . Further, we assume that each point of γ is tangent to a circle-of fixed radius r_0 lying entirely inside Ω . The function $\varphi \in C_{1,1}(\gamma)$, that is, its first derivative with respect to s satisfies a Lipschitz condition.

As is known ([5], p. 257) a solution of (2.1) that is continuous in $\bar{\Omega}$ has its first derivatives uniformly bounded in Ω . We investigate the higher derivatives of the function u . We construct a circular disk K of radius $r_{\smile}$, lying in Ω , whose boundary is tangent to γ at a certain fixed point M . We introduce a system of polar coordinates ρ, θ , with the origin at the center of the disk and the polar axis on the line from the center to the point M . Let n_M be the normal to γ at the point M . Then

$$v = u + a_M x + b_M y + c_M \quad , \tag{2.2}$$

where the constants a_M, b_M, c_M are so chosen that

$$v|_M = v_s^{(1)}|_M = (v-u)_{n_M}^{(1)}|_M = 0 \tag{2.3}$$

We represent the function v on the disk K by the Poisson integral

$$v(\rho,\Theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(r_o^2 - \rho^2)v(r_o,\omega)}{r_0^2 - 2r_o \rho \cos(\omega-\Theta) + \rho^2} d\omega \quad . \tag{2.4}$$

In view of (2.2), (2.3) and the uniform boundedness in Ω of the first derivatives of u ,

$$|v(r_o,\omega)| \le c'\omega^2 \quad , \quad |\omega| \le \pi \tag{2.5}$$

where c' is a certain constant not depending on the choice of the point $M\in\gamma$. By differentiating in the interior of the disk K under the integral sign in (2.4) both with respect to Θ and to ρ and applying some elementary transformations, we derive in view of (2.5) and (2.2) the following inequalities:

$$|u^{(2)}_{\rho^2}||_{\rho=r_o-t,\ \Theta=0} \le c'_2 \int_0^{\pi} \frac{(t + \omega^2)d\omega}{\sigma_t^2(\omega)} \le c_2^* \ ,$$

$$|(\frac{1}{\rho} u_\Theta^{(1)})_\rho^{(1)}||_{\rho=r_o-t,\ \Theta=0} \le c'_2 \int_0^{\pi} \frac{d\omega}{\sigma_t(\omega)} \le c_2^*(\ln \frac{r_o}{t} + 1) \ , \tag{2.6}$$

$$u^{(k)}_{\rho^k}||_{\rho=r_o-t,\ \Theta=0} + \left|\left(\frac{1}{\rho} u_\Theta^{(1)}\right)^{(k-1)}_{\rho^{k-1}}\right||_{\rho=r_o-t,\ \Theta=0}$$

$$\le c'_k \int_0^{\pi} \frac{d\omega}{\sigma_t^{k-1}(\omega)} \le c_k^* t^{2-k} \quad , \quad k > 2 \quad , \tag{2.7}$$

where u is the solution of problem (2.1), $0 < t \leq r_o$, and c'_ν, c^*_ν are constants independent of t and the choice of the point M. Also, $\sigma_t(\omega) = \omega + tr_o^{-1}$.

In this manner, in particular, it is established that the second derivative of u computed along an arbitrary normal to y, at an arbitrary point not farther than r_o from y, remains bounded by a constant that does not depend on the choice of the normal. Consequently, by virtue of Laplace's equation, on an arbitrary normal in a neighborhood of the boundary, the second derivative computed in a direction perpendicular to the normal is also bounded. Thus (1.7) holds.

Inside the region at a distance from y exceeding r_o, all derivatives of the function u with respect to the variables x and y are bounded by constants depending on the order of the derivative; see [3],§3. Hence from (2.6), (2.7), and the fact that u is harmonic it also follows that

$$\max_{0\leq m\leq 2} |u^{(2)}_{x^m y^{2-m}}| \leq \tilde{c}_2(|\ln t| + 1) , \tag{2.8}$$

$$\max_{0\leq m\leq k} |u^{(k)}_{x^m y^{k-m}}| \leq \tilde{c}_k t^{2-k} , \quad k > 2 , \tag{2.9}$$

where t is the distance of the current point from y, and $\tilde{c}_\nu$ are constants not depending on t.

The estimates (2.7)-(2.9) cannot be improved in the degree of dependence on t, since they are achieved for the function (3.1).

We construct a square net by the lines $x,y = 0, \pm h, \pm 2h, \ldots$. We denote by Ω_h the set of nodes of the net lying in Ω and having the property that all interior points of the segments of net connecting them with the four neighboring nodes lie in Ω . All other nodes that lie in Ω are assigned to the set γ_h . We introduce on Ω_h the averaging operator A ,

$$Au(x, y) \equiv (u(x+h, y) + u(x-h,y) + u(x,y+h) + u(x,y-h))/4 \quad .$$

At the point P of γ_h we construct the interpolating operator I ,

$$Iu \equiv \varphi_o/(1+\delta) + u_1\delta/(1+\delta) \ ,$$

where u_1 is the value of u at a point $P_1 \in \Omega_h \cup \gamma_h$; φ_o is the value of φ at the point P_o at the intersection of y with the line passing through the point P_1 and the point $P \in \gamma_h$; δ is the ratio of the lengths of the segments P_oP and PP_1 . We assume that $\delta > 2$, that the point P lies on the segment P_oP_1 , that the length of the segment P_oP_1 does not exceed 3h , and, moreover, that the segment P_oP_1 forms with the tangent to y at the point P_o an angle larger than a certain fixed positive value, for example $\pi/30$, and that all the interior points {of P_oP_1 ? -- GEF} belong to Ω .

Let $u_h = \varphi$ on γ . The following system of difference equations has a unique solution and approximates the problem (2.1):

$$u_h = Au_h \quad \text{on} \quad \Omega_h \quad , \quad u_h = Iu_h \quad \text{on} \quad \gamma_h \quad . \qquad (2.10)$$

Theorem 2.1. _If_ $\gamma \in C_{1,1}$ and$_{1,1}(\gamma\ )$, _then_

$$\max_{\Omega_h \cup \gamma_h} |u_h - u| < ch^2(|\ln h| + 1) \qquad , \qquad (2.11)$$

where u is the solution of (2.1), u_h is the solution of the system (2.10), and c is a constant independent of h and of the rectangular coordinate system used to construct the net.

Proof. Taking into account (2.8) and using the remainder term in Taylor's formula in the integral form (cf. [6], §3) using the second derivative, we derive the inequality*

$$|u-Iu| \leq c_1 h^2(|\ln h| + 1) \quad \text{on } \gamma_h \ . \tag{2.12}$$

Let Ω_h^1 be the subset of Ω_h consisting of the nodes whose distance from y does not exceed $3h/2$, and let $\Omega_h^2 = \Omega_h \backslash \Omega_h^1$. Analogous to (2.12) we can derive the inequality

$$|u-Au| \leq c_2 \ h^2(|\ln h| + 1) \quad \text{on } \Omega_h^1 \ . \tag{2.13}$$

From (2.9) for k = 4 and Taylor's formula follows the inequality

$$|u-Au| \leq c_3 \ h^4/t^2 \quad \text{on } \Omega_h^2 \quad ,$$

where t is the distance of the current node of the set Ω_h^2 from y .

In view of the comparison theorem ([7], p. 594)

$$|u_h-u| \leq \bar{\epsilon}_h \quad \text{on } \Omega_h \cup \gamma_h \quad , \tag{2.15}$$

where

$$\bar{\epsilon}_h = \bar{\epsilon}_h^1 + \bar{\epsilon}_h^2 + \bar{\epsilon}_h^3 \quad , \tag{2.16}$$

$$\bar{\epsilon}_h^1 = A\bar{\epsilon}_h^1 \ \text{on } \Omega_h \ , \ \bar{\epsilon}_h^1 = I\bar{\epsilon}_h^1 + |u-Iu| \ \text{on } \gamma_h \ , \tag{2.17}$$

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* Here and below c_p (P = 1,2,...) will denote constants not depending on the factor standing to their right.

$$\bar{\epsilon}_h^1 = A\bar{\epsilon}_h^1 + |u-Au| \quad \text{on } \Omega_h^1 ,$$

$$\bar{\epsilon}_h^2 = A\bar{\epsilon}_h^2 \quad \text{on } \Omega_h^2 , \quad \bar{\epsilon}_h^2 = I\bar{\epsilon}_h^2 \quad \text{on } \gamma_h , \tag{2.18}$$

$$\bar{\epsilon}_h^3 = A\bar{\epsilon}_h^3 \quad \text{on } \Omega_h^1 , \quad \bar{\epsilon}_h^3 = A\bar{\epsilon}_h^3 + |u-Au| \quad \text{on } \Omega_h^2 ,$$

$$\bar{\epsilon}_h^3 = I\bar{\epsilon}_h^3 \quad \text{on } \gamma_h , \tag{2.19}$$

and $\bar{\epsilon}_h^\nu = 0$ on y for $\nu = 1,2,3$.

In view of (2.12) and the estimates of [8],

$$\max_{\Omega_h \cup \gamma_h} \bar{\epsilon}_h^1 \leq c_4 h^2(|\ln h| + 1) \quad . \tag{2.20}$$

BY the method expounded in [9], p. 1074, and in [8], using (2.13), we easily establish the following inequality:

$$\max_{\Omega_h \cup \gamma_h} \bar{\epsilon}_h^2 \leq c_5 h^2(|\ln h| + 1) \tag{2.21}$$

and, in addition, the inequality

$$\max_{\Omega_h \cup \gamma_h} \bar{\epsilon}_h^3 \leq c_6 \max_{\Omega_h^2} \bar{\epsilon}_h^4 , \tag{2.22}$$

where

$$\bar{\epsilon}_h^4 = A\bar{\epsilon}_h^4 + |u-Au| \text{ on } \Omega_h^2 , \quad \bar{\epsilon}_h^4 = 0 \text{ on } \gamma_h \cup \Omega_h^1 . \tag{2.23}$$

On the basis of (2.14) and a lemma from [1], §1 (see also [1], §2, 3 and [10], §2, 3) which, as is easily shown, is applicable to the system (2.23), we derive

$$\max_{\Omega_h^2} \bar{\epsilon}_h^4 \leq c_7 h^2(|\ln h| + 1) \qquad (2.24)$$

Using the fact that the constants in inequalities (2.12)-(2.14), (2.20)-(2.22), and (2.24) may be chosen so as not to depend on the system 'of rectangular coordinates in which the net is built, from (2.20)-(2.22), (2.24), (2.16), (2.15) we are led to inequality (2.11). Theorem 2.1 has been proved.

Observation. In the special case where the boundary y has a curvature which everywhere satisfies a Hölder condition with a positive exponent, for the derivation of inequality (2.24) we may exploit the majorant

$$\psi = a_1 h^2 \ln (1 + (1 - \mu^2(x,y) - \nu^2(x,y))/a_2 h) ,$$

where μ,ν are the real and imaginary parts of the function which conformally maps the region Ω onto the unit circle, and a_1 and a_2 are certain positive constants [11].

§3. Lower bounds for the error

We prove that the estimate (2.11) is sharp with respect to h . We consider on the disk $\Omega = \Omega\{(x-1)^2 + y^2 < 1)$ the harmonic function

$$u = \operatorname{Im} \{z^2 \ln z\} , \tag{3.1}$$

where $z = x+iy$. The boundary values of this function satisfy the conditions of Theorem 2.1. We construct a net by means of two families of lines:

$$\frac{x+y}{\sqrt{2}} = 0, \pm h, \pm 2h, \ldots . \quad \frac{x-y}{\sqrt{2}} = 0, \pm h, \pm 2h, \ldots .$$

Let $0 < h < 1/50$, $P = P(h/\sqrt{2}, h/\sqrt{2})$, $P_0 = P_0(0,0)$, $P_1 = P_1(\sqrt{2}\, h, \sqrt{2}\, h)$. Here $P \in \gamma_h$, $P_1 \in \Omega_h$. We have at the point P

$$u-Iu = h^2(\ln h - 2 \ln 2h) > h^2(|\ln h| + 1)/2 . \tag{3.2}$$

Since

$$(u_h-u)\,|_P = \frac{1}{2}\,(u_h-u)|_{P_1} - (u-Iu)|_P \quad ,$$

it follows therefore, using also (3.2), that for $0 < h < 1/50$

$$\max_{\Omega_h \cup \gamma_h} |u_h-u| > \frac{h^2}{3}\,(|\ln h| + 1) \tag{3.3}$$

and, consequently, estimate (2.11) cannot be improved by any order of magnitude with respect to h .

We now prove that the speed of convergence of the system (2.10) can be worse than h^2 , even if the boundary values have a continuous second derivative. Indeed, for any function g(x) such that the ratio

$g(x)/\ln x$ is equal to unity for $x = 2$ and is strictly monotonically decreasing for increasing x $(x \geq 0)$, going to zero at infinity, there exists a function u, harmonic in the disk $\Omega\{(x-1)^2 + y^2 < 1\}$ with boundary values in $C_2(\gamma)$ such that on the above-considered net

$$\max_{\Omega_h \cup \gamma_h} |u_h - u| > c^* h^2 g(h^{-1}) \qquad , \tag{3.4}$$

where u_h is the approximate value of the function u, derived from the system (2.10), $0 < h \leq h^*$, and- c^* is a positive constant independent of h. For example, we may take $g(x) = \ln^{1-\alpha} 2 \ln^{\alpha} x$, where α is a constant with $0 < \alpha < 1$.

Let $\eta(x)$ be the function $g(x)/\ln x$, and let $\xi(x)$ be its inverse function $(\xi(\eta(x)) \equiv x$, $x \geq 2)$. Then let

$$u(x,y) = \sum_{n=1}^{\infty} \frac{1}{n^2} v_n(x,y) \qquad , \tag{3.5}$$

where $v_n = \operatorname{Im}\{z_n^2 \operatorname{In} z_n\}$, and

$$z_n(x,y) = x + \frac{1}{90\ \xi(n^{-1})} + i\left(y + \frac{1}{90\ \xi(n^{-1})}\right) \quad .$$

By Weierstrass, Theorem the function u of (3.5) is harmonic on the disk Ω and continuous on $\bar{\Omega}$; moreover, in view of the uniform (with respect to n) boundedness of the maximum of the moduli of the second derivatives of the boundary values of the functions v_n, the boundary values of the function u are twice continuously differentiable $(u \in C_2(\gamma)$, where y is the boundary of the disk Ω).

Since for $0 < x < e^{-3}$

$$\frac{d^2(x^2 \ln x)}{dx^2} = 2 \ln x + 3 \leq \ln x ,$$

then for $0 < h < e^{-3}/3$ at the point $P \in \gamma_h$ considered in the preceding example

$$v_n - Iv_n > -\frac{h^2}{2} \ln \left(2h + \frac{\sqrt{2}}{90\,\xi(n^{-1})}\right) > 0 ,$$

where $n = 1,2,\ldots$. In particular, at the point P for $m \geq m_h = [\eta^{-1}(h^{-1})]+1$, we have

$$v_m - Iv_m > -\frac{h^2}{2} \ln 3h > \frac{h^2}{4} \ln h^{-1} .$$

Therefore at the point P , for $0 < h < e^{-3}/3$, we have

$$u - Iu > \frac{h^2}{4} \ln h^{-1} \sum_{m=m_h}^{\infty} \frac{1}{m^2} > \frac{h^2}{4m_h} \ln h^{-1} > \frac{h^2}{8} \ln h^{-1} \eta(h^{-1})$$

$$= \frac{h^2}{8} g(h^{-1}) \qquad (3.6)$$

Since

$$(u_h - u)\big|_P = \frac{1}{2}(u_h - u)\big|_{P_1} - (u - Iu)\big|_P ,$$

where u_h is the solution of the system (2.10) for boundary values coinciding with the boundary values of the function (3.5), and u is the function (3.5), then in view of (3.6)

$$\max_{P \cup P_1} |u_h - u| > \frac{h^2}{12} g(h^{-1}) .$$

Consequently, inequality (3.4) is satisfied, where u is the function of (3.5), and $c^* = 1/12$ $(h^* = e^{-3}/3)$. That is, for Dirichlet's problem on the disk Ω , the solution of the system (2.10) for boundary values coinciding with those of (3.5) $(u \in C_2(\gamma))$, converges with a speed not better than $h^2 g(h^{-1})$. On the other hand, in view of Theorem 2.1 the estimate (2.11) holds.

Remark. By methods analogous to those of the examples considered, one can prove there exist functions harmonic in Ω with boundary values in $C_{1,1}(\gamma)$ and $C_2(\gamma)$, for which for sufficiently small h are satisfied, respectively, inequality (3.3) and (3.4) for an arbitrary choice of coordinate systems in which the net is constructed.

§4. A method with accuracy $O(h^2)$.

We assume that $\gamma \in C_{1,1}$. We introduce on $\gamma_h \cup \Omega_h^1$ a special interpolation operator I^* of the following form. Through the point $P \in \gamma_h \cup \Omega_h^1$ we draw the normal to γ (at the point P_o) and extend it to the point P_1 lying on some diagonal of the nearest net square which is at a distance not less than $h/2$ from γ (see the figure on the next page). In a special case one of the vertices of this square may coincide with the point P ; then the point P_1 must lie on the 'diagonal not containing P . We denote by $\mu h/\sqrt{2}$ the distance from P_1 to the center of the chosen square, and by δ the ratio of the length of the segment P_oP to the length of the segment PP_1 . At the point P

$$I^*u \equiv \frac{\varphi_o}{1+\delta} + \delta \, \frac{(1+\mu)^2 u_2 + (1-\mu)^2 u_4 + (1-\mu^2)(u_3+u_5)}{4(1+\delta)} ,$$

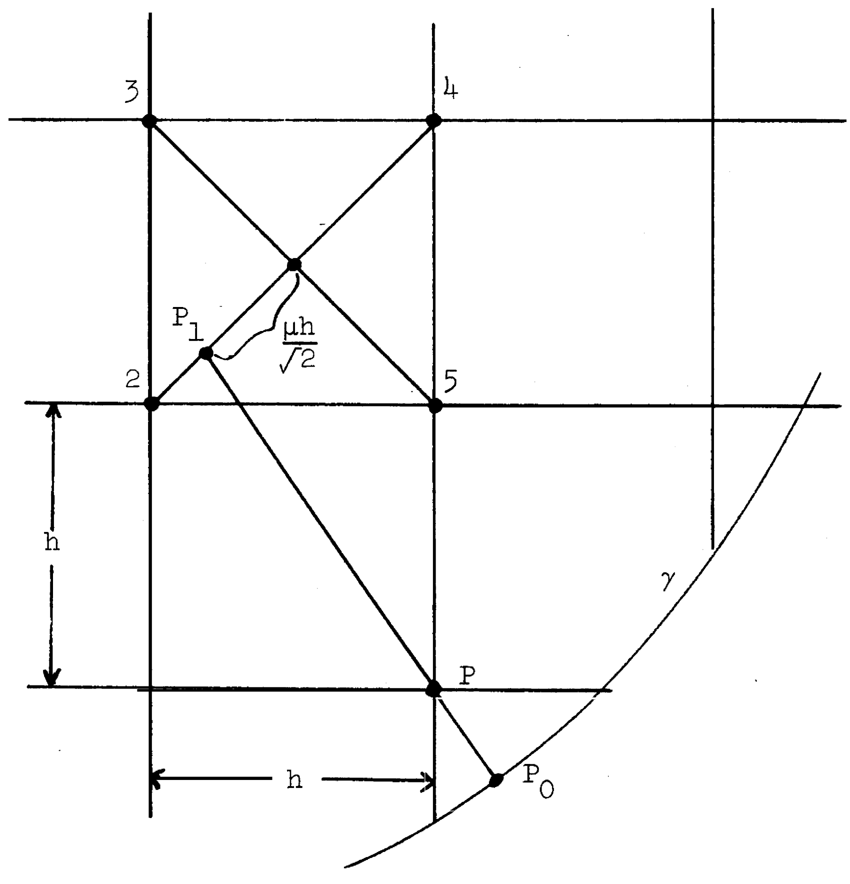
where φ_o is the value of u at the point P_o and u_k is the value of u at the point numbered k (see the figure). On Ω_h^2 we introduce the averaging operator A^* :

$$A^*u(x,y) \equiv (u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h))/5$$
$$+ (u(x+h,y+h) + u(x+h,y-h) + u(x-h,y+h) + u(x-h,u-h))/20 .$$

We consider the following system of difference equations:

$$u_h^* = A^* u_h^* \quad \text{on } \Omega_h^2 \quad , \quad u_h^* = I^* u_h^* \quad \text{on } \gamma_h \cup \Omega_h^1 , \qquad (4.1)$$

which have a unique solution (cf. [7], p. 594).


3
4
P1
μh/√2
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5
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γ
P
h
P0

Theorem 4.1. If $\gamma \in C_{1,1}$ and $\varphi \in C_{1,1}(\gamma)$, then

$$\max_{\Omega_h \cup \gamma_h} |u_h^* - u| \leq ch^2 \quad , \tag{4.2}$$

where u is the solution of u_h^*rok is the .1), s o l u t i o n o f the system (4.1), and c is a constant not depending on h nor on the choice of rectangular coordinate system for constructing the net.

Proof. Let $U_{P_1^3}$ be the maximum of the maximum moduli of all possible third derivatives with respect to x and y of the harmonic function on the closed net square containing P_1. Then the operator I^* is the composition of an interpolation operator at the point P_1 in terms of points 2, 3, 4, 5 (see figure) with a local error not exceeding in modulus the quantity $c_8 h^3 U_{P_1^3}$ and the operator of linear interpolation at P along the normal between points P_0 and P_1 Hence, because of (2.6) and (2.9),

$$|u - I^* u| \leq c_9 h^2 \quad \text{on } \gamma_h \cup \Omega_h^1 \ , \tag{4.3}$$

where the constant c_9 may be chosen independently not only of h, but also of the position of the rectangular system of coordinates in which the net is constructed. Moreover, by using Taylor's theorem with a remainder term computed with the eighth derivative and by using (2.9) with k = 8, it is easy to establish the inequality

$$|u - A^* u| < c_{10} h^8 / t^6 \quad \text{on } \Omega_h^2 \ , \tag{4.4}$$

where c_{10} is a constant independent of h, t, or the choice of rectangular coordinates. Furthermore, by use of (4.3) and (4.4), and relying on a lemma from [1], §1 and the method of [8, we prove Theorem 4.1

analogously to the proof of Theorem 2.1.

Observation. The estimate (4.2) cannot be improved in its order of magnitude with respect to h ; see the introduction to [1].

§5. On the necessity of the conditions on the boundary and boundary values.

We prove first that in the statement of Theorems 2.1 and 4.1 the conditions on the boundary in terms of the classes $C_{k,\lambda}$ cannot be removed. We consider a special finite region Ω, whose boundary contains the piece described by the equation

$$x = |y|^{1+\lambda} \quad , \quad |y| \le 1 \; ,$$

where λ is a number, $\frac{1}{2} < \lambda < 1$. On the whole boundary γ, except the point at the origin of coordinates, we assume the curvature to be continuously differentiable. Moreover, we assume that the region Ω lies entirely on one side of the curve $x = |y|^{1+\lambda}$, $|y| \le \infty$, for example on the right-hand side. Obviously, $\gamma \notin C_{1,1}$, but in any case the angle made by the tangent to γ with the x-axis satisfies a Hölder condition with exponent λ; i.e., $\gamma \in C_{1,\lambda}$, $\frac{1}{2} < \lambda < 1$.

We consider the function

$$u = \rho^{1+\lambda} \cos(1+\lambda)\theta - \rho \cos\theta \cos\frac{\pi(1+\lambda)}{2} \; , \tag{5.1}$$

which is harmonic in Ω, where $\rho = |z|$, $\theta = \arg z$, and $z = x+iy$. The boundary values of the function u are twice differentiable, and their second derivative on γ satisfies a Hölder condition with positive exponent. We have

$$u^{(4)}_{x^4}\Big|_{y=0} = u^{(4)}_{y^4}\Big|_{y=0} > (1-\lambda)\rho^{\lambda-3} \; . \tag{5.2}$$

Moreover, on Ω

$$|u^{(5)}_{x^5}| + |u^{(5)}_{y^5}| < c_{11}\ \rho^{\lambda-4} \quad . \tag{5.3}$$

Construct a net with the lines $x,y = 0, \pm 1, \pm 2, \ldots$ and choose h_0 so that for all h $(0 < h \leq h_0)$, the point $P(h(2 + [\,5c_{11}/(1-\lambda)\,],0)\in\Omega_h$. By expanding the function u in the neighborhood of the point P according to Taylor's formula with a remainder term involving the fifth derivative, in view of (5.2) and (5.3), we obtain the inequality

$$|u - Au|\Big|_P > c^* h^{1+\lambda}, \quad 0 < h \leq h_0 ,$$

where $c^* = (1-\lambda)(2+[5c_{11}/(1-\lambda)])^{\lambda-3}/48$. Hence, since $u_h - u = A(u_h-u)-(u-Au)$ on Ω_h, where u_h is the solution of the corresponding system (2.10),

$$\max_{\Omega_h \cup \gamma_h} |u_h - u| > \frac{c^*}{2} h^{1+\lambda} \quad , \quad 0 < h \leq h_0 \quad . \tag{5.4}$$

Analogously it can be established that the use of the scheme (4.1) for the function (5.1) on the region considered above gives convergence with a speed not better than $h^{1+\lambda}$.

The impossibility of weakening the requirements on the boundary values in terms of the classes $C_{k,\lambda}$ in the hypotheses of Theorems 2.1 and 4.1 was already implied in the introduction to [1]. Moreover, if in the function (5.1) one eliminates the second term, which in view of the linearity in x and y does not affect the error of the schemes (2.10) and (4.1), and considers u on the disk $\Omega = \Omega\{(x-1)^2 + y^2 < 1\}$, then the inequality (5.4) is still satisfied. But here $u\in C_{1,\lambda}(\gamma)$, where y is the boundary of the disk.

§6. Observations

1. Theorem 2.1 can be generalized in a natural way to multiply connected multidimensional regions with smooth boundaries in the class $C_{1,1}$.

2. In the two-dimensional case Theorem 2.1 can be generalized to a bounded region with N $(N < \infty)$ corners with angles less than $\pi/2$, when the pieces γ_j , $j = 1, \ldots, N$, connecting adjacent vertices of the corners belong to the class $C_{1,1}$, and when the boundary values are continuous on the boundary and, moreover, belong to $C_{1,1}(\gamma_j)$, $j = 1,2,\ldots,N$.

3. The method in [12] for obtaining a numerical majorant of the error in the form of the solution of an auxiliary system of difference equations can be applied also to the cases considered in the present paper. For that it is necessary to develop in more detail the estimates of the derivatives in (2.6)-(2.9), giving the numerical values of the exhibited constants.

4. There remains the open question, whether under the hypotheses of Theorem 2.1 it would accelerate the convergence, if in scheme (2.10) at γ_h the difference equations were not used with the operator I , but instead with the five-point difference operator approximating the Laplace operator on a non uniform net (see [7], number 3, page 591). In this case for the examples considered in §2 one does not succeed in deriving an estimate from below for the maximal error that is worse than $O(h^2)$. On the other hand, the inequality (2.11) is satisfied.

5. In [13] are considered regions that are rectangular parallelepipeds, and under certain conditions on the boundary values on the faces of the parallelepipeds that are weaker than Lipschitz conditions, there is derived a uniform estimate of order $h^2 \ln h^{-1}$ for the error in the method of finite differences for the Dirichlet problem for Laplace's equation, when the net is determined by planes parallel to the faces.

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