CS 102

INTEGER PROGRAMMING OVER A CONE

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TECHNICAL REPORT NO. CS 102
JULY 12, 1968

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0. Introduction

The properties of a special form integer programming problem are discussed. We restrict ourselves to optimization over a cone (a set of n constraints in n unconstrained variables) with a square matrix of positive diagonal and non positive off-diagonal elements. (Called a bounding form by F. Glover[3]).

It is shown that a simple iterational process gives the optimal integer solution in a finite number of steps,

It is then shown that any cone problem with bounded rational solution can be transformed to the bounding form and hence solved by the outlined method.

Some extensions to more than n constraints are discussed and a numerical example is shown to solve a bigger problem.

The research reported here was supported in part by the Advanced Research Projects Agency of the Office of the Secretary of Defense (SD-183).

1. Notation and Formulation

Our main concern in this paper will be with linear programming problems of the form

(1) Maximize $Z = C^T X$

Subject to $MX \leq b$

Where C, X, b are n dimensional column vectors, M is a rectangular $m \times n$ matrix, and all components of M, C, b are assumed to be integral. X is unconstrained and any positivity requirements are explicitly incorporated into the system $MX \leq b$.

We will refer to the linear program (1) as the rational program, because it possesses in general a rational solution X .

If we add to (1) the requirement that the solution X be integral, we obtain the integral program, whose solution gives in general a lower value of the objective function Z .

As is known, the notion of optimal solution to (1) is meaningful only when $m > n$, since otherwise unbounded rational solutions exist and most often unbounded integral solutions. Thus we will be interested in this case only.

An additional requirement imposed is one of complete non degeneracy, i.e. All square $n \times n$ submatrices -of M should be non singular.

The special case $m = n$ is called a rational (integral) cone program.

2. Properties of a Bounding Cone

A cone program (1) is in bounding form if the following conditions are met:

a) $C \geq 0, C \neq 0$

b) $M_{ij} \leq 0$ for i# j

$M_{ii} > 0$ for $i = 1, \ldots n$

Thus M can be written as M = D-A where D is a positive diagonal matrix and A is a non-negative off-diagonal matrix.

c) There exists a positive row vector $\alpha = (\alpha_1, \ldots \alpha_n)$, $\alpha_i > 0$ such that $\alpha M > 0$ or for each component

$$\sum_{i=1}^{n} \alpha_i M_{ij} > 0 \qquad j = 1, 2, \ldots n.$$

Since M is not singular by requirement, there exists M^{-1}. And we have

Lemma 1 $M^{-1} \geq 0$

Let $u = (U_1, \ldots U_n)$ be any row vector such that $UM \geq 0$. Consider the quantities $U_1/\alpha_1, U_2/\alpha_2, \ldots U_n/\alpha_n$ where α_i are those given in condition c). Let k be the one for which U_k/α_k is minimal i.e.

$$(2) \qquad U_k/\alpha_k \leq U_i/\alpha_i \qquad \text{for} \quad i = 1, \ldots n .$$

Inspecting the k^{th} inequality of $UM \geq 0$ we get

$$0 \leq \sum_{i=1}^{n} U_i M_{ik} = \sum_{i=1}^{n} \left(\frac{U_i}{\alpha_i} \right) (\alpha_i M_{ik}) \leq \left(\frac{U_k}{\alpha_k} \right) \sum_{i=1}^{n} \alpha_i M_{ik}$$

Where replacing each $\frac{U_i}{\alpha_i}$ $i \neq k$ by $\frac{U_k}{\alpha_k}$ only increases its value because of (2) and the fact that $\alpha_i M_{ik} < 0$. From condition c) we know that $\sum_{i=1}^{n} \alpha_i M_{ik} > 0$ therefore we have $U_k/\alpha_k \geq 0$ and also

$$U_i/\alpha_i \geq U_k/\alpha_k \geq 0 \quad \text{for } i = 1, \ldots n .$$

Thus we conclude that U is non negative or

$$UM \geq 0 \Rightarrow U \geq 0$$

Since all rows of the inverse matrix M^{-1} satisfy the left side of this implication it follows that $M^{-1} \geq 0$

<u>Observation</u>: Given that $M^{-1} = N \geq 0$ it is easily shown that condition c) is satisfied by taking $\alpha_i = \sum_{k=1}^{n} N_{ki}$ (Let α be the sum of the rows of M^{-1}).

Hence given a) and b) condition c) is equivalent to $M^{-1} > 0$.

We show that a bounding cone always possesses a bounded rational solution given by $X_R = M\text{-}1b$.

Let X be any feasible point satisfying $MX \leq b$.

Subtracting the equation $MX_R = b$ from this we get

$$M(X - X_R) \leq 0 .$$

Since M^{-1} is non negative we may multiply both sides by M^{-1} without distorting the inequalities implied to get

$$X - X_R < 0 \qquad \text{or}$$

$$(3) \qquad X \leq X_R \quad \text{for any feasible } X .$$

Since $C \geq 0$ we get $C^T X \leq C X_R^T$ and hence no feasible solution may have higher value for Z than X_R .

It can be shown that any non degenerate cone contains an integral feasible point.

We will outline now a method for deriving the optimal -integral solution. Define the following sequence:

$$(4) \qquad X_0 = [X_R]$$

$$X_{n+1} = [D^{-1}(b+AX_n)]$$

Where [ ] is the truncation operation i.e. taking the largest integer not larger than the argument, applied to each component. D and A are the positive decomposition of M = D-A into diagonal and off diagonal matrices. Clearly $X_0 \leq X_R$

From which by multiplications of positive matrices and additions we get $D^{-1}(b+AX_0) \leq D^{-1}(b+AX_R) = X_R$

Truncating both sides we get

$$X_1 = [D^{-1}(b+AX_0)] < [X_R] = X_0 \qquad \text{or}$$

$$X_1 \leq X_0$$

Assuming inductively that $X_n \leq X_{n-1}$

We prove similarly that

$$[D^{-1}(b+AX_n)] < [D^{-1}(b+AX_{n\ 1})] \qquad \text{or}$$

$$(5) \qquad X_{n+1} \leq X_n \qquad \text{which holds therefore for all } n \text{ .}$$

Let Y_0 be any integral-feasible point.

By (3) it follows that $Y_0 \leq X_R$ and since Y_0 is integral also

$$Y_0 \leq X_0$$

Since Y_0 is feasible $Y_0 < D^{-1}(b+AY_0)$ Furthermore it is integral

$$Y_0 \leq [D^{-1}(b+AY_0)] \ (6)$$

Apply the operator $T(X) = [D^{-1}(b+AX)])$ to both sides of

$Y_0 \leq X_0$ and then use (6) to get

$Y_0 \leq X_1$

which can be inductively extended to

(7) $Y_0 < X_n$.

We proved our sequence X_n to be non increasing in all components and bounded below by any integral feasible point Y_0. There must exist therefore n such that $X_{n+1} = X_n$ which implies

$$X_n = [D^{-1}(b+AX_n)] < D^{-1}(b+AX_n)$$

Le. that X_n is a feasible integral point,

Furthermore it is not lower in any component than any other integral feasible point and hence is an optimal integral solution.

We summarize this chain of arguments in the following:

Theorem1

The sequence defined by (4) converges in a finite number of steps to an optimal integral solution of any bounding cone program.,

3. General Applicability of the Method

The contents of this chapter is a theorem showing that any cone program possessing a bounded rational solution can be transformed to a special bounding form and solved by the above suggested method,

When transforming integral programs one must be careful to preserve integrality of the feasible points, The only permissible transformations are change-of-variable transformations (column operations on the matrix M and the row C^T) which map integral points on integral points and do not map any non integral point on an integral point.

These are unimodular transformations and may be built up of the following elementary operations:

a) Reverse the sign of a column,

b) Add to a column a multiple (integral in our case) of another column.

Let us indicate a sequence of increasingly complex procedures that can be executed using these twc operations.

c) Interchange two columns.

Given two columns U and V, the following sequence interchanges their contents.

```
U:= U+V;
V:= V-U;
V:= -V;
U:= U-V;
```

d) Given two elements in a given row, use column operation on the two corresponding columns which result in a 0 being placed in the first elements location and the greatest common divisor of the two elements in the second location, (If both are initially 0 leave them that way).

We start by sign modifications to make both elements positive. By successive subtractions of the smaller from the bigger we cause the numbers to decrease until one or the other becomes 0 . If this happens to the second element, interchange columns. The remaining non zero element is the G.C.D.

e) Given a row segment by a specified row i and last column $k \leq n$, transform it to a row containing 0 in locations $M_{i,1}$ to $M_{i,k-1}$

and the G.C.D. of this segment (if not zero) at location $M_{i,k}$.

Apply procedure d) to the first two elements producing 0 in the first column and the G.C.D. of the pair in the second column.

Repeat successively between 2nd and 3rd, 3rd and 4th, etc., until after applying it to the k-1 and the k^{th} columns, the row segment attains the desired form.

f) Given a square subma-trix by specifying k as the last row and column to be included, transform it to an upper triangular matrix 'having positive diagonal and non positive super-diagonal elements. (Assume the submatrix to be non singular).

Apply procedure e) to the last row segment for columns 1 to k. producing zeroes everywhere except for $M_{k,k}$ which gets a positive value.

Apply procedure e) again to the k-1 st row for columns 1 to k-1. Since the k^{th} row contains non zero element in the k^{th} column only, it will not be changed. Raving produced zeroes in columns 1 to k-2 and a positive number in column k-1, we proceed to subtract that column sufficiently many times from column kto make $M_{k-1,k} \leq 0$.

Similar application to the previous rows finally produce the desired form.

g) Given a square submatrix of order k , transform the whole matrix to a form in which the $k+1^{st}$ column has negative entries in all of the first k rows,

Transform first the submatrix to an upper triangular form by prodcedure f). Subtract a large multiple of the k^{th} column from the $k+1^{st}$ to make $M_{k,k+1} \leq 0$. Subtract now a large multiple of the $k\text{-}1^{st}$

column from the k+1st column to make $M_{k-1,k+1} \leq 0$. Notice that since $M_{k,k-1} = 0$ we did not destroy the negativity property of $M_{k,k+1}$. Proceeding in this fashion we finally subtract the 1 - element first column from the k+1st column to make $M_{1,k+1} \leq 0$, not disturbing the rest of this column,

Before we proceed to procedure h) which effects the complete transformation we derive a necessary result,

We assumed our cone problem to have a bounded rational. solution. 'The only one possible is $X_R = M^{-1}b$.

In order for it to be optimal it is necessary that for any other feasible solution:

$$MX \leq b \Rightarrow C^T X \leq C^T X_R$$

or

$$(8) \qquad MY \leq 0 \Rightarrow C^T Y \leq 0 \qquad \text{if we let } X-X_R = Y .$$

by Farkas Lemma [1] it follows that

$$(9) \qquad C^T = \beta M$$

where β is a non negative row vector.

Now we are ready to outline the final procedure.

h) Given a cone problem which is rationally solvable, it can be transformed to a bounding cone problem of the following special type: (to be called Pl form).

$$C = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ C_n \end{bmatrix} \quad C_n > 0 \qquad M = \begin{bmatrix} E & F \\ & \\ 0 & G \end{bmatrix}$$

$$F < 0$$

E is an upper triangular bounding matrix, i.e.,

$$e_{ii} > 0 \quad e_{ij} \leq 0 \quad \text{for} \quad i<j$$

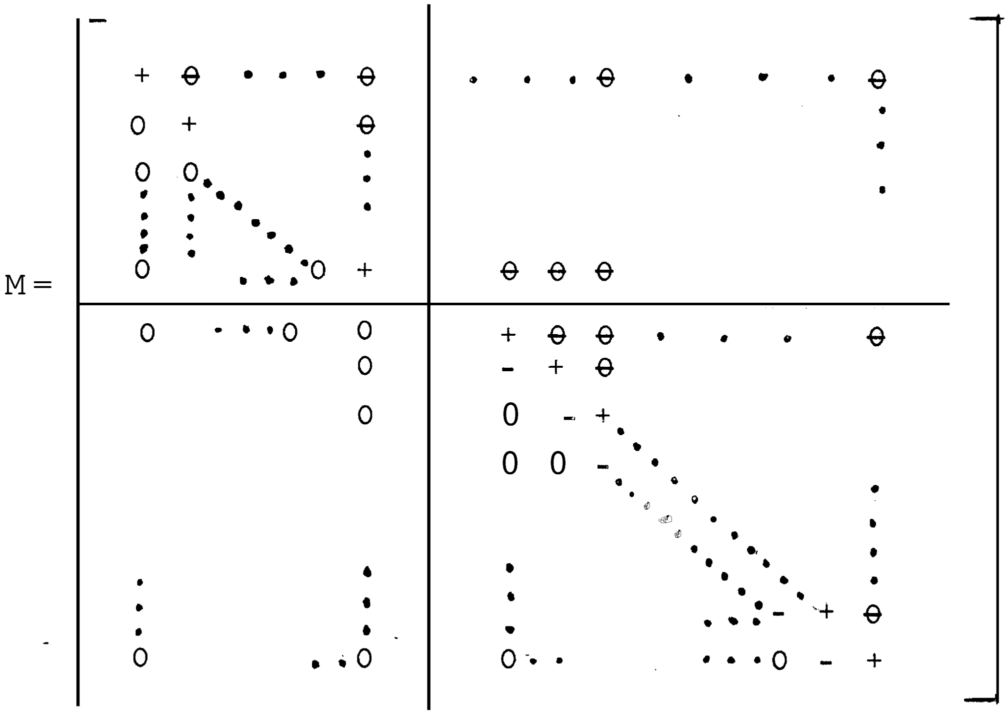
$$e_{ij} = 0 \text{ for } i > j$$

G is an upper sub-triangular bounding matrix with negative (non zero) elements on the first lower sub diagonal,

i.e. $g_{ii} > 0 \quad g_{ij} \leq 0$ for $i < j$

$g_{i+1,i} < 0$

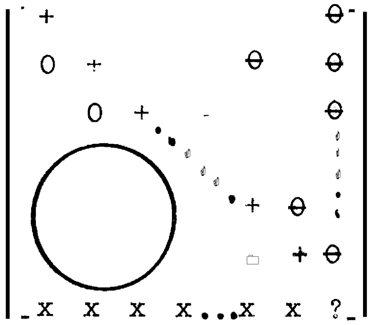
$g_{ij} = 0$ for $i > j+1$



θ stands for a non positive element.

For convenience we place the C^T row as an additional n+1st row of the matrix M.

Start by applying procedure 3) to row n+1 up to column n , getting C^T into the above indicated form, Apply now procedure g) to the submatrix of order n-1 making the first n-1 entries of the n^{th} column non positive, (One can always find a non singular (n-1) x (n-1) submatrix among the first n-1 columns), The matrix at this stage looks as shown:



$$C^T = (0, 0 \ldots \quad 0 \ 0 \ +)$$

The " ?" stands for $M_{n,n}$ whose sign is unknown, Consider the fact that a non negative combination of the n rows should result in the last C^T row. This could be true for the last column only if $M_{n,n} > 0$ and the n^{th} row actively participates in that combination i.e. $\beta_n > 0$.

2). Inspect now row n , columns 1 to n-1 (where we drew X to signify we do not have any information about their sign pattern). If all entries are 0 the matrix has been brought to Pl form with G of order 0. (M then consists of E only).

If any non zero entry is present, apply procedure e) to this row segment. All entries will be 0 now except $M_{n,n-1}$ which is positive. Reverse this column sign to make $M_{n,n-1} < 0$.

Repeat now and apply procedure g) to the submatrix of order n-2, making all elements of column n-1 and rows 1 to n-2 non positive. The matrix now has the following form

$$\begin{bmatrix} + & \ominus & & \ominus & \ominus \\ 0 & + & & \ominus & \ominus \\ & \ddots & \ominus & \vdots & \vdots \\ & & \ddots\; + & \ominus & \ominus \\ x & x\cdots & \cdots x & ? & \ominus \\ 0\cdots & & 0 & - & + \end{bmatrix}$$

$$C^T = (0, \ldots\ldots\ldots\ldots 0 \; +)$$

Inspecting column n-1 we find non positive entries in rows 1 to n-2, a negative entry which participates actively in the non negative combination, and 0 as the value of that combination. It must therefore be that $M_{n-1,n-1} > 0$ and row n-1 also participates actively in the combination.

Inspect now row n-1 (where the Xs are). If all entries are 0 we are done. Otherwise go back to step 2) with row n-1 and shorter segment 1 to n-2.

Proceeding in this way we either stop because such a row segment has all zero entries, at which stage we have a non trivial E, or we carry this process through up to row 1, in which case E is of dimension 0 and G is all of M .

Observe that all rows of G participate actively in the positive combination, A complete Algol program effecting this transformation is attached as an appendix.,

We have thus shown that any cone problem can be brought to Pl form* In order to complete the proof of the theorem we have to show that

$$M^{-1} > 0 \text{ . for } M \text{ in the Pl form.}$$

We start by showing that $G^{-1} \geq 0$. Let G be of order ℓ .

From the observation about all rows of G actively participating in the Farkas combination we conclude the existence of

$$\alpha = (\alpha_1, \ldots \alpha_\ell) \qquad \text{Such that } \alpha_i > 0$$

$$Z_j = \sum_{i=1}^{\ell} \alpha_i g_{ij} \geq 0 \text{ for } j=1,\ldots \ell-1$$

and

$$Z_\ell = \sum_{i=1}^{\ell} \alpha_i g_{i\ell} \geq C_n > 0$$

Let now $U = (U_1,\ldots U_\ell)$ be any row vector such that $UG > 0$.

Consider the sequence U_i/α_i and let k be the largest index such that $U_k/\alpha_k \leq U_i/\alpha_i$ for $i=1, 0.e\ \ell$.

Consider the k^{th} inequality of $UG > 0$, as before we can derive

$$0 \leq \sum_{i=1}^{\ell} U_i g_{ik} = \sum_{i=1}^{\ell} \left(\frac{U_i}{\alpha_i} \right) (\alpha_i g_{ik}) \leq \left(\frac{U_k}{\alpha_k} \right) \sum_{i=1}^{\ell} (\alpha_i g_{ik}) = \frac{U_k}{\alpha_k} \cdot Z_k$$

If $k = \ell$ then $Z_\ell > 0$ and we conclude that $U_k/\alpha_k \geq 0$.

If $k \neq \ell$ and the inequalities on the way were strict inequalities we have $0 < \frac{U_k}{\alpha_k} \cdot Z_k \Rightarrow \frac{U_k}{\alpha_k} > 0$.

In order for the inequalities to become equalities it is necessary that all $\frac{U_i}{\alpha_i}$ $i \neq k$, for which $g_{ik} \neq 0$ be equal to $\frac{U_k}{\alpha_k}$. But since $g_{k+1,k} \neq 0$ it follows that $\frac{U_{k+1}}{\alpha_{k+1}} = \frac{U_k}{\alpha_k}$ which violates

our requirement that k be the largest index for which U_k/α_k is minimal. Hence as proven in Lemma 1

$$(UG \geq 0 \Rightarrow U \geq 0) \Rightarrow G^{-1} \geq 0 .$$

$$M^{-1} = \begin{bmatrix} E & F^{-1} \\ 0 & G \end{bmatrix} = \begin{bmatrix} E^{-1} & -E^{-1}FG^{-1} \\ 0 & G^{-1} \end{bmatrix} > 0$$

E^{-1} being the inverse of a triangular matrix with positive diagonal and non positive superdiagonal elements is positive. $-F \geq 0$ and so is its product with positive matrices $-E^{-1}FG^{-1} \geq 0$:

We summarize these results in the following:

Theorem 2

Any non singular, rationally solvable cone problem can be transformed to a Pl bounding form,

4. Extensions

The natural extension to a bounding cone is a bounding program which. may be defined as follows:

$$\text{Maximize } Z = C^T X$$

Subject to $MX \leq b$ C, M, b, X integers, M completely non degenerate.

With the conditions:

a) $C \geq 0$

The rows indices may be partitioned into exclusive non empty sets I_i $i=1,\ldots n$ such that

b) For $j \in I_i$ $M_{jk} \leq 0$ $k \# i$

$M_{ji} > o$

c) There exists a row α $\alpha \geq 0$ such-that $\alpha M > 0$. In other words there should be a basic bounding cone and additional constraints, each having exactly one positive coefficient.

The iterational process is defined now as

$$X_0 = [X_R]$$

$$(8) \qquad X^i_{n+1} = \min \left[\frac{b_j - \sum_{k \neq i} M_{jk} X^i_n}{M_{ji}} \right]$$

By using this sequence one may convert all proofs for the cone case to proofs for the more general case, thus showing the iterational process to converge provided an integral feasible point exists.

Existence of a feasible integral point is assured in the case of a bounding non degenerate cone.

In the case of a more general bounding program the following procedure is suggested:

1) Look first for a bounded rational solution to the minimum program $\quad$ Minimize $Z = C^T X$ $\quad$ S/T $\quad$ $MX \leq b$

If no bounded solution exists and the program is completely non degenerate, an integral feasible point must exist and hence the iterational process must converge, If a bounded rational solution exists, record its value. At each generation of Xn compare $C^T X_n$ against the minimal value. If it gets beiow the minimal value, no feasible integral point exists.

It is thus ascertained that in a finite number of steps one either proves the non existence of integral solution or finds it.

Unfortunately no general procedure is known which will transform a general problem to a bounding form except for the case of n = 2.

Another possible extension is using the bounding form to derive a suboptimal solution rather than an optimal one. We use again a sequence defined in the same way as (7) or (8). The only difference is in taking as X_0 any feasible integer point, This sequence is guaranteed to converge to a suboptimal solution with objective functions value Z not lower than the one for X_0 $Z_0 = C^T X_0$.

In order to get a complete integer programming algorithm one may use the following scheme:

1) Identify the active cone (the one which tightly bounds the rational optimal solution) and solve it.

2) If some other constraints are still unsatisfied add to the set of constraints the n new constraints

   $X_i \leq h_i \cdot i=1,\ldots n.$

   Where h is the integral solution of the currently solved cone!.

Return to step 1.

5. P2 Form

During computation it has been observed that the matrices obtained in form P1 are not in general diagonal dominant. Since diagonal dominance may enhance convergence, we show now how any bounding cone can be brought to a P2 form, which is also a bounding form with the additional requirement that

$$M_{ii} > |M_{ij}| \qquad j \neq i \ .$$

This transformation is accomplished by column additions only 'and hence preserve condition c) validity.

Assume M to be in bounding form and let α be the row vector, $\alpha > 0$ such that

$$\alpha M = \beta > 0$$

Suppose there are a row i and a column j such that

$$M_{ii} \leq |M_{ij}| \ i \neq j \ .$$

We apply then a basic step of adding the i^{th} column to the j^{th} column. 'In order to show that this does not destroy the bounding form of M , we focus our attention on the principal minor formed out of rows and columns i and j. (Without loss of generality assume i < j).

$$\begin{array}{ccccc} & \vdots & & \vdots & \\ \cdots & M_{ii} & \cdots & M_{ij} & \cdots \\ & \vdots & & \vdots & \\ \cdots & M_{ji} & \cdots & M_{jj} & \cdots \\ & \vdots & & \vdots & \end{array}$$

Because any other elements in these columns are non positive it follows that

$$\alpha_1 M_{11} + \alpha_j M_{ji} = \delta_i > 0$$

$$\alpha_i M_{ij} + \alpha_j M_{jj} = 6 j > 0$$

Since α_i, α_j are positive and the sign pattern of the minor is the bounding one, one considers the solution of this 2 X 2 system and infers that

$$M_{ii}M_{jj} - M_{ij}M_{ji} > 0$$

If now $M_{ii} \leq |M_{ij}|$ it must follow that $M_{jj} > |M_{ji}|$. Adding the i^{th} column to the j^{th} column is going therefore to leave M_{ij} non positive, M_{jj} positive and all other element in the altered column non positive.

To show that this process of repeated column additions must terminate in a finite number of steps observe the following:

Adding column i to column j transforms matrix M to M' and accordingly $\alpha M' = \beta' > 0$ where quite obviously

$$\beta_j' = \beta_j + \beta_i$$

Hence

$$\sum_k \beta_k' > \sum_k \beta_k$$

This implies that the sum of all the matrix elements weighted by positive weight vector α has increased. This transformation, however, never increases any positive elements value, and must therefore decrease the weighted sum of the negative elements in absolute magnitude. This sum should decrease by at least 1 (assume α_i to be integral) at each such step. Since it cannot decrease below 0 the process must terminate; termination means that no i and j exist such that $M_{ii} \leq |M_{ij}|$ $i \neq j$, and P2 form has been attained.

6. Numerical Example

Trying different numerical examples we adopted the following computational procedure.

1) Accept a general program (larger than a cone).

2) Solve first to get a rational minimal solution for bounding Z from below,

3) Solve next to get a rational maximal solution - XR . This process (using dual simplex method) singles out a basic cone of tight or active constraints.

4) Transform this cone to P1 and later to P2 bounding form, modifying XR on the way to be expressed in the new variables.

5) Search among the rest of the transformed program for additional constraints that are in bounding form and add them to form an extended cone problem.

6) Solve the extended cone problem by the iterational method.

7) Check if the optimal solution attained satisfies the non-participating constraints, If it does not - print an error message. Return to step 1 to try the next problem.

In most of the cases tried, no error message was printed, which means that the solution to the active cone is also a solution to the complete program.

As an illustrative numeric&l. example we chose the following fixed-charge problem [2].

$$\text{Maximize } Z = X_3 + X_4 + x_5$$

S/T

$$\begin{array}{rcl}
 & -X_5 & \leq 0 \\
-8X_2 & + x_4 & < 0 \\
-6X_1 & + x_3 & \leq 0 \\
3X_1 + 2X_2 + 2X_3 + X_4 + 2X_5 & & < 25 \\
2X_1 + 3X_2 + X_3 + 2x4 + 2X_5 & & \leq 19 \\
- X_1 & & \leq 0 \\
- X_2 & & \leq 0 \\
- x_3 & & < 0 \\
- X_4 & & < 0
\end{array}$$

The constraints have been ordered in such a way that the first five are the tight constraints and yield the following **rational** solution

$$Z_R = 11 \frac{27}{41}$$

$$X_1 = 1\frac{16}{41} \quad X_2 = \frac{17}{41} \quad X_3 = 8\frac{14}{41} \quad X_4 = 3\frac{13}{41} \quad X_5 = 0 \quad .$$

Under transformation to new variables we get the following rule of substitution:

$$\begin{array}{l}
X_1 = 1 + y_1 \qquad y_3 + y_4 - y_5 \\
X_2 = y_1 - y2 \\
X_3 = 3 + 5y1 - 3y_2 - 2{;}3 + 3y_4 - 7y_5 \\
X_4 = -5 + 5y_1 - 2y_2 - 2y3 - 5y_4 - y_5 \\
X_5 = 9 - 10y_1 + 5y_2 + 4y_3 + 2y_4 + 9y_5
\end{array}$$

$$Z = 7 + y_5$$

Where the y_i are constrained to satisfy:

1. $10y_1 - 5y_2 - 4y_3 - 2y_4 - 9y_5 \leq 9$

2. $-3y_1 + 6y_2 - 2y_3 - 5y_4 - Y5 \leq 5$

3. $-y_1 - 3y_2 + 4y_3 - 3y_4 - y_5 \leq 3$

4. $y_3 + 8y_4 \leq 3$

5. $y_4 + 7y_5 \leq 6$

6. $-y_1 + y_2 < 0$

7. $-y_1 + y_3 - y_4 + y_5 \leq 1$

8. $-5y_1 + 3y_2 + 2y_3 - 3y_4 + 7y_5 \leq 3$

9. $-5y_1 + 2y_2 + 2Y_3 + 5y_4 + y_5$ f-5

Notice that the first 5 constraints are in the P2 form, i.e., positive diagonal, non positive off diagonal, and diagonal greater in absolute value than any other element in its row. Also note that inadvertently constraint 6 is also of bounding form and should be added to the five when generating the sequence.

The initial solution achieved by suitably truncating the transformed rational solution is listed below together with the iterations:

Initial	1st	2nd	3rd	4	5	6	7	8	9	
169	168	167	166	165	164	163	162	162	162	y_1
169	168	167	166	165	164	163	162	162	162	y_2
189	188	187	186	185	184	183	182	181	181	y_3
24	24	23	23	23	23	23	23	23	23	y_4
4	4	4	4	4	4	4	4	4	4	y_5

Substituting these final values of y into the expressions for x we get the final solution as

$$X_1 = 1, \quad X_2 = 0, \quad x_3 = 6, \quad X_4 = 0, \quad X_5 = 5$$

and the optimal value is $Z = 11$.

It was verified of course that this cone solution satisfies also the other constraints as well,

7. Relation to Previous Work

Even though independently derived, the suggested method is in some respects similar to F. Glovers "Bound Escalation Method" [3]. Once a bounding form is achieved, the process of attaining the optimal integral solution to this partial problem is quite smilar.

The main differences are in justification of the iterational method and in the proof of its convergence.

While the bound escalation method strongly relies on the assumption that all variables are restricted to be positive, and hence is restricted in the range of admissible transformations for creation of a bounding form, our method has no such limitation. Convergence is proved without any positivity assumptions,,

The-possibility of simultaneously transforming a complete cone into either P1 or P2 bounding form is considered and proved here for the first time.

APPENDIXA

Following are two ALGOL-60 procedures which transform solvable cone problems into P1 and P2 forms respectively.

```
        PROCEDURE P1(M,n,m ); VALUE m,n;
INTEGER ARRAY M; INTEGER n,m;
        BEGIN COMMENT M is the matrix to be transformed.  Assume column
              n+1 to contain the negated right hand side of the inequali-
              ties, row n+1 contains the objective function coefficients
              and M n+1,n+1 = 0.  m is the total number of constraints to
              be transformed together with the cone. m > n+1;
              INTEGER b,e,q;


              PROCEDURE NEGATE(a); VALUE a;
              INTEGER a;
              BEGIN COMMENT Reverse sign of column a;
                    FOR b:=1 STEP 1 UNTIL m DO
                    M[b,a]:= -M[b,a]
              END;
              PROCEDURE MULADD(c,d,p); VALUE c,d,p;
              INTEGER c,d,p;
              BEGIN COMMENT Multiply column c by p and add to column d;
                    FOR b:=1 STEP 1 UNTIL m DO
                    M[b,d,]:= M[b,d] + p × M[b,c]
              END;
              PROCEDURE XG(c,d); VALUE c,d;
              INTEGER c,d;
```

```
BEGIN COMMENT Interchange contents of columns c and d;
      INTEGER t;
      FOR b:=1 STEP 1 UNTIL m DO
BEGIN     M[b,c]; M[b,c]:= M[b,d]; M[b,d]:=t
END
END;
PROCEDURE GCD1 (a,c,d); VALUE a,c,d;
INTEGER a,c,d;
BEGIN COMMENT Set M[a,c] to 0 and M[a,d] to the G.C.D.
      of the two elements;
      IF M[a,c] < 0 THEN NEGATE(c);
      IF M[a,d] < 0 THEN NEGATE(d);
    L:IF M[a,c] ≠ 0 THEN
      BEGIN IF M[a,d] = 0 THEN XG(c,d) ELSE
            BEG? IF M[a,c] < M[a,d] THEN
                  MULADD(C,d,-M[a,d] ÷ M[a,c]) ELSE
                  MULADD(d,c,-M[a,c] ÷ M[a,d]);
                  GO TO L
            END
      END
END;
PROCEDURE GCD2(a,c); VALUE a,c;
INTEGER a,c;
BEGIN COMMENT SET M[a,1], M[a,2], . . . UNTIL M[a,c-1] TO 0
      and M[a,c] to the G.C.D. of the two elements;
      FOR e:=1 STEP-1 UNTIL    D O
          GCD1 (a,e,e+1);
      IF M[a,c] < 0 THEN NEGATE(c)
END;
PROCEDURE TRIANG(k); VALUE k;
INTEGER k;
BEGIN COMMENT  Transform submatrix of order k into upper tri-
    angular bounding form.  Use if necessary  $k+1^{st}$ row for
    interchanges;
    INTEGER ERC,i,j,t     ;
    ERC:=0;
    FOR i:=k STEP -1 UNTIL 1 DO
    BEGIN I2: GCD2(i,i);
        IF M[i,i]=0 THEN
        BEGIN IF ERC ≠ 0 THEN GO TO ERROR1;
            COMMENT ERROR1 is an error exit for cases
                 of singular matrix;
            ERC:= 1;
            FOR j:=1 STEP 1 UNTIL n+1 DO
            BEGIN t:= M[k+1,j];
                M[k+1,j]:= M[i,j];
                M[i,j]:= t
            END;
            COMMENT in case of temporary singularity
                 exchange row with $k+1^{st}$ row;
            GO TO I2
        END;
        FOR j:=i+1 STEP 1 UNTIL k+1 DO
        MULADD(i,j,-(M[i,j] + M[i,i]-1) ÷ M[i,i]);
```

```
          COMMENT make jth column negative by subtracting
                ith column. Note that j ranges until
                k+1;
      END FORi;
  END;
  COMMENT main body of procedure;
  GCD2  (n+1,n);
  FOR q:= n STEP -1 UNTIL 2 DO
  BEGIN TRIANG(q-1);
      GCD(q,q-1);
     IF M[q,q-1]=0 THEN GO TO LEND;
     NEGATE(q-1)
  END;
  LEND:
END;
  PROCEDURE P2(M,n,m); VALUE m,n;
  INTEGER ARRAY M; INTEGER m,n;
  BEGIN COMMENT transform matrix Mto form P2;
      INTEGER i,j,k,p;
      P1(M,n,m);
    LB:FOR i:= 1 STEP1 UNTIL n DO
      FOR j:= 1 STEP 1 UNTIL n+1 DO
      IF (i# j) A (abs(M[i,j]) ≥ M[i,i]) THEN
      BEGIN p:= abs(M[i,j]) ÷ M[i,i];
```

```
          FOR k:=1 STEP 1 UNTIL m DO
          M[k,j]:= M[k,j] + p X M[k,i];
          GO TO LB
      END
END
```

No attempt has been made to code these procedures efficiently,

ACKNOWLEDGEMENTS

I wish to thank the Computer Science Department *at* Stanford University for making my stay there hospitably possible, and for placing at my disposal computing facilities which enabled me to test and verify this work,

I also want to express deep gratitude to Dr. Gordon H. Bradely for initiating me into the field of integer programming, I owe him the benefit of illuminating discussions and constructive criticsm without which this work could not be made possible,

June 14,1968

Weizmann Institute of Science, Israel

On a year's visit with Stanford University.

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