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# FIXED POINTS OF ANALYTIC FUNCTIONS

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## Abstract

A continuous mapping of a simply connected, closed, bounded set of the euclidean plane into itself is known to have at least one fixed point. It is shown that the usual condition for the fixed point to be unique, and for convergence of the iteration sequence to the fixed point, can be relaxed if the mapping is defined by an analytic function of a complex variable.

We consider the problems of the existence and of the construction of solutions of the equation

$$z = F(z) \,, \tag{1}$$

where the function F is analytic in some domain S of the complex plane. Such solutions are called *fixed points* of F. By standard results in real numerical analysis, it follows immediately that F has at least one fixed point if S is bounded and simply connected, F is continuous on the closure S' of s, and $F(S') \subset S'$. If the mapping defined by F is contracting, then there is a unique fixed point, and the *iteration sequence* defined by

$$z_{n+1} = F(z_n) \,, \; n = 0, 1, 2, \ldots, \tag{2}$$

converges to the fixed point for every choice of $z_0 \in S'$. If S is convex, a necessary and sufficient condition for the mapping to be contracting is that the derivative F' of F satisfies

$$|F'(z)| \leqq k \,, \; z \in S \,, \tag{3}$$

where $k < 1$.

It is the purpose of this note to show that the hypothesis that F is contracting can be dispensed with due to the analyticity of F. The argument provides an opportunity to apply some basic facts of complex variable theory in a constructive setting.

THEOREM. *Let S denote the interior of a Jordan curve* $\Gamma$*, let F be analytic in S and continuous on* $S \cup \Gamma$*, and let* $F(S \cup \Gamma) \subset S$*. Then F has exactly one fixed point, and the iteration sequence defined by (2) con-*

verges to the fixed point for arbitrary $z_0 \in S \cup \Gamma$ .

Clearly, there are functions F satisfying the hypotheses for which $|F'|$ is arbitrarily large, e.g., $F(z) = \frac{1}{2} z^{100}$ in $|z| \leq 1$ .

Proof. We first prove the Theorem in the case where S is the unit disk. Here the hypothesis implies

$$r := \max_{|z| \leq 1} |F(z)| < 1 \quad . \tag{4}$$

The point s is a fixed point if and only if it is a zero of $z - F(z)$. To prove the existence of a zero, we apply Rouché's theorem ([1], p. 124) with z in the role of the "big" function and $F(z)$ in the role of the "small" function. The essential hypothesis of Rouché's theorem is satisfied in view of (4). It follows that $z - F(z)$ has exactly as many zeros inside $|z| = 1$ as z , namely one.

Let s denote the unique fixed point. In order to prove the convergence of the iteration sequence, let

$$t(z) = \frac{z - s}{1 - z\bar{s}} \quad .$$

This is a linear transformation which maps $|z| \leq 1$ onto itself and sends s into 0 . Hence the function $G = t \circ F \circ t^{-1}$ has the fixed point 0 . It is continuous and maps $|z| \leq 1$ onto a closed subset of $|z| < 1$ , hence

$$k := \sup_{|z| \leq 1} |G(z)| < 1$$

We may assume that $k > 0$ , for otherwise G , and consequently F , is constant, and convergence takes place in one step. The function $k^{-1}G$ vanishes at 0 and is bounded by 1, hence by the Lemma of Schwarz ([1], p. 110),

$k^{-1}|G(z)| \leqq |z|$ and consequently,

(5) $$|G(z)| \leqq k|z|$$

for all $z$ such that $|z| \leqq 1$. Let $w_n = t(z_n)$. Since

$w_{n+1} = t(z_{n+1}) = f(F(z_n)) = t(F(t^{-1}(w_n))) = G(w_n)$, it follows from (5) that

$$|w_{n+1}| \leqq k\,|w_n|$$

and hence that $|w_n| \leqq k^n|w_0|$, implying that $w_n \to 0$. Hence $z_n = t^{-1}(w_n) \to t^{-1}(0) = s$.

To prove the Theorem for an arbitrary Jordan domain S, we require a less elementary tool, the Osgood-Carathéodory theorem ([2], p.92-98) stating the existence of a function $g$ that maps S conformally onto $|z| < 1$ and $S \cup \Gamma$ continuously and one-to-one onto $|z| \leqq 1$. The function $H = g \circ F \circ g^{-1}$ is easily seen to satisfy the hypotheses of the Theorem for the unit disk. Furthermore, if the points $z_n$ are defined by (2) and $w_n = g(z_n)$, then $w_{n+1} = H(w_n)$. Thus the validity of the Theorem for the unit disk implies the validity for a general S.

In line with the paedagogical nature of this note, we add some problems amplifying its content.

1) Show that $k \leqq \dfrac{2r}{1 + r^2}$.

2) In the case where S is the unit disk, show that

$$|z_n - s| \leqq (1 + r)\,k^n,\ n = 0, 1, 2, \ldots.$$

3) Let $F'(s) = F''(s) = \ldots = F^{(m-1)}(s) = 0$, $F^{(m)}(s) \neq 0$ for some integer $m > 1$. If S is the unit disk, establish the following error estimate showing superlinear convergence:

$$|z_n - s| \leqq (1 + r)\, k^{1} \text{[illegible]}^{2} + \cdots + m^{n-1}, \quad n = 1, 2, \ldots .$$

Research problem. Can similar results be established for systems of analytic equations?

## REFERENCES


[1] L. Ahlfors, Complex Analysis, 1st edition. McGraw-Hill, New York 1953.

[2] C. Caratheodory, Theory of functions of a complex variable, vol. 2 (English edition). Chelsea, New York 1960.