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COLLECTIVELY COMPACT OPERATOR APPROXIMATIONS

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Lectures presented July - August 1967

Notes prepared by

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Abstract

A general approximation theory for linear and nonlinear operators on Banach spaces is presented. It is applied to numerical integration approximations of integral operators. Convergence of the operator approximations is pointwise rather than uniform on bounded sets, which is assumed in other theories. The operator perturbations form a collectively compact set, i.e., they map each bounded set into a single compact set. In the nonlinear case, Frkchet differentiability conditions are also imposed. Principal results include convergence and error bounds for approximate solutions and, for linear operators, results on spectral approximations.

Chapter I

APPROXIMATE SOLUTIONS OF EQUATIONS

1. Introduction

Consider a Fredholm integral equation of the second kind

$$\lambda\, x(s) - \int_0^1 k(s,t)\, x(t)\, dt = y(s), \quad 0 \leq s \leq 1 \tag{1.1}$$

where x(s), y(s), and k(s,t) are continuous, real or complex valued functions for $0 < s, t < 1$ and $\lambda \neq 0$.

In a classical method of approximate solution based on numerical integration, the integral in (1.1) is replaced by a summation to obtain

$$\lambda\, x_n(s) - \sum_{j=1}^{n} w_{nj} k(s,t_{nj}) x_n(t_{nj}) = y(s), \quad 0 \leq s \leq 1. \tag{1.2}$$

If we replace the free variable by subdivision points we get a finite linear algebraic system

$$\lambda\, x_n(t_{ni}) - \sum_{j=1}^{n} w_{nj} k(t_{ni},t_{nj}) x_n(t_{nj}) = y(t_{ni}), \quad i = 1,2,\dots,n. \tag{1.3}$$

The two equations (1.2) and (1.3) are effectively equivalent. Certainly if $x_n(s)$ satisfies (1.2) then the $x_n(t_{ni})$, $i = 1,2,\dots,n$, satisfy (1.3). Conversely, if (1.3) is satisfied, then (1.2) determines $x_n(s)$ explicitly in terms of the $x_n(t_{ni})$ -- in effect, (1.3) serves as an interpolation formula.

The technique of replacing an integral equation by a finite system goes at least back to Fredholm [26]. Hilbert [28] gave convergence proofs for approximate solutions using the rectangular quadrature formula. The idea of using (1.2) goes at least back to Nyström [35] in the mid 1920's. The advantage is that equations (1.1) and (1.2) are defined in the same space. In particular cases this technique has been studied by Bückner [24a, 24b], Kantorovitch [29], Mysovskih [30, 31, 32], Wielandt [37], and Brakhage [22, 23]. In all cases rather particular assumptions were made.

Basic problems to be considered are the solvability of the equations, convergence of the solutions, error bounds, and the eigenvalue problems associated with these operators. Also to be considered are integral equations with discontinuous and even singular kernels, problems in higher dimensions, unbounded domains and nonlinear integral equations.

Applications can be made in the field of radiative transfer. The transport equation is an integrodifferential equation which yields a system of ordinary differential equations when the integral is replaced by a sum. This problem can be cast so that it can be treated by the theory to be presented here (see [5, 7, 34]).

An abstract theory concerning equations in a Banach space will be presented. Thus all the applications will simply be special cases of the general theory. The abstract theory has been developed within the last four or five years. Other principal contributors to the theory have been R. H. Moore, T. W. Palmer and K. E. Atkinson.

2. Banach Space Fundamentals

Let X denote the Banach space of real or complex continuous functions x(t), $0 \leq t \leq 1$, with the uniform norm $\|x\| = \max|x(t)|$. The unit ball will be denoted by $\mathcal{B}$. Thus,

$$\mathcal{B} = \{x \in X: \|x\| \leq 1) \quad . \tag{2.1}$$

The symbol [X] will represent the space of bounded linear operators on X into X with the usual operator norm, $\|K\| = \sup_{x \in \mathcal{B}} \|Kx\|$ for $K \in [X]$.

Lemma 2.1 Let K and K_n be elements of [X]. Then $\|K_n - K\| \to o$ iff $\|K_n x - Kx\| \to 0$ for all $x \in X$ uniformly for $x \in \mathcal{B}$ (or any bounded set).

In the general theory as well as the applications to be discussed, the convergence of the operator approximations is only pointwise and not uniform in the operator norm. Nevertheless, for the sake of motivation, suppose we have convergence in the operator norm, that is, $\|K_n - K\| \to 0$ as $n \to \infty$. Then we have the following lemma.

Lemma 2.2 Let $\|K_n - K\| \to o$. Then there exists $(\lambda - K)^{-1} \in [X]$ iff for all n sufficiently large there exist $(\lambda - K_n)^{-1} \in [X]$ bounded uniformly in n.

In either case,

$$\|(\lambda - K_n)^{-1} - (\lambda - K)^{-1}\| \to 0 \text{ as } n \to \infty \ . \tag{2.2}$$

A constructive proof of lemma 2.2 also provides error bounds. In order to prove the lemma however, let us first state two further results.

Lemma 2.3 Let $T \in [X]$ and $\|T\| < 1$. Then there exists $(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$ t[X] and

$$\|(I-T)^{-1}\| \leq 1/(1-\|T\|). \tag{2.3}$$

This is the well known Neumann series for $(I-T)^{-1}$.

Lemma 2.4 Assume there exists $T^{-1} \in [X]$ and $\Delta = \|T^{-1}\| \cdot \|T-S\| < 1$. Then there exists $S^{-1} \in [X]$ with

$$\|S^{-1}\| \leq \frac{\|T^{-1}\|}{1-\Delta} \tag{2.4}$$

and furthermore --.

$$\|S^{-1}-T^{-1}\| \leq \frac{\|T^{-1}\|^2 \cdot \|T-S\|}{1-\Delta} . \tag{2.5}$$

To prove this we can write (since T^{-1} exists)

$$S = T-(T-S) = T[I-T^{-1}(T-S)] . \tag{2.6}$$

Hence, there exists

$$S^{-1} = [I-T^{-1}(T-S)]^{-1} T^{-1} , \tag{2.7}$$

and an application of Lemma 2.3 yields (2.4).

To prove (2.5) we write

$$S^{-1}-T^{-1} = S^{-1}(T-S)T^{-1} . \tag{2.8}$$

Taking the norm of (2.8) and using (2.4) gives (2.5) immediately.

Now we can easily show (2.2) and give the error bound by using (2.5) with $T = (h-K)$ and $S = (\lambda-K_n)$. We obtain

$$\|(\lambda-K_n)^{-1}-(\lambda-K)^{-1}\| \leq \frac{\|(\lambda-K)^{-1}\|^2 \, \|K_n-K\|}{1-\Delta_n} , \tag{2.9}$$

where

$$\Delta_n = \|(\lambda-K)^{-1}\| . \|K_n-K\| . \tag{2.10}$$

Since' $\|K_n-K\| \to 0$ and $\|(\lambda-K)^{-1}\|$ is bounded, (2.2) follows.

Notation

The notation $K_n \to K$ will be used for pointwise convergence. That is, $K_n \to K$ iff $\|K_n x-Kx\| \to 0$ for all $x \in X$.

Lemma 2.5 If $K_n \to K$ for bounded linear operators K_n and K, then the sequence $\{K_n\}$ is bounded. That is, there exists a bound, $b < \infty$, such that $\|K_n\| \leq b$ for all n.

This is an application of the principle of uniform boundedness.

Lemma 2.6 If $K_n \to K$, then the convergence is uniform on each compact set in X.

Before proving Lemma 2.6 a brief discussion relating the concepts of compact, sequentially compact and totally bounded is in order. A compact set is a set such that any open cover has a finite subcover. A set S is sequentially compact if, given any infinite sequence taken from S, it has a convergent subsequence (the limit may or may not be in S). A set S is totally bounded iff for each $\epsilon > 0$ there exists a finite set (an e-net), $x_1,\ldots,x_m$, such that for any $x \in S$, $\min_{1 \leq i \leq m}\|x-x_i\| < \epsilon$. These three concepts are closely related. First, in a complete metric space, sequentially compact and totally bounded are the same. Secondly, compactness is the same as either one or the other concepts if the set **S** is also closed. And thirdly, a set S is totally bounded (or sequentially compact) iff the closure of S is compact. The third concept, total boundedness, is the one to be used in the following discussion.

Lemma 2.6 can be stated for either compact or totally bounded sets and the two statements are equivalent. The proof is now given for totally bounded sets.

Let $T_n = K_n - K \to 0$ and let S be a totally bounded set. Since we maymultiply by a scalar we may, without loss of generality, assume $\|T_n\| \le 1$ for all n. Fix $\epsilon > 0$. Then there exists a finite e-net, $x_1, \ldots, x_m$, such that $\min_{1 \le i \le m} \|x - x_i\| < \epsilon$ for any $x \in S$. Since $\|T_n\| \to 0$ pointwise, there exists N such that $\|T_n x_i\| < \epsilon$ for $i = 1, \ldots, m$ and $n > N$. Hence, by the triangle inequality we have

$$\|T_n x\| \le \|T_n x_i\| + \|T_n (x - x_i)\| < 2\epsilon \tag{2.11}$$

which proves Lemma 2.6.

Lemma 2.6 is a special case of a more general proposition. Here we had pointwise convergence of uniformly bounded operators. Recall that an operator is bounded iff it is continuous. Similarly, a set of operators is uniformly bounded iff it is an equicontinuous set of functions. In much the same fashion it can be proved that pointwise convergence of equicontinuous functions from one metric space to another is always uniform on totally bounded sets.

In the approximation theory and applications to follow, results will be obtained for pointwise convergence which are quite analogous to those which hold for operator norm convergence.

3. Collectively Compact Sets of Operators

The integral operators to be used are compact (or completely continuous). Some definitions and theorems concerning this class of operators will now be given.

Definition An operator $K \in [X]$ is compact iff K maps $\mathcal{B}$, the unit ball, into a totally bounded set; equivalently, the closure of $K\mathcal{B}$ is compact.

Definition A set of operators, $\mathcal{K} \subset [X]$, is collectively compact iff the set $\mathcal{K}\mathcal{B} = \{Kx: K \in \mathcal{K}, x \in \mathcal{B}\}$ is totally bounded (or has compact closure).

It will be shown later that approximations to integral operators defined by sums form a collectively compact sequence.

Theorem 3.1 Let $T_n, T \in [X]$ and $T_n \to T$. Then, for each compact operator K,

$$\|(T_n - T)K\| \to 0 . \tag{3.1}$$

Moreover, the convergence is uniform for $K \in \mathcal{K}$, where $\mathcal{K}$ is any collectively compact set.

To prove this, recall that convergence in norm is the same as pointwise convergence uniformly for $x \in \mathcal{B}$. Consider

$$(T_n - T)Kx \to 0 . \tag{3.2}$$

If x is some point in $\mathcal{B}$ and K is either fixed or is some element of a collectively compact set, then the argument, Kx, is some element of the set $\mathcal{KB}$ which is totally bounded by definition. But, pointwise convergence is always uniform on a totally bounded set so in (3.2) the convergence is uniform when both K and x vary.

This is an important theorem in that pointwise convergence has been used to give us a form of convergence in norm. Two corollaries follow.

Corollary 3.2 If $K_n \to K$ and $\{K_n - K\}$ is collectively compact, then

$$\|(K_n - K)^2\| \to 0 \ . \qquad (3.3)$$

Corollary 3.3 If $K_n \to K$ and $\{K_n\}$ is a collectively compact set, then

i) K is compact and $\{K_n - K\}$ is collectively compact,

ii) $\|(K_n - K)K\| \to 0$,

iii) $\|(K_n - K)K_n\| \to 0$.

The general theory will be continued in more detail later. First, examples from the field of integral equations are presented.

4. Integral Operators

Now the integral equations setting will be shown to be a valid application of the theory. As before, we will assume our functions are in real or complex C = C[0,1], with the maximum norm, $\|x\| = \max_{0 \le t \le 1} |x(t)|$. Thus, $\|x_n - x\| \to 0$ iff $x_n(t) \to x(t)$ uniformly.

Define bounded linear functionals φ and φ_n, $n \geq 1$, by

$$\varphi x = \int_0^1 x(t)dt, \tag{4.1}$$

$$\varphi_n x = \sum_{j=1}^{n} w_{nj} x(t_{nj}) . \tag{4.2}$$

It is necessary to assume that

$$\varphi_n \to \varphi . \tag{4.3}$$

This convergence holds for most of the usual quadrature formulas such as the rectangular and trapezoidal and those of Simpson, Weddle, Gauss, and Chebyshev. The Newton-Cotes quadrature rule however, does not satisfy (4.3).

Lemma 4.1 Given φ_n as above, there exists $B < \infty$ such that

$$\|\varphi_n\| = \sum_{j=1}^{n} |w_{nj}| \leq B \text{ for } n = 1,2,\ldots . \tag{4.4}$$

This follows from the principle of uniform boundedness and an elementary calculation

The convergence in (4.3) is pointwise convergence which is automatically uniform on a totally bounded set, The Arzelà-Ascoli lemma tells us that a totally bounded set in $C[0,1]$ is a bounded equicontinuous family of functions. In other words, $\varphi_n x \to \varphi x$ uniformly for x in a bounded equicontinuous family of functions.

To illustrate this uniform convergence, suppose that we have a family of differentiable functions that satisfy a Lipschitz condition,

$$|x'(s)-x'(t)| \leq m\,|s-t| . \tag{4.5}$$

Then, for the trapezoidal rule for example, the error in numerical integration is given by

$$|\varphi_n x - \varphi x| \leq \frac{m}{12n^2} . \qquad (4.6)$$

It follows, since x does not occur on the right side of (4.6), that the convergence $\varphi_n \to \varphi$ is uniform for functions satisfying the hypotheses, which form an equicontinuous family,

It is easy to demonstrate that we do not have convergence in norm.

<u>Lemma 4.2</u> $\|\varphi_n - \varphi\| \not\to 0$.

To show this consider a function which is identically zero except in the neighborhoods of the subdivision points. Then the numerical integral and the integral are far apart.

Now consider an integral operator with a kernel, $k(s,t)$, which is continuous for $0 \leq s,\ t \leq 1$ and let

$$M = \max_{0 \leq s,\ t \leq 1} |k(s,t)| .$$

A formal definition of the integral operator and the approximate operator is as follows,

<u>Definition</u> $K, K_n \in [C!]$ are defined by

$$(Kx)(s) = \int_0^1 k(s,t)x(t)dt , \qquad (4.7)$$

$$(K_n x)(s) = \sum_{j=1}^{n} w_{nj} k(s,t_{nj}) x(t_{nj}) . \qquad (4.8)$$

These operators are bounded,

Lemma 4.3

$$\|K\| = \max_{0 \le s \le 1} \int_0^1 |k(s,t)|\,dt \le M \text{ , and} \qquad (4.9)$$

$$\|K_n\| = \max_{0 \le s \le 1} \sum_{j=1}^{n} |w_{nj} k(s,t_{nj})| \le MB \qquad (4.10)$$

for n = 1,2,... , where B is from (4.4).

Proposition 4.4. K_n converges to K pointwise. That is,

$$K_n \to K \text{ , but } \|K_n - K\| \not\to 0 \text{ (unless } K = 0). \qquad (4.11)$$

Define $k_s(t) = k(s,t)$. Then equations (4.7) and (4.8) yield

$$(Kx)(s) = \varphi(k_s x) \text{ and} \qquad (4.12)$$

$$(K_n x)(s) = \varphi_n(k_s x) \text{ .} \qquad (4.13)$$

Since k(s,t) is a continuous function on the unit square, the family of functions $\{k_s: 0 \le s \le 1\}$ is equicontinuous and the products $k_s x$ share this property. Pointwise convergence is uniform on such a family so we have uniform convergence for each fixed x in (4.11).

It is easy to show that $K\mathcal{B}$ and $\{K_n x: n \ge 1, x \in \mathcal{B}\}$ are bounded and equicontinuous. Thus, we have

Proposition 4.5. K is compact and $\{K_n\}$ is collectively compact.

These are basic properties needed in the approximation theory. In particular, they yield the following results involving norm convergence.

Proposition 4.6 In the integral equations case we have

$$e_n \geq \|(K_n-K)K\| \to 0 \quad \text{and} \tag{4.14}$$

$$Be_n > \|(K_n-K)K_n\| \to 0 . \tag{4.15}$$

where

$$e_n = \max_{0 \leq s,t \leq 1} |(\varphi_n-\varphi)[K(s,u)K(u,t)]| \to 0 \text{ and} \tag{4.16}$$

φ_n and φ operate with respect to u.

The estimate, e_n, comes from the numerical integration and shows that $\|(K_n-K)K\| \to 0$ and $\|(K_n-K)K_n\| \to 0$ independently of the abstract theory. The quantity, $[K(s,u)K(u,t)]$ is an equicontinuous family of functions of u parameterized by s and t , so $\varphi_n \to \varphi$ uniformly on that set. Hence, the maximum goes to zero, and $e_n \to 0$. If an error formula for the numerical integration is known, it can give a ~~computable~~ estimate for e_n.

5. Abstract Approximation Theorems

Again consider bounded linear operators $K_n, K \in [X]$, where X is a Banach space. The principal hypotheses are:

(1) Pointwise convergence, $K_n \to K$,

(2) $\{K_n\}$ collectively compact. As before, we infer

(3) K is compact.

We wish to compare the equations

$$(\lambda-K)x = y, \tag{5.1}$$

$$(\lambda-K_n)x_n = y , \tag{5.2}$$

and the inverse operators

$$(\lambda-K)^{-1}\in[X] \quad , \tag{5.3}$$

$$(\lambda-K_n)^{-1}\in[X] \quad . \tag{5.4}$$

The Fredholm alternative asserts that $(\lambda-K)^{-1}$ exists iff $(h-K)X = X$. If $(\lambda-K)^{-1}$ exists, it is automatically bounded.

The type of result we will obtain is illustrated by the next theorem.

Theorem 5.1 Let $K, K_n \in [X]$ for $n \geq 1$. Assume (1) and (2) hold and $\lambda \neq 0$. Then

(a) $(\lambda-K)^{-1}\in[X]$ exists iff

(b) for all n sufficiently large $(\lambda-K_n)^{-1}\in[X]$ exists and is bounded uniformly with respect to n.

In either case,

(c) $(\lambda-K_n)^{-1} \to (h-K)-?$

Proof Assume (b). Then $(\lambda-K)x = 0$ implies $\|x\| \leq \|(\lambda-K_n)^{-1}\|\cdot\|(\lambda-K_n)x\| \to 0$ which in turn implies that $x = 0$. Hence (b) implies (a) by the Fredholm alternative. Now assume (b) fails. Then there exist $\{n_i\}$ and $\{x_{n_i}\}$ such that

$$\|x_{n_i}\| = 1 \ , \quad (X-K,_{i})x_{n_i} \to 0 \ .$$

Since $\{K_{n_i}\}$ is collectively compact, there exists $\{n_{i_j}\}$ and $y\in X$ such that $K_{n_{i_j}} x_{n_{i_j}} \to y$. Then $x_{n_{i_j}} \to y/\lambda$, $y \neq 0$, and

$$(\lambda-K_{n_{i_j}})x_{n_{i_j}} \to (\lambda-K)y/\lambda = 0 \ .$$

Thus (a) fails. Hence, (a) implies (b). Since

$$(\lambda-K_n)^{-1}-(\lambda-K)^{-1} = (\lambda-K_n)^{-1}(K_n-K)(\lambda-K)^{-1} ,$$

we also have that either (a) or (b) implies (c).

We shall give another proof which yields error bounds. First recall that

$$\|(K_n-K)K\| \to 0 , \tag{5.5}$$

$$\|(K_n-K)K_n\| \to 0 . \tag{5.6}$$

The following auxiliary theorem will be needed.

<u>Theorem 5.2</u> Let $S,T \in [X]$. Assume $(\lambda-T)^{-1}\in[X]$ exists and

$$\Delta = \|(\lambda-T)^{-1}\|\cdot\|(S-T)S\| < |\lambda| . \tag{5.7}$$

Then $(\lambda-S)^{-1}$ exists,

$$\|(\lambda-S)^{-1}\| \le \frac{\|I+(\lambda-T)^{-1}S\|}{|\lambda| - \Delta} , \tag{5.8}$$

and for any $y\in(\lambda-S)X$,

$$\|(\lambda-S)^{-1}y-(\lambda-T)^{-1}y\| \le \frac{\|(\lambda-T)^{-1}\|\cdot\|Sy-Ty\|+\Delta\ \|(\lambda-T)^{-1}y\|}{|\lambda| - \Delta} . \tag{5.9}$$

To prove this we first consider the following identity where we use the resolvent operator to express $(\lambda-T)^{-1}$.

$$\lambda^{-1}[I + (\lambda-T)^{-1}T](\lambda-T) = I . \tag{5.10}$$

We want to express the inverse of (h-S). If we consider $\lambda^{-1}[I + (\lambda-T)^{-1}S]$ as an approximate inverse of (h-S) and substitute the expression for $(\lambda-T)^{-1}$ from (5.10) we obtain

$$\lambda^{-1}[I+(\lambda-T)^{-1}S](\lambda-S) = I-\lambda^{-1}(\lambda-T)^{-1}(S-T)S \quad . \tag{5.11}$$

By the hypothesis (5.7) the operator on the right has a bounded inverse and we have an expression for $(\lambda-S)^{-1}$:

$$(\lambda-S)^{-1} = \lambda^{-1}[I-\lambda^{-1}(\lambda-T)^{-1}(S-T)S]^{-1}\cdot[I+(\lambda-T)^{-1}S]. \tag{5.12}$$

Taking norms in (5.12) we see that (5.8) holds. By subtracting the expression for $(\lambda-T)^{-1}$ given by (5.10) from (5.12) we obtain, after some manipulation,

$$\begin{aligned}(\lambda-S)^{-1}-(\lambda-T)^{-1} &= \lambda^{-1}[I-\lambda^{-1}(\lambda-T)^{-1}(S-T)S]^{-1}\cdot \\ &\quad \cdot(\lambda-T)^{-1}[(S-T)+(S-T)S(\lambda-T)^{-1}] \quad .\end{aligned} \tag{5.13}$$

Application of this operator on $y\in(\lambda-S)X$ and taking the norm yields (5.[illegible]).

Theorem 5.2 can now be used to prove Theorem 5.1 by substituting K_n for S and K for T. Thus we obtain

$$\|x_n-x\| \to 0. \tag{5.14}$$

Moreover we obtain the error estimate

$$\|x_n-x\| \le \frac{\|(\lambda-K_n)^{-1}\|\cdot\|K_ny-Ky\|+\Delta_n\|x_n\|}{|\lambda| - \Delta_n} \to 0 \; , \tag{5.15}$$

whenever

$$\Delta_n = \|(\lambda-K_n)^{-1}\|\cdot\|(K_n-K)K\| < |\lambda| \quad . \tag{5.16}$$

The roles of K and K_n may be interchanged to obtain an inequality similar to (5.15).

The convergence to 0 in (5.15) follows from the boundedness of $\|(\lambda-K_n)^{-1}\|$ and $\|x_n\|$ and the convergence to 0 of $\|K_n y-Ky\|$ and Δ_n. The convergence $\Delta_n \to 0$ follows from the boundedness of $\|(\lambda-K_n)^{-1}\|$ and (5.5) or (5.6) if the roles of K and K_n are interchanged,

The estimate given by (5.15) is computable in the integral equations case if the error in numerical integration is known. The only quantity not due to error in numerical integration is $\|(\lambda-K_n)^{-1}\|$ which can be estimated as follows.

Lemma 5.3

$$\|(\lambda-K_n)^{-1}\| \leq |\lambda|^{-1}(1+\|K_n\|\cdot\|A_n^{-1}\|) \quad . \tag{5.17}$$

Here $\|K_n\|$ is given by (4.10) and $\|A_n^{-1}\|$ is the maximum row sum of the inverse of the coefficient matrix, A_n, from the system of equations (1.3).

The following chapter will apply the abstract theory to integral operators with discontinuous or singular kernels. It will be shown that these operators do satisfy the hypotheses so that the desired conclusions can be drawn.

In Chapter III, the abstract theory will be extended to the eigenvalue problems and more general spectral properties of operators. Nonlinear problems will be treated in Chapter IV. This involves combining the linear theory with the abstract Newton's method.

Chapter II

INTEGRAL OPERATORS WITH DISCONTINUOUS OR SINGULAR KERNELS

1. Introduction

Consider a Fredholm integral equation with functions in real or complex $C = C[0,1]$, with the maximum norm, $\|x\| = \max_{0\le t\le 1} |x(t)|$. We have

$$(Kx)(s) = \int_0^1 k(s,t)x(t)dt, \quad 0\le s\le 1 \; , \tag{1.1}$$

for the integral operator and

$$(K_n x)(s) = \sum_{j=1}^{n} w_{nj} k(s,t_{nj})x(t_{nj}) \; , \quad 0\le s\le 1 \; , \tag{1.2}$$

for the approximate operators defined by numerical integration.

The kernel, $k(s,t)$, is assumed to be bounded and possibly discontinuous. We include Volterra and other "mildly discontinuous" kernels which are discontinuous on a finite number of continuous curves $t = t(s)$ in the unit square, and bounded uniformly. More general classes of kernels will be defined explicitly later.

With these discontinuous kernels, the integral operator, K, maps C into C but the approximate operators, K_n, do not map continuous functions into continuous functions. That is,

$$KC \subset C \text{ but } K_n C \not\subset C \; . \tag{1.3}$$

Therefore we cannot regard K and K_n on the same space C as was the case for continuous kernels. To circumvent this problem we define a new and larger space.

Definition Let R denote the normed linear space of proper Riemann integrable functions x(t), $0 \leq t \leq 1$, with the supremum norm, $\|x\| = \sup|x(t)|$.

Lemma 1.1 R is complete; hence R is a Banach space.

To show this note that $x \in R$ iff x is bounded and x is continuous almost everywhere. From these two facts the completeness follows immediately.

Lemma 1.2 C is a closed subspace of R.

The space R is chosen since it is a rather minimal extension of C which includes step functions and other piecewise continuous functions.

We will show that the operators K and K_n map R into R, that $K_n \to K$ pointwise, that $\{K_n\}$ is collectively compact and that K is compact. Hence, the general theory of Chapter I will apply as well as the approximate spectral theory in Chapter III.

2. The Quadrature Formula

To examine the quadrature formula we introduce linear functionals expressing integration,

$$\varphi x = \int_0^1 x(t)dt \quad , \text{ for } x \in R, \tag{2.1}$$

and numerical integration,

$$\varphi_n x = \sum_{j=1}^{n} w_{nj} x(t_{nj}) \quad , \text{ for } x \in R. \tag{2.2}$$

We assume that the weights in the quadrature formula are all non-negative. Thus,

$$w_{nk} > 0 \ , \ 1 \leq k \leq n \ . \tag{2.3}$$

So we have bounded linear functionals cp, $\varphi_n \in R^*$ with norms

$$\|\varphi\| = 1, \ \|\varphi_n\| = \sum_{j=1}^{n} w_{nj} \ . \tag{2.4}$$

In addition to (2.3) we hypothesize that

$$\varphi_n \to \varphi \text{ as } n \to \infty \ , \text{ on } C \ , \tag{2.5}$$

and note that for the usual quadrature rules (New-ton-Cotes excepted) these two assumptions hold.

Before proving that the hypotheses also hold on the space R we show that the norm, $\|\varphi_n\|$ is bounded. Since $\varphi_n \to \varphi$ on C,

$$\sum_{j=1}^{n} w_{nj} = \varphi_n 1 \to \varphi 1 = 1. \tag{2.6}$$

The sum of the weights is bounded uniformly in n since it converges. Thus we have the inequality

$$\|\varphi_n\| = \sum_{j=1}^{n} w_{nj} \leq B < \infty . \tag{2.7}$$

The next lemma states that φ and φ_n are positive and monotone linear functionals.

Lemma 2.1 For $x, y \in R$,

$$x \geq 0 \text{ implies } \varphi x \geq 0 \text{ and } \varphi_n x \geq 0 , \tag{2.8}$$

$$x \geq y \text{ implies } \varphi x \geq \varphi y \text{ and } \varphi_n x \geq \varphi_n y . \tag{2.9}$$

We also have for complex functions the following fact.

Lemma 2.2 $x \in R$ iff Re x, Im $x \in R$.

The next lemma will be used to extend (2.5) to the space R.

Lemma 2.3 A real function x is in R iff for any $\varepsilon > 0$ there exist real functions x_ε, $x^\varepsilon \in C$ such that

$$x_\varepsilon \leq x \leq x^\varepsilon , \tag{2.10}$$

$$\varphi x^\varepsilon - \varphi x_\varepsilon < \varepsilon . \tag{2.11}$$

This follows easily from the usual definition of R in terms of upper and lower integrals. By Lemmas 2.1 and 2.3 we have

$$\varphi x^\varepsilon \to \varphi x \text{ and } \varphi x_\varepsilon \to \varphi x \text{ as } \varepsilon \to 0 . \tag{2.12}$$

Now it can be proved that the numerical integral converges to the integral on R.

Proposition 2.4 If $x \in R$ then

$$\varphi_n x \to \varphi x \ , \ x \in R \ . \tag{2.13}$$

Proof By Lemmas 2.1 and 2.3 we have

$$\varphi_n x_\varepsilon \leq \varphi_n x \leq \varphi_n x^\varepsilon \ , \tag{2.14}$$

$$\varphi x^\varepsilon - \varepsilon < \varphi x < \varphi x_\varepsilon + \varepsilon \ . \tag{2.15}$$

Subtracting (2.15) from (2.14) Yields

$$\varphi_n x_\varepsilon - \varphi x_\varepsilon - \varepsilon < \varphi_n x - \varphi x < \varphi_n x^\varepsilon - \varphi x^\varepsilon + \varepsilon \ . \tag{2.16}$$

Since x_ε, $x^\varepsilon \in C$ and (2.5) holds, it is easy to see that $\varphi_n x - \varphi x \to 0$ as $n \to \infty$.

For our theory to apply, it is necessary to know for what classes of functions in R there is uniform convergence in (2.13) A standard answer is that on any totally bounded set $\varphi_n \to \varphi$ uniformly, However, in the present case this can be extended. To this end, we introduce the following concept.

Definition 2.1 A set, S, of real functions such that $S \subset R$ is regular iff for each $x \in S$ and each $\varepsilon > 0$ there exist real functions x_ε, $x^\varepsilon \in C$ such that (2.10) and (2.11) hold and, for each fixed $\varepsilon > 0$, the sets

$$S_\varepsilon = \{x_\varepsilon : x \in S\} \text{ and } S^\varepsilon = \{x^\varepsilon : x \in S\} \tag{2.17}$$

are totally bounded (or bounded and equicontinuous -- by the Arzelà-Ascoli lemma). An arbitrary set, $S \subset R$, is regular iff Re S and Im S comprise regular sets.

Using this definition the following theorem holds.

Theorem 2.5 The convergence in (2.13) is uniform on each regular set $S \subset R$.

Proof By (2.5) and (2.17), for each fixed ε,

$$\varphi_n x_\varepsilon \to \varphi x_\varepsilon , \quad \varphi_n x^\varepsilon \to \varphi x^\varepsilon \text{ uniformly for } x \in S. \tag{2.18}$$

Now (2.16) and (2.18) imply that the convergence in (2.13) is uniform for $x \in S$.

To illustrate the concepts just developed, consider the following examples.

Example 2.1 The set of all characteristic functions of intervals in [0,1] is a regular set but not totally bounded.

This follows from the fact that this set can be approximated in the sense of Definition 2.1 by sets of trapezoidal functions which are bounded and equicontinuous.

Any regular set is bounded since S_ε and S^ε are bounded. The converse is false as shown by the next example,

Example 2.2 Let $x_n(t) = \cos(2\pi nt)$. Then $\{x_n : n = 1,2, \ldots\}$ is bounded but not regular.,

Proof For an indirect proof, use the rectangular quadrature rule:

$$\varphi_n x = \frac{1}{n} \sum_{k=1}^{n} x(k/n) \; . \tag{2.19}$$

Then $\varphi_n x_n = 1$ and $\varphi x_n = 0$, $n \geq 1$. Therefore, by Theorem 2.5, the set $\{x_n : n \geq 1\}$ is not regular.

Based on these two examples, we can now state the following result.

Proposition 2.6 Given a set $S \subset R$,

$$S \text{ totally bounded} \Rightarrow S \text{ regular}, \qquad (2.29)$$

$$S \text{ regular} \Rightarrow S \text{ bounded}, \qquad (2.21)$$

but neither reverse implication holds.

Without proving it here, we state that regular sets may be very much larger than totally bounded sets. Regularity is essentially a requirement of compactness or total boundedness in one dimension only. Any regular set is totally bounded with respect to the $\mathcal{L}_1$ semi-norm but the converse is false. We observe that the pointwise convergence is uniform on much larger sets when the operators are positive than when they are not.

From the definition of regular sets we have the following.

Lemma 2.7 If S_1 and S_2 are regular sets then $S_1 \cup S_2, S_1 + S_2$, and $S_1 S_2$ are regular sets.

Lemma 2.8 If S is a regular set then $|S|$ is a regular set.

Hence, regular sets behave much like totally bounded or compact sets and may be combined and operated in much the same way. A convex combination of regular sets is also regular.

By using these properties we may obtain further examples of regular sets. For example, regular classes of step functions and of piecewise continuous functions may be constructed from the set of all characteristic functions.

There are several equivalent forms of the definition of a regular set. For example, in Definition 2.1 the functions x_ε and x^ε could be required to be Riemann integrable instead of continuous. Secondly, the requirement that S_ε and S^ε be totally bounded can be replaced by requiring S_ε and S^ε to be finite.

The preceding remarks are a special case of an abstract theory. If we work in any partially ordered Banach space and φ and φ_n are positive linear functionals which converge pointwise, then we can define q-regular in precisely the same manner as we defined regular, and pointwise convergence is uniform on any q-regular set.

3. Integral Operators

Let K be a linear integral operator on R and consider the equation

$$(Kx)(s) = \int_0^1 k(s,t)x(t)dt \ , \ x \in R, \ 0 \leq s \leq 1 \ . \tag{3.1}$$

Definition 3.1 A real kernel $k(s,t)$ is uniformly t-integrable iff for each $\varepsilon > 0$ there exist real continuous kernels $k_\varepsilon(s,t)$ and $k^\varepsilon(s,t)$ such that

$$k_\varepsilon(s,t) \leq k(s,t) \leq k^\varepsilon(s,t) \ , \ 0 \leq s,t \leq 1 \ , \tag{3.2}$$

$$\int [k^\varepsilon(s,t) - k_\varepsilon(s,t)]dt < \varepsilon \ , \ 0 \leq s \leq 1 \ . \tag{3.3}$$

An arbitrary kernel $k(s,t)$ is uniformly t-integrable iff Re $k(s,t)$ and Im $k(s,t)$ are uniformly t-integrable.

Examples of uniformly t-integrable kernels are continuous kernels, continuous kernels for the Volterra operator, and mildly discontinuous kernels.

In what follows we use the notation:

$$k_s(t) = k(s,t),$$
$$k_s^{\varepsilon}(t) = k^{\varepsilon}(s,t),$$
$$k_{\varepsilon s}(t) = k_{\varepsilon}(s,t).$$

Theorem 3.1 Let the kernel k(s,t) be uniformly t-integrable. Then

$$\{k_s : 0 \le s \le 1\} \text{ is a regular set in } R, \tag{3.4}$$

$$\varphi(\|k_s - k_{s'}\|) \to 0 \text{ as } s-s' \to 0, \text{ uniformly for } 0 \le s,s' \le 1 . \tag{3.5}$$

Proof. A check of the definition gives (3.4). To prove (3.5) define functions f, f^{ε} such that

$$f, f^{\varepsilon} : [0,1] \to \mathcal{L}_1(0,1), \tag{3.6}$$

$$f(s) = k_s , \quad f^{\varepsilon}(s) = k_s^{\varepsilon} . \tag{3.7}$$

Then f^{ε} is continuous for each $\varepsilon > 0$ and

$$f^{\varepsilon} \to f \text{ uniformly as } \varepsilon \to 0 . \tag{3.8}$$

Thus f is the uniform limit of continuous functions so f is continuous, proving (3.5).

The properties of any uniformly t-integrable kernel, given by Theorem 3.1, allow us to describe a larger class of kernels which we can deal with.

Proposition 3.2 Let k(s,t) be a kernel such that (3.4) and (3.5) hold. Then $KR \subset C$, K is compact and $\|K\| = \max_{0 \le s \le 1} \varphi(|k_s|) = \max_{0 \le s \le 1} \|k_s\|_1$.

Proof. By (3.1),

$$(Kx)(s) = \varphi(k_s x) . \tag{3.9}$$

By (3.4), $k_s \in R$ for all s so $\varphi(k_s x)$ exists. Secondly,

$$|(Kx)(s)| \le \max_{0 \le s \le 1} \|k_s\|_1 \cdot \|x\| , \tag{3.10}$$

where the maximum exists because in (3.7) f is continuous on a compact set. Thirdly, consider

$$|(Kx)(s) - (Kx)(s')| \le \|k_s - k_{s'}\|_1 \cdot \|x\| . \tag{3.11}$$

By (3.5), the quantity $\|k_s - k_{s'}\|_1 \to 0$ and we have $KR \subset C$. For $x \in \mathcal{B}$, the unit ball, (3.5) and (3.11) imply that the functions (Kx)(s) are bounded and equicontinuous. So by the Arzela-Ascoli lemma, K is a compact operator.

To sketch an alternate proof, consider Definition 3.1. This proof is for the real case in that definition.

Define the integral operator K^ε with the kernel k^ε. Then K^ε compact and $\|K^\varepsilon - K\| < \varepsilon$ imply K compact as follows.

Since

$$\|K^\varepsilon x - Kx\| < \varepsilon \text{ for all } x \in \mathcal{B}, \tag{3.12}$$

$K^\varepsilon \mathcal{B}$ is totally bounded and is also an e-net for $K\mathcal{B}$. Hence $K\mathcal{B}$ is totally bounded and K is compact.

4. Operator Approximations

Consider the operators K_n on R defined by

$$(K_n x)(s) = \sum_{j=1}^{n} w_{nj} k(s,t_{nj}) x(t_{nj}) \ , \ 0 \le s \le 1 \ , \tag{4.1}$$

where k(s,t) satisfies (3.4) and (3.5) and

$$k(s,t) \text{ is s-integrable for } 0 \le t \le 1 \ . \tag{4.2}$$

The quadrature formula satisfies the conditions in Section 2, Since dim $K_n R < \infty$, each K_n is compact.

Theorem 4.1 The operators K_n satisfy

$$K_n R \subset R \ , \tag{4.3}$$

$$K_n \to K \ , \tag{4.4}$$

$$\{K_n\} \text{ collectively compact.} \tag{4.5}$$

Proof From (4.2) it follows that (4.3) holds. From (3.4) we have that

$$(K_n x)(s) - (Kx)(s) = \varphi_n(k_s x) - \varphi(k_s x) \to 0 \text{ uniformly in s.} \tag{4.6}$$

This proves (4.4). To prove (4.5), let x vary in $\mathcal{B}$ and note that

$$|(K_n x)(s) - (K_n x)(s')| \le \varphi_n(|k_s - k_{s'}|) \ . \tag{4.7}$$

But by (3.4) we have

$$\varphi_n(|k_s - k_{s'}|) \to \varphi(|k_s - k_{s'}|) \text{ uniformly in } s,s', \tag{4.8}$$

and by (3.5) we have

$$\varphi(|k_s - k_{s'}|) \to 0 \text{ as } s-s' \to 0 . \tag{4.9}$$

Now, for each $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that

$$|(K_n x)(s) - (K_n x)(s')| < \varepsilon \text{ if } n \geq N, \ |s-s'| < 6, \text{ and } x \in \mathcal{B}. \tag{4.10}$$

We already know that

$$|(K_n x)(s)| \leq M \text{ for } x \in \mathcal{B}, 0 \leq s \leq 1 \text{ and } n = 1,2,3,\ldots . \tag{4.11}$$

It follows that $(K_n x: n \geq N, x \in \mathcal{B}\}$ has a finite e-net of step functions. Since each K_n is compact, $\bigcup_{n=1}^{N-1} \{K_n x: x \in \mathcal{B}\}$, also has a finite e-net. Therefore tne set $\{K_n x: n \geq 1, x \in \mathcal{B}\}$ has a finite s-net. By definition then, the set $\{K_n\}$ is collectively compact, proving (4.5).

Since (4.4) and (4.5) hold, the general approximation theory concerning convergence and error bounds applies to this case.

Consider

$$(\lambda - K)x = y \ , \ (\lambda - K_n)x_n = y \tag{4.12}$$

with $\lambda \neq 0$ and $y \in C$. Suppose $(\lambda-K)^{-1}$ and $(\lambda-K_n)^{-1}$ exist. Since $KR \subset C$, $x = \lambda^{-1}(Kx + y) \in C$ and $(\lambda-K)^{-1}C \subset C$. But $x_n \notin C$ in general since $K_n C \not\subset C$ for discontinuous kernels. That is, if the given function in an integral equation is in C, the solution will be in C. The approximate solutions would only be in R. However, we have the familiar situation of discontinuous functions converging uniformly to continuous functions:

$$x_n(\text{discontinuous}) \to x \ (\text{continuous}) \text{ uniformly.} \tag{4.13}$$

An abstract generalization of the result, $K_n \to K$, and $\{K_n\}$ collectively compact, can be given after verifying one additional property, namely

$$\|K_n\| \to \|K\| . \tag{4.14}$$

This follows from $\|K\| = \max_{0 \leq s \leq 1} \varphi(|k_s|)$ and $\|K_n\| = \sup_{0 \leq s \leq 1} \varphi_n(|k_s|)$. For the case involving a kernel, $\overline{k}(\overline{s},t)$, which is uniformly t-integrable, there exist continuous kernels k_ε and k^ε such that $k_\varepsilon \leq k \leq k^\varepsilon$. Considering the continuous kernel, k^ε, we can define the corresponding integral operator K^ε and we can use numerical integration to define the approximate operators

$$(K_n^\varepsilon x)(s) = \sum_{j=1}^{n} w_{nj} k_n^\varepsilon(s,t_{nj}) x(t_{nj}). \tag{4.15}$$

By (3.3) and (4.14),

$$\|K_n^\varepsilon - K_n\| = \|(K^\varepsilon - K)_n\| \to \|K^\varepsilon - K\| < \varepsilon. \tag{4.16}$$

In the abstract setting we now have

Theorem 4.2 If

$$K_n^\varepsilon \to K^\varepsilon \text{ as } n \to \infty, \text{ for each } \varepsilon > 0 , \tag{4.17}$$

$$\{K_n^\varepsilon : n \geq 1\} \text{ collectively compact, for each } \varepsilon > 0, \tag{4.18}$$

K_n compact, for $n = 1, 2, \ldots$, (4.19)

$$\|K_n^{\varepsilon} - K_n\| \to \|K^{\varepsilon} - K\| < \varepsilon \text{ as } n \to \infty , \tag{4.20}$$

then

$$K_n \to K , \tag{4.21}$$

$\{K_n\}$ collectively compact. (4.22)

<u>Proof</u> $K_n \to K$ by the triangle inequality. Fix $\varepsilon > 0$. Then there exists $N = N(\varepsilon)$ such that $\|K_n^{\varepsilon}x - K_nx\| < \varepsilon$ for all $n \geq N$ and $x \in \mathcal{B}$. Hence the set $\{K_n^{\varepsilon}x: n \geq N, x \in \mathcal{B}\}$ is a totally bounded s-net for $\{K_nx: n \geq N, x \in \mathcal{B}\}$. It follows from (4.19), by an argument similar to the one used in the proof of Theorem 4.1, that $\{K_n\}$ is collectively compact.

This abstract version of the theorem is of interest since it indicates a way to extend the theory. For example, suppose we have a theory for integral equations with continuous kernels. Then we can extend the theory so it holds for neighboring objects in some well defined sense, This could be used to extend the theory to integral equations in several dimensions with other kinds of kernels without repeating the detailed analysis necessary to the development of the initial theory.

5. <u>Weakly Singular Kernels</u>

The material in this section is adapted from Atkinson [19].

For $x \in C$ consider

$$(Kx) = \int_0^1 k(s,t)x(t)dt, \quad 0\le x\le 1 , \tag{5.1}$$

where $k_s(t) = k(s,t)$ satisfies

$$k_s \in \mathcal{L}_1(0,1) \text{ for all } s , \tag{5.2}$$

$$\|k_s - k_{s'}\|_1 \to 0 \text{ as } s - s' \to 0 . \tag{5.3}$$

As in Section 3 of this chapter, the quantity $\max_s \|k_s\|_1$ exists, and the convergence in (5.3) is uniform for $0\le s, s'\le 1$. Conditions (5.2) and (5.3) imply

$$KC \subset C, \ K \text{ compact}, \ \|K\| \doteq \max_s \|k_s\|_1 . \tag{5.4}$$

The continuous and discontinuous kernels treated above satisfy (5.2) and (5.3). Another example is

$$k(s,t) = r(s,t)|s-t|^{-\alpha} , \tag{5.5}$$

where $r(s,t)$ is continuous for $0\le s,t\le 1$ and $0<\alpha<1$. More generally, suppose

$$k(s,t) = r(s,t)\sigma(s,t) , \tag{5.6}$$

$$r(s,t) \text{ continuous for } 0\le s,t\le 1 , \tag{5.7}$$

and $\sigma_s(t) = \sigma(s,t)$ satisfies (5.2) and (5.3). Then k_s satisfies (5.2) and (5.3), $KC \subset C$ and K is compact. As in the example with $\sigma(s,t) = |s-t|^{-\alpha}$, the "singular part" of a kernel often can be isolated in a simple explicit form. Now we have

$$(Kx)(s) = \int_0^1 [r(s,t)x(t)]\sigma(s,t)dt \ .$$

Suppose we have operators $A_n \in [C]$ such that $A_n x \to x$ for all $x \in C$ as $n \to \infty$. Then we define

$$(K_n x)(s) = \int_0^1 \{A_n [r(s,t)x(t)]\}\sigma(s,t)dt \ , \tag{5.8}$$

where A_n operates with respect to t.

For example, suppose $A_n x$ is the piecewise linear interpolation of x with subdivision points $t_{nj} = j/n$, $j = 0,1,\ldots,n$. Then $(K_n x)(s)$ reduces to

$$(K_n x)(s) = \sum_{j=o}^{n} w_{nj}(s) r(s,t_{nj}) x(t_{nj}) \ , \tag{5.9}$$

where

$$w_{nj}(s) = \frac{1}{n}\int_{\frac{j-1}{n}}^{j/n} (t - \frac{j-1}{n})\sigma(s,t)dt \tag{5.10}$$

$$+ \frac{1}{n}\int_{j/n}^{\frac{j+1}{n}} (\frac{j}{n} - t)\sigma(s,t)dt$$

and $\sigma(s,t) \equiv 0$ for $t \notin [0,1]$ to make the expressions for $w_{no}(s)$ and $w_{nn}(s)$ correct. Note that we must be able to integrate $\sigma(s,t)$ and $t\sigma(s,t)$ with respect to t in closed form in order to obtain an explicit expression for $(K_n x)(s)$. If $A_n x$ is a piecewise polynomial interpolation of x, then $(K_n x)(s)$ has the form (5.9) with $w_{nj}(s)$ defined in terms of integrals of $\sigma(s,t)$, $t\sigma(s,t)$, $t^2\sigma(s,t)$, etc.

Again consider the general situation.

Lemma 5.1

$$A_n[r(s,t)x(t)] \to r(s,t)x(t) \text{ uniformly in } s,t \ . \tag{5.11}$$

This follows from the fact that $\{r(s,t)x(t)\}$ is a bounded equicontinuous family of functions of t.

Lemma 5.2 The set $\{A_n[r(s,t)x(t)]: n > 1)$ is bounded and equicontinuous.

In general, if F_n and F are continuous functions, and $F_n \to F$ uniformly, then $\{F_n: n \geq 1\}$ is equicontinuous.

Proposition 5.3 The following two facts hold.

$$\{K_n\} \text{ is collectively compact,} \tag{5.12}$$

$$K_n \to K . \tag{5.13}$$

Proof BY (5.2), (5.3), Lemma 5.2 and a simple triangle inequality argument,

$$\{K_n x: n \geq 1, x \in \mathcal{B}\} \text{ is bounded and equicontinuous .} \tag{5.14}$$

Hence $\{K_n\}$ is collectively compact. Let $E_n = A_n - I$. Then $E_n \to 0$, and

$$\|K_n x - Kx\| \leq \sup_s \|E_n[r(s,t)x(t)]\| \sup_s \|\sigma_s\|_1 \to 0 \text{ as } n \to \infty . \tag{5.15}$$

Thus $K_n \to K$.

In view of Proposition 5.3, the general approximation theory applies.

Chapter III

SPECTRAL APPROXIMATIONS

1. General Properties of Collectively Compact Sets

Again let X be a real or complex Banach space. Recall $\mathcal{K} \subset [X]$ is collectively compact iff $\mathcal{K}\mathcal{B}$ is totally bounded. If $\mathcal{K}$ is collectively compact then each $K \in \mathcal{K}$ is compact and $\mathcal{K}$ is bounded. Finite unions and sums of collectively compact sets are also collectively compact.

Proposition 1.1 Let $\mathcal{K}$ be collectively compact. Then each of the following sets is collectively compact:

(a) $\mathcal{K}\mathcal{M}$ for each bounded $\mathcal{M} \subset [X]$;

(b) $\mathcal{M}\mathcal{K}$ for each totally bounded $\mathcal{M} \subset [X]$;

(c) the strong and norm closures of $\mathcal{K}$;

(d) $\{\sum_{n=1}^{N} \lambda_n K_n : K_n \in \mathcal{K}, \sum_{n=1}^{N} |\lambda_n| \leq b\}$ for each $b < \infty$, $N \leq \infty$;

(e) $\{\int_{\Gamma} K(\lambda) d\lambda : K(\lambda) \in \mathcal{K}, \ell(\Gamma) < b\}$ for each $b < \infty$,

where Γ is an interval or rectifiable arc of finite length $\ell(\Gamma)$ and the integral is the limit in operator norm of the usual approximating sums.

We shall study operators in [X] such that

$$T_n \to T, \{T_n - T) \text{ collectively compact.} \tag{1.1}$$

The special case,

$$T_n \to T, \{T_n\} \text{ collectively compact,}$$

includes the integral equations examples.

Lemma 1.2 Let T, $T_n \in [X]$. Then

$$T_n \to T \ , \ \{T_n\} \text{ collectively compact} \tag{1.2}$$

iff

$$T_n \to T \ , \ \{T_n - T\} \text{ collectively compact, } T \text{ compact.} \tag{1.3}$$

2. Resolvent sets and spectra

Let $T \in [X]$. Recall:

(i) $\lambda \in \rho(T)$, the resolvent set, iff there exists $(\lambda-T)^{-1} \in [X]$;

(ii) the spectrum $\sigma(T)$ is the complement $P(T)$;

(iii) $a(T) \supset \{\text{eigenvalues}\}$ (for example, if T is compact, the eigenvalues of T form a finite set or an infinite sequence converging to 0);

(iv) if $|\lambda| > \|T\|$ then $\lambda \in \rho(T)$, $(\lambda-T)^{-1} = \sum_{n=o}^{\infty} \frac{T^n}{\lambda^{n+1}} \in [X]$

and

$$\|(\lambda-T)^{-1}\| \leq \frac{1}{|\lambda| - \|T\|} \tag{2.1}$$

(consequently $|\lambda| \leq \|T\|$ for all $\lambda \in \sigma(T)$);

(v) $\rho(T)$ open, $a(T)$ closed and bounded (compact);

(vi) the map $\lambda \to (h-T)^{-1}$ is continuous on $o(T)$ and is uniformly continuous on each closed set in $p(T)$;

(vii) $\{(\lambda-T)^{-1}: \lambda \in A)$ is totally bounded for each closed $\Lambda \subset \rho(T)$.

The following identity will be used several times.

$$(\lambda-S)^{-1}-(\lambda-T)^{-1} = (\lambda-S)^{-1}(S-T)(\lambda-T)^{-1}. \tag{2.2}$$

Lemma 2.1 If $K\in[X]$ and $\|K^2\| < 1$, then $(I-K)^{-1}\in[X]$ and

$$(I-K)^{-1} = (I-K^2)^{-1}(I + K) , \tag{2.3}$$

$$\|(I-K)^{-1}\| \leq \frac{\|I + K\|}{1-\|K^2\|} . \tag{2.4}$$

Theorem 2.2 Assume $T_n \to T$, $\{T_n-T\}$ collectively compact, and λ arbitrary. Then

(a) $\lambda\in\rho(T)$

iff

(b) there exists N such that $\lambda\in\rho(T)$ for all $n > N$ and $\{(\lambda-T_n)^{-1}: n \geq N\}$ is bounded.

Either (a) or (b) implies

(c) $(h-T_n)^{-1} \to (\lambda-T)^{-1}$.

Proof Assume (a). Then verify

$$\lambda-T_n = (I-K_n)(\lambda-T) , \tag{2.5}$$

$$K_n = (T_n-T)(\lambda-T)^{-1} , \tag{2.6}$$

$$K_n \to 0 , \{K_n\} \text{ collectively compact.} \tag{2.7}$$

From Theorem 5.1 of Chapter I, there exists N such that

$$(I-K_n)^{-1}\in[X] \text{ for } n \geq N, \tag{2.8}$$

$$\{(I-K_n)^{-1}: n \geq N) \text{ is bounded,} \tag{2.9}$$

$$(I-K_n)^{-1} \to I. \qquad (2.10)$$

Therefore,

$$(\lambda-T_n)^{-1} = (\lambda-T)^{-1}(I-K_n)^{-1} \qquad (2.11)$$

and (a) implies (b), (c).

To obtain error bounds, note

$$\|K_n^2\| \to 0 \ . \qquad (2.12)$$

Whenever $\|K_n^2\| < 1$, (2.11) holds,

$$\|(\lambda-T_n)^{-1}\| \le \frac{\|(\lambda-T)^{-1}\| \cdot \|I + K_n\|}{1 - \|K_n^2\|} \quad , \qquad (2.13)$$

$$(\lambda-T_n)^{-1}-(\lambda-T)^{-1} = (\lambda-T_n)^{-1}K_n \ , \qquad (2.14)$$

$$\|(\lambda-T_n)^{-1}x-(\lambda-T)^{-1}x\| \le \|(\lambda-T_n)^{-1}\| \cdot \|K_n x\| \to 0 \ . \qquad (2.15)$$

Now assume (b). Then $(\lambda-T)x = 0$ implies

$$\|x\| \le \|(\lambda-T_n)^{-1}\| \cdot \|(\lambda-T_n)x\| \to 0 \qquad (2.16)$$

which implies $x = 0$. Hence, $(\lambda-T)^{-1}$ exists. For $n \ge N$,

$$\lambda-T = (I-L_n)(\lambda-T_n), \qquad (2.17)$$

$$L_n = (T-T_n)(\lambda-T_n)^{-1} \text{ compact.} \qquad (2.18)$$

Hence, $(\lambda-T)^{-1} \in [X]$ by the Fredholm alternative. Thus (b) implies (a).

To obtain error bounds, note that $\{L_n\}$ is bounded and

$$L_n = (L_n - I)K_n \, , \tag{2.19}$$

$$L_n \to 0 \, , \; \{L_n\} \text{ collectively compact}, \tag{2.20}$$

$$\|L_n^2\| \to 0. \tag{2.21}$$

For $\|L_n^2\| < 1$,

$$(\lambda - T)^{-1} = (\lambda - T_n)^{-1}(I - L_n)^{-1}, \tag{2.22}$$

$$\|(\lambda - T)^{-1}\| \leq \frac{\|(\lambda - T_n)^{-1}\| \cdot \|I + L_n\|}{1 - \|L_n^2\|} \, , \tag{2.23}$$

$$\|(\lambda - T_n)^{-1}x - (\lambda - T)^{-1}x\| \leq \| (\lambda - T)^{-1}\| \, . \, \|L_n x\| \to 0. \tag{2.24}$$

<u>Theorem 2.3</u> Assume $T_n \to T$ and $\{T_n - T\}$ collectively compact. Let Λ be closed and $\Lambda \subset \rho(T)$. Then there exists N such that

(a) $\Lambda \subset \rho(T)$ for $n \geq N$,

(b) $\{(\lambda - T_n)^{-1}: \lambda \in \Lambda, \, n > N)$ bounded,

(c) for each $x \in X$, $(\lambda - T_n)^{-1}x \to (\lambda - T)^{-1}x$ uniformly for $\lambda \in \Lambda$.

<u>Proof</u> In the proof of Theorem 2.1 write $K_n(\lambda)$ for K_n. Thus

$$K_n(\lambda) = (T_n - T)(\lambda - T)^{-1} \tag{2.25}$$

and

$$\|[K_n(\lambda)]^2\| \to 0, \; \|K_n(\lambda)x\| \to 0 \text{ for all } x \in X. \tag{2.26}$$

These functions of λ are equicontinuous on Λ. Hence the convergence is uniform for $\lambda \in \Lambda$, and the desired results follow as in the proof of Theorem 2.1.

The next theorem is essentially a corollary of Theorem 2.3.

Theorem 2.4 Assume $T_n \to T$ and $\{T_n - T\}$ collectively compact. Let Ω be open and $\sigma(T) \subset \Omega$. Then there exists N such that

$$\sigma(T_n) \subset \Omega \text{ for all } n > N. \qquad (2.27)$$

Proof Let Ω = complement Λ in Theorem 2.3.

To illustrate, suppose we have operators K_n and K such that $K_n \to K$, $\{K_n\}$ collectively compact. Then the following apply.

Lemma 2.5 Assume $K_n x_n = \mu_n x_n$, $\mu_n \to \mu \neq 0$, $\|x_n\| = 1$. Then there exists a Subsequence $\{n_i\}$ and an x such that

$$x_{n_i} \to x, \; Kx = \mu x, \; \| x \| = 1. \qquad (2.28)$$

The proof is similar to that of Theorem 5.1 in Chapter I.

Lemma 2.6 If in addition, x is unique, then

$$x_n \to x. \qquad (2.29)$$

This follows from the facts that $\{x_n\}$ has a convergent subsequence and has at most one limit point.

3. Functions of Operators; Projections

For further details on the material to appear in this section see [1] and [3], for example. Let X be a complex Banach space. For each $T \in [X]$ let

$$\mathcal{F}(T) = \{f: f \text{ locally analytic on an open domain } \mathcal{D}(f) \supset \sigma(T)\}. \qquad (3.1)$$

For each $f \in \mathfrak{F}(T)$ there exists a contour $\Gamma \subset \Delta(f)$ with $a(T)$ inside Γ. Define

$$f(T) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - T)^{-1} d\lambda \tag{3.2}$$

as the limit in norm of the usual approximating sums.

<u>Lemma 3.1</u> $f(T)$ is independent of Γ.

<u>Examples:</u> $f(T) = I, T, T^n$, polynomials in T.

<u>Lemma 3.2</u> If $f, g \in \mathfrak{F}(T)$ then

$$(f + g)(T) = f(T) + g(T), \tag{3.3}$$

$$(fg)(T) = f(T)g(T) . \tag{3.4}$$

If in addition

$$f_n(\lambda) \to f(\lambda) \text{ uniformly on } \Gamma \tag{3.5}$$

then

$$\|f_n(T) - f(T)\| \to 0 . \tag{3.6}$$

<u>Example</u> Limits of polynomials.

<u>Theorem 3.3</u> Assume $T_n \to T$ and $\{T_n - T\}$ collectively compact. Let $f \in \mathfrak{F}(T)$. Then there exists N such that

(a) $f \in \mathfrak{F}(T_n)$ for all $n > N$,

(b) $f(T_n) \to f(T)$,

(c) $\{f(T_n) - f(T): n \geq N\}$ collectively compact.

<u>Proof</u> Theorem 2.4 implies (a). For $n \geq N$,

$$f(T_n)-f(T) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)[(\lambda-T_n)^{-1}-(\lambda-T)^{-1}]d\lambda$$

$$= \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda-T)^{-1}(T_n-T)(\lambda-T_n)^{-1}d\lambda. \tag{3.10}$$

Theorem 2.3 (c) then implies (b). Proposition 1.1 (a), (b), (e) implies (c).

Definition 3.1 Complementary spectral sets σ, σ' associated with T are disjoint closed sets σ, σ' such that $\sigma \cup \sigma' = \sigma(T)$.

Lemma 3.4 There exists a contour Γ with σ inside and σ' outside. Conversely, each $\Gamma \in \rho(T)$ determines complementary spectral sets σ and σ'.

With this notation let

$$E = E_\Gamma(T) = \frac{1}{2\pi i} \int_\Gamma (\lambda-T)^{-1}d\lambda. \tag{3.11}$$

Note that

$$E = e(T), \tag{3.12}$$

where

$$e \in \mathcal{F}(T),\ e \equiv 1 \text{ on } \sigma,\ e \equiv 0 \text{ on } \sigma'. \tag{3.13}$$

Lemma 3.5 $e^2 = e \Rightarrow E^2 = E$. Thus, E is a projection.

Definition 3.2 EX is the spectral subspace associated with T and σ (or Γ).

Example If σ consists of a single isolated eigenvalue, $\sigma = \{\mu\}$, and if T is compact and $\mu \neq 0$ we may have

$$EX = \mathfrak{n}(\mu-T), \text{ an eigenmanifold,} \tag{3.14}$$

or

$$EX = \mathfrak{n}[(\mu - T)^{\nu}], \text{ a generalized eigenmanifold,} \tag{3.15}$$

where h(T) is the null space of T.

Lemma 3.6 Let E' = I-E. Then

$$X = EX \oplus E'X, \tag{3.16}$$

$$TEX \subset EX, \quad TE'X \subset E'X. \tag{3.17}$$

If in addition we let $T_E = T|_{EX}$, $T_{E'} = T|_{E'X}$, then

$$\sigma(T_E) = \sigma, \quad \sigma(T_{E'}) = \sigma'. \tag{3.17}$$

The next theorem is a specialization of Theorem 3.3 to operators which are projections.

Theorem 3.7 Assume $T_n \to T$ and $\{T_n - T\}$ collectively compact. Let Γ be a contour in p(T) around a spectral set σ. Then there exists N such that $\Gamma \subset \rho(T_n)$ for all $n \geq N$. The part σ_n of $\sigma(T_n)$ inside Γ is a spectral set for T_n. Let $E = E_\Gamma(T)$ and $E_n = E_\Gamma(T_n)$. Then

(a) $E_n \to E$,

(b) $\{E_n - E\}$ collectively compact,

(c) $\dim E_n X = \dim EX$ (finite or $+\infty$)

. for all n sufficiently large.

Proof Theorem 3.3 implies all but (c). We assert, for projections that (a), (b) imply (c). We also assert that T_n, $T \in [X]$, $T_n \to T$ imply

$$\dim T_n X \geq \dim TX \text{ eventually.} \tag{3.19}$$

To prove (3.19) let $\{Tx_j : j = 1, \ldots, m)$ be linearly independent and define $C = \{\sum_{j=1}^{m} c_j x_j : \max|c_j| = 1\}$. Then C and TC are compact, so $T_n \to T$ uniformly on C and $\min_{x\in C}\|T_x\| > 0$." So eventually $\min_{x\in C}\|T_n x\| > 0$ and $\{T_n x_j : j=1, \ldots, m\}$ is linearly independent. The result, (3.19), follows. Now we show $\leq$ in (c). Without loss of generality dim EX < ∞. Then E is compact and $\{E_n\}$ is collectively compact. Suppose that $\dim E_n X \geq m$ for $n \geq 1$. By the Riesz lemma, there exist linearly independent sets $\{x_{nk} : k = 1, \ldots, m\} \subset E_n x$, $n \geq 1$, such that

$$\|x_{nk}\| = 1, \ \|x_{nk} - \sum_{j=1}^{k-1} c_j x_{nj}\| \geq 1 \tag{3.20}$$

for all n, k and $\{c_j\}$. Since $x_{nk} = E_n x_{nk} \in \{E_n\}\mathcal{B}$, which is precompact, there exist a subsequence $\{n_i\}$ and elements $x_k \in X$ such that $x_{n_i k} = E_{n_i} x_{n_i k} \to x_k$ for $k = 1, \ldots, m$. Then

$$\|x_k\| = 1, \ \|x_k - \sum_{j=1}^{k-1} c_j x_j\| \geq 1 \tag{3.21}$$

for all k and $\{c_j\}$, so $\{x_k : k = 1, \ldots, m\}$ is linearly independent. Now $E_n \to E$ implies $E_{n_i} x_{n_i k} \to Ex_k$, so that $x_k = Ex_k \in EX$ for all k. Thus

$$\dim E_n X \geq m \text{ for all } n \Rightarrow \dim EX \geq m. \tag{3.22}$$

Apply this result to an arbitrary subsequence of $\{E_n\}$ to conclude that

$$\dim E_n X \leq \dim EX \text{ eventually.} \tag{3.23}$$

Since we now have (3.19) and the reverse inequality (3.23), (c) follows.

In Theorem 3.7, let $\sigma = \{\mu\}$, $\sigma_n = \{\mu_n\}$. Then $\mu_n \to \mu$ by Theorem 2.4. If dim $E\mathfrak{X} = 1$, then $E_n \to E$ implies convergence of eigenvectors as follows. Suppose $Tx = \mu x$ and $\|x\| = 1$. Then $Ex = x$. Let $x_n = E_n x$. Then $T_n x_n = \mu_n x_n$ and $x_n \to x$.

Proposition 3.8 For some n, let $\Gamma \subset \rho(T) \cap \rho(T_n)$. Define $E = E_\Gamma(T)$ and $E_n = E_\Gamma(T_n)$. Let σ and σ_n be the parts of $\sigma(T)$ and $\sigma(T_n)$ inside Γ. Assume

$$T_n x_n = \mu_n x_n, \qquad \mu_n \in \sigma_n, \qquad \|x_n\| = 1.$$

then

$$\|x_n - Ex_n\| \leq r_n \cdot \frac{\ell(\Gamma)}{2\pi} \max_{\lambda \in \Gamma} \frac{\|(\lambda - T)^{-1}\|}{|\lambda - \mu_n|} \|\mu_n x_n - Tx_n\| .$$

Now assume $r_n < 1$. Then $Ex_n \neq 0$, $E \neq 0$ and σ is nonvoid. Let $y_n = Ex_n / \|Ex_n\|$. Then $y_n \in E\mathfrak{X}$, $\|y_n\| = 1$ and

$$\|y_n - x_n\| \leq 2r_n.$$

Proof Note that

$$E_n - E = \frac{1}{2\pi i} \int_\Gamma (\lambda - T)^{-1} (T_n - T) (\lambda - T_n)^{-1} d\lambda,$$

$(\lambda - T_n)^{-1} x_n = (\lambda - \mu_n)^{-1} x_n$ for $\lambda \in \Gamma$, and $E_n x_n = x_n$.

Hence,

$$x_n - Ex_n = \frac{1}{2\pi i} \int_\Gamma \frac{(\lambda - T)^{-1}}{\lambda - \mu_n} d\lambda \, (\mu_n x_n - Tx_n)$$

and $\|x_n - Ex_n\| \leq r_n$. For $r_n < 1$,

$$\|y_n - x_n\| \leq \|y_n - Ex_n\| + \|Ex_n - x_n\|$$

$$\leq |1 - \|Ex_n\|| + r_n$$

$$\leq \|x_n - Ex_n\| + r_n \leq 2r_n.$$

In Proposition 3.8 suppose that dim $EX < \infty$ (e.g., T is compact and 0 is not inside Γ). Then y_n is an eigenvector of T. The corresponding eigenvalue μ_n lies inside Γ and is determined by $Ty_n = \mu_n y_n$.

Now assume dim $EX < \infty$ and that the hypotheses of Theorem 3.8 are satisfied for all $n > N$. By Theorem 3.4 there is a $\delta > 0$ such that $|\lambda-\mu_n| \geq \delta$ for all $\lambda \in \Gamma$ and $n \geq N$. Note that

$$\|\mu_n x_n - Tx_n\| = \|(T_n-T)E_n x_n\| \leq \|(T_n-T)E_n\| .$$

Since E is compact and $\{E_n-E\}$ is collectively compact, $\{E_n\}$ is collectively compact. Hence, $\|(T_n-T)E_n\| \to 0$ and

$$r_n \to 0,$$

$$\|x_n - Ex_n\| \to 0,$$

$$\|y_n - x_n\| \to 0.$$

In order to estimate r_n, we may use the inequality (2.23) for $\|(\lambda-T)^{-1}\|$ in terms of $\|(\lambda-T_n)^{-1}\|$.

As an application of Theorem 3.8, suppose that, by computational evidence, certain eigenvalues μ_n of T_n seem to converge to some value near λ_0 as $n \to \infty$. Fix n and $\epsilon > 0$ such that $|\mu_n - \lambda_o| < \epsilon$ and

$$\Gamma = \{\lambda: \quad |\lambda-\lambda_0| = \epsilon\} \subset \rho(T) \cap \rho(T_n).$$

Then $\ell(\Gamma) = 2\pi\epsilon$ and

$$r_n \leq \epsilon \max_{\lambda \in \Gamma} \frac{\|(\lambda-T)^{-1}\|}{|\lambda-\mu_n|} \|\mu_n x_n - Tx_n\|.$$

If dim EX $< \infty$ and $r_n < 1$, there is an eigenvalue μ of T with $|\mu-\lambda_o| < \epsilon$. The calculation of r_n presents a problem when ϵ is small, since then $\|(\lambda-T)^{-1}\|$ is large and $|\lambda-\mu_n|$ is small for $\lambda \in \Gamma$. Thus, ϵ should not be taken too small. This limits the practicality of Theorem 3.8. For further details, see Atkinson [21].

In Theorem 3.7, let $\sigma = \{\mu\}$, $EX = \mathfrak{N}[(\mu-T)^\nu]$, where ν is minimal. Then $\sigma_n = \{\mu_{nk}: k = 1, \ldots, k_n\}$, $\max_k |\mu_{nk} - \mu| \to 0$, and

$$E_n X = \bigoplus_{k=1}^{k_n} \mathfrak{N}[(\mu_{nk}-T_n)^{\nu_{nk}}], \tag{3.24}$$

where the ν_{nk} are minimal. Let

$$P(\lambda) = (\mu-\lambda)^\nu, \; P_n(\lambda) = \prod_{k=1}^{k_n} (\mu_{nk}-\lambda)^{\nu_{nk}}. \tag{3.25}$$

Then

$$EX = \mathfrak{N}[P(T)], \; E_n X = \mathfrak{N}[P_n(T_n)]. \tag{3.26}$$

Let ν_n = degree $P_n = \sum_{k=1}^{k_n} \nu_{nk}$.

Theorem 3.9 There exists N such' that $\nu_n \geq \nu$ for all $n \geq N$.

Proof $EP(T) = 0$ and $P(h)$ divides $Q(\lambda)$ for each polynomial Q such that $EQ(T) = 0$. Similarly, $E_n P_n(T_n) = 0$. Suppose

$$\nu_{n_1} = \alpha \text{ for some } \{n_i\} \subset \{n\}. \tag{3.27}$$

Then

$$0 = E_{n_i} P_{n_i}(T_{n_i}) \to E(\mu-T)^{\alpha} = 0,$$

so $\alpha > \nu$. The result follows.

Theorem 3.10 There exists N such that $\dim \mathfrak{N}(\mu_{nk}-T_n) \leq \dim \mathfrak{N}(\mu-T)$ for all $n \geq N$ and for all k.

Proof See [17, p. 12].

Theorem 3.10 There exists N such that for all $n \geq N$

$$\dim \mathfrak{N}[\sum_{k=1}^{k_n} (\mu_{nk}-T_n)^{\alpha_{nk}}] \leq \dim \mathfrak{N}[(\mu-T)^{\alpha}] \tag{3.28}$$

-whenever

$$0 \leq \alpha_{nk} \leq \nu_{nk} \text{ and } \sum_{k=1}^{k_n} \alpha_{nk} = a$$

Proof See [17].

Chapter IV

FURTHER TOPICS

1. An Alternative Method

Again let X be a real or complex Banach space. Consider $K, K_n \in [X]$ for $n \geq 1$, with

$$K_n \to K, \{K_n\} \text{ collectively compact, } K \text{ compact.} \tag{1.1}$$

As noted before, the first two conditions imply the third. We wish to solve

$$(I-K)x = y \tag{1.2}$$

or to determine $(I-K)^{-1}$.

The basic idea of the present method is to find operators $T, L \in [X]$ such that $T^{-1} \in [X]$, L is compact, and

$$T(I-K) = I-KL, \tag{1.3}$$

$$I-K = T^{-1}(I-KL). \tag{1.4}$$

Then the operator I–KL is approximated by $I-K_nL$. By Theorem 3.1 of Chapter I,

$$\|K_nL-KL\| \to 0. \tag{1.5}$$

Therefore, the standard approximation theory given in Section 1 of Chapter I applies. Thus, $(I-KL)^{-1}$ exists iff $(I-K_nL)^{-1}$ exists and is uniformly bounded for n sufficiently large, in which case

$$\|(I-K_nL)^{-1}-(I-KL)^{-1}\| \to 0, \qquad (1.6)$$

and there are error bounds.

Clearly, $(I-K)^{-1}$ exists iff $(I-KL)^{-1}$ exists, in which case

$$(I-K)^{-1} = (I-KL)^{-1}T, \qquad (1.7)$$

$$\|(I-K_nL)^{-1}T-(I-K)^{-1}\| \leq \|(I-K_nL)^{-1}-(I-KL)^{-1}\|\cdot\|T\|, \qquad (1.8)$$

$$\|(I-K_nL)^{-1}T-(I-K)^{-1}\| \to 0, \qquad (1.9)$$

and error bounds are available.

Such operators T and L exist. They can be determined in a variety of ways. For example, if $(I + K)^{-1}$ exists, then

$$(I + K)\ (I - K) = I - K^2, \qquad (1.10)$$

$$I - K = (I + K)^{-1}(I - K^2). \qquad (1.11)$$

Thus, $T = I + K$ and $L = K$ in this case.

More generally, let

$$T = I + K + \ldots + K^{p-1} \qquad (p \geq 2). \qquad (1.12)$$

Then

$$T(I - K) = I - K^p, \qquad (1.13)$$

$$T(I - K) = I - KL, \qquad L = K^{p-1}. \qquad (1.14)$$

We show that T^{-1} exists if p is a sufficiently large prime. Without loss of generality, X is complex; otherwise extend T to the space x + ix. Note that

$$T = \prod_{q=1}^{p} (K-\alpha_{pq}I), \tag{1.15}$$

where the α_{pq} are the nontrivial p^{th} roots of unity. For p prime, the α_{pq} are distinct numbers of absolute value one. Since K is compact, the eigenvalues of K form a finite set or an infinite sequence converging to zero. Therefore, only a finite number of the α_{pq} can be eigenvalues and

$$T^{-1} = \prod_{q=1}^{p} (K-\alpha_{pq})^{-1} \tag{1.16}$$

for p sufficiently large. Usually $p \leqq 5$ will suffice.

Another possibility is

$$T = I + K + cK^2, \tag{1.17}$$

where the constant c is chosen such that T^{-1} exists. Then

$$T(I - K) = I - KL, \quad L = (1-c)K + cK^2. \tag{1.18}$$

If K and L are integral operators on C[0,1] with continuous kernels, and K_n is defined by means of numerical integration, then the determination of $(I-K_nL)^{-1}$ is equivalent to a matrix problem (cf. [14]). Each matrix element is an integral over [0,1]. This contrasts with the method of Chapter I, where the matrix elements were simply values of given functions. The two methods also differ in that

$$\|(I - K_nL)^{-1}-(I-KL)^{-1}\| \to 0, \tag{1.19}$$

whereas there is merely pointwise convergence of

$$(I - K_n)^{-1} \to (I - K)^{-1}. \tag{1.20}$$

Thus, the present method requires more work but gives stronger results.

Integral equations of the form $(I - KL)x = z$ sometimes arise directly from physical problems. For examples in mechanics, electromagnetic theory, and radiative transfer, see [5, 7, 8, 9, 14, 34].

In such cases, we can proceed directly to the approximations $I - K_nL$.

2. Collectively Compact and Totally Bounded Sets of Operators

We have shown in Chapters I and III that operators T, $T_n \in [X]$ such that

$$T_n \to T, \quad \{T_n-T\} \text{ collectively compact}, \tag{2.1}$$

have many of the properties of operators for which $\|T_n-T\| \to 0$. Since the analysis simplifies in the latter case it is important to determine when $T_n \to T$ but $\|T_n-T\| \not\to 0$. It is easy to prove

Lemma 2.1 $\|T_n-T\| \to 0$ iff $T_n \to T$ and $\{T_n-T\}$ is totally bounded (equivalently, sequentially compact).

Thus, the theory presented above is intended mainly for operators such that $T_n \to T$, $\{T_n-T\}$ is collectively compact, but $\{T_n-T\}$ is not totally bounded. We shall compare collectively compact and totally bounded sets in $[X]$.

Proposition 2.2 Every totally bounded set $\mathcal{K}$ of compact operators in $[X]$ is collectively compact.

Proof Fix $\varepsilon > 0$. Then there exist $K_i \in \mathcal{K}$, $i=1,\ldots,m$, such that $\min_i \|K-K_i\| < \varepsilon$ for each $K \in \mathcal{K}$. Hence,

$$\min_i \|Kx-K_ix\| < \varepsilon \text{ for all } K\in\mathcal{K},\ x\in\mathcal{B}. \tag{2.2}$$

It follows that $S = \bigcup_{i=1}^{m} K_i\mathcal{B}$ is an ε-net for $\mathcal{K}\mathcal{B}$. Since each K_i is compact, S is totally bounded. Therefore, $\mathcal{K}\mathcal{B}$ is totally bounded and $\mathcal{K}$ is collectively compact.

The next example shows that the converse of Proposition 2.2 is false.

Example Let $\mathcal{K}$ be the set of operators on ℓ^2 such that

$$K_n(x_1,\ldots,x_n,\ldots) = (x_n,0,0,\ldots). \tag{2.3}$$

Then $\mathcal{K}$ is collectively compact. Since $\|K_m-K_n\| = \sqrt{2}$ for $m \neq n$, $\mathcal{K}$ is not totally bounded.

It was proved in [16] that the converse of Proposition 2.2 holds for any set $\mathcal{K}$ of self-adjoint operators on a Hilbert space. The proof involved the spectral theorem. More generally, it was established that:

Theorem 2.3 Let $\mathcal{K}$ be a set of compact normal operators on a Hilbert space. Then $\mathcal{K}$ is totally bounded iff both $\mathcal{K}$ and $\mathcal{K}^*$ are collectively compact, where $\mathcal{K}^* = \{K^*: K\in\mathcal{K}\}$.

From this, it follows that:

Theorem 2.4 Let $\mathcal{K}$ be a set of compact operators on a Hilbert space. Then $\mathcal{K}$ is totally bounded iff both $\mathcal{K}$ and $\mathcal{K}^*$ are collectively compact.

Later, the same result was obtained in [13] for any set $\mathcal{K}$ of compact operators from one normed linear space to another such that $\{\dim KX: K \in \mathcal{K}\}$ is bounded, In [18] this was extended to other sets in [X] by means of spectral theory. Finally, Palmer [36] recently found a quite direct proof of Theorem 2.4 for an arbitrary set of operators from one Banach space to another. In fact a somewhat stronger result was obtained.

3. Nonlinear Operator Approximations

Consider a nonlinear operator equation

$$Tx = 0, \tag{3.1}$$

where T maps a Banach space X into X. For example, this might be a Hammerstein integral equation on C[0,1]:

$$(Tx)(s) \equiv x(s)+\int_0^1 k(s,t)f(t,x(t))dt - z(s) = 0 \, . \tag{3.2}$$

Assume that T is Fréchet differentiable on X. Thus, there exists the unique Fréchet derivative $T'(x) \in [X]$ for each $x \in X$ which satisfies

$$\frac{\|T(x+y)-Tx-T'(x)y\|}{\|y\|} \to 0 \text{ as } \|y\| \to 0. \tag{3.3}$$

Under reasonable conditions on k(s,t) and f(t,u) in the example, T'(x) is the linear integral operator

$$[T'(x)y](s) = y(s)+\int_0^1 k(s,t)\frac{\partial}{\partial u} f(t,x(t))y(t)dt. \tag{3.4}$$

Consider $Tx = 0$ in the Banach space setting. Suppose $Tx^* = 0$, $\|x^*-x_0\|$ is small and $T'(x_0)^{-1} \in [X]$ exists. Then

$$T'(x_0)(x^*-x_0) \cong Tx^*-Tx_0 = -Tx_0, \tag{3.5}$$

$$x^* \cong x_1, \; x_1 = x_0 - T'(x_0)\, Tx_0. \tag{3.6}$$

Newton's method is based on

$$x_{m+1} = x_m - T'(x_m)^{-1} Tx_m, \; m = 0, 1, 2, \ldots, \tag{3.7}$$

provided the --inverse operators exist. The Kantorovitch theorem [29], gives sufficient conditions for the existence of the iterates x_m, for the existence of a locally unique solution x^* of $Tx = 0$, and for $\|x_m-x^*\| \to 0$. It also provides error bounds.

To apply Newton's method we must solve a linear problem or invert a linear operator at each iteration. In the integral equation example, and more generally, a second approximation method is needed to deal with these linear problems. R. H. Moore [33a, 33b] has combined Newton's method with the theory developed in Chapters I - III for linear operators to obtain an approximation theory for nonlinear operator equations in Banach spaces.

As Moore indicates, it is equivalent and somewhat more convenient to first introduce nonlinear operator approximations T_n, say with $\dim T_n X < \infty$, and then to solve $T_n x_n = 0$ by Newton's method. For example, T_n can be defined by numerical integration when T is an integral operator.

Theorem 3.1 For some $x_o \in X$ and $r > 0$ let

(1) $\|T_n x - Tx\| \to 0$ for $\|x - x_o\| < r$;

(2) $\{T_n\}$ equidifferentiable at x_0, i.e., the limit in the definition of $T_n'(x_o)$ is uniform in n;

(3) $\{T_n\}$ collectively compact, i.e., $\{T_n x: n \geq 1, \|x\| \leq b\}$ is totally bounded for each $b < \infty$.

Then

(4) $T_n'(x_o) \to T'(x_o)$;

(5) $\{T_n'(x_o)\}$ collectively compact;

(6) $T'(x_o)$ compact.

The hypotheses are satisfied under reasonable conditions for the Hammerstein operator, For the proof and further theory and applications, see [33a, 33b].

4. Collectively Compact Sets of Gradient Mappings

This material is adapted from [25] by James W. Daniel.

Let X be a real reflexive Banach space and E_1 the real field regarded as a Banach space with the absolute value norm. Suppose that $f: X \to E_1$ is Fréchet differentiable on some domain $\mathcal{D} \subset X$. Then $f'(x) \in X^*$ for all $x \in \mathcal{D}$. The map $\nabla f: \mathcal{D} \to X^*$ defined by $(\nabla f)(x) = f'(x)$ is the gradient of f.

Now let $\mathcal{F}$ be a family of such maps f.

Theorem 4.1 If $\{\nabla f; f \in \mathcal{F}\}$ is collectively compact then $\mathcal{F}$ is weakly equicontinuous on each bounded convex set.

For a proof, see [25].

Weak equicontinuity plays an important role in the approximate solution of variational problems. This is indicated by the following result.

Theorem 4.2 Let f and f_n be weakly lower semi-continuous functionals such that $f_n(x) \to f(x)$ for all $x \in B$, a closed and bounded set in X. Assume that $\{f_n - f\}$ is weakly equicontinuous on B. For each n, let $x_n \in B$ and $f_n(x_n) \leq \inf_{x \in B} f_n(x) + \varepsilon_n$, where $\varepsilon_n > 0$ and $\varepsilon_n \to 0$. Then every weak limit point x' of $\{x_n\}$ minimizes f on B.

For a proof and a number of related results, see [25].

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