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ABSTRACT

Let an entire function F(z) of finite genus have infinitely many zeros which are all positive, and take real values for real z. Then it is shown how to give two-sided bounds for all the zeros of F in terms of the coefficients of the power series of F, and of coefficients obtained by Graeffe's algorithm applied to F. A simple numerical illustration is given for a Bessel function.

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Graeffe's Method for Eigenvalues

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In several problems of mathematical physics the eigenvalues are positive and they are the zeros of an entire function of finite genus, It will be shown in what follows that in such a case a slight modification of the Graeffe process is ideally efficient: For any eigenvalue, it yields both a lower and an upper bound at each step, both bounds are improved by the next step, and these bounds converge to the desired eigenvalue with the rapidity well-known in the case of polynomials.

The remarks underlying this result are actually very simple, I develop them first in their simplest form in Section 1 and postpone the statement of the full result to Section 3. A simple numerical example, for the inclusion of which I am obliged to Professor George Forsythe, is given in Section 4.

1. Sums of Like Powers,

Let $\gamma_1, \gamma_2, 7_3, \dots$ be a finite or infinite sequence of increasingly ordered positive numbers, We set $\gamma_1 = 7$ so that

$$0 < \gamma = 71 \leq \gamma_2 \leq \gamma_2 \leq \dots \qquad . \tag{1.1}$$

We assume that there are at least two different terms in the sequence, that is, that there exists an ℓ such that $7 < \gamma_\ell$. Thanks to this assumption we avoid those (rather uninteresting) cases in which our inequalities degenerate into equations, We define

$$s_n = \Sigma \gamma_k^{-n} \tag{1.2}$$

for n = 1, 2, 3, ... the sum is extended over the whole sequence and is supposed to be convergent for n = 1 (and therefore also for $n > 1$) if the sequence of the γ_k is infinite,

Obviously

$$\gamma^{-n} < s_n , \tag{1.3}$$

$$s_{2n} = \Sigma \gamma_k^{-n} \gamma_k^{-n} < \gamma^{-n} s_n . \tag{1.4}$$

By a well-known inequality

$$(s_{2n})^{1/2n} < s_n^{1/n} ; \tag{1.5}$$

see 2, p.28, Theorem 19. (Underlined numbers like 2 refer to the bibliography at the end of this paper.)

By Hölder's inequality (see 2, p. 22, Theorem 11)

$$s_{2n} = \Sigma (\gamma_k^{-n})^{2/3} (\gamma_k^{-4n})^{1/3} < s_n^{2/3} s_{4n}^{1/3}$$

and so

$$\left(\frac{s_{2n}}{s_{4n}}\right)^{1/2n} < \left(\frac{s_n}{s_{2n}}\right)^{1/n} . \tag{1.6}$$

From (1.3), (1.4), (1.5), and (1.6) we conclude that

$$\left(\frac{1}{s_n}\right)^{1/n} < \left(\frac{1}{s_{2n}}\right)^{1/2n} < \gamma < \left(\frac{s_{2n}}{s_{4n}}\right)^{1/2n} < \left(\frac{s_n}{s_{2n}}\right)^{1/n} \tag{1.7}$$

and so

$$\left(\frac{1}{s_1}\right) < \left(\frac{1}{s_2}\right)^{1/2} < \left(\frac{1}{s_4}\right)^{1/4} < \ldots < \gamma < \ldots < \left(\frac{s_4}{s_8}\right)^{1/4} < \left(\frac{s_2}{s_4}\right)^{1/2} < \frac{s_1}{s_2} \quad . \qquad (1.8)$$

It is obvious from (1.2) that

$$\lim_{n \to \infty} \left(\frac{1}{s_n}\right)^{1/n} = \lim_{n \to \infty} \left(\frac{s_n}{s_{2n}}\right)^{1/n} = \gamma \quad . \qquad (1.9)$$

Let us summarize: If we are given the quantities $s_1, s_2, s_4, s_8, \ldots$ and we wish to compute 7 , we form, at the m-th step, the interval whose end points are

$$\left(\frac{1}{s_n}\right)^{1/n} \quad , \quad \left(\frac{s_n}{s_{2n}}\right)^{1/n} \qquad \text{with } n = 2^{m-1} .$$

This interval contains 7 in its interior, it contains also the next interval formed at the (m+1)-st step, and its end points converge to 7 as m tends to[1)] ∞ .

2. On entire functions of finite genus.

Let F(z) denote an entire function of genus p subject to the following restrictions:

(I) F(z) has infinitely many zeros which are all positive.

(II F(z) takes real values for real z .

(III) F(0) = 1 .

Such a function is of the form[2)]

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1) The symbols s_n, 7, γ_k are used in the same meaning in 3, p. 199-201, but the process of computation given there is different from the process offered here, and converges more slowly,

2) See e.g. 1, especially p.18-23.

$$F(z) = e^{Q(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\alpha_k}\right) \exp\left(\frac{z}{\alpha_k} + \frac{z^2}{2\alpha_k^2} + \cdots + \frac{z^p}{p\alpha_k^p}\right) \tag{2.1}$$

$$= \sum_{h=0}^{\infty} (-1)^h a_h z^h$$

where $\exp(u)$ stands for e^u,

$$0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \ldots , \tag{2.2}$$

$$\alpha_1^{-p-1} + \alpha_2^{-p-1} + \alpha_3^{-p-1} + \ldots$$

is convergent, $Q(z)$ is a polynomial with real coefficients of degree $\leq p$, and $Q(0) = 0$. Finally, $a_0, a_1, a_2, \ldots a$ are real, and $a_0 = 1$.

We consider the coefficients a_k as given, the genus p as known, and we want to compute the zeros α_k. For this purpose we consider the positive integer n, we define

$$e^{2\pi i/n} = \omega , \tag{2.4}$$

and we put

$$f(z)\, f(\omega z)\, f(\omega^2 z) \ldots f(\omega^{n-1} z) = \sum_{h=0}^{\infty} (-1)^h a_{n,h} z^{nh} \tag{2.5}$$

The coefficients $a_{n,h}$ can be computed in terms of the coefficients a_h (are polynomials in a_h). This computation is most convenient in the well-known practical case when n is a power of 2. Yet, for the moment, we need not restrict the integer n to any particular form.

Provided that

$$n > p \tag{2.6}$$

we have the simple expression

$$f(z)\ f(\omega z) \dots f(\omega^{n-1} z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^n}{\alpha_k^n}\right) . \tag{2.7}$$

By comparing (2.5) and (2.7) we obtain finally that, for $h > 1$,

$$a_{n,h} = \sum \left(\frac{1}{\alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_h}} \right)^n \tag{2.8}$$

where the sum is extended over all combinations of the subscripts $i_1, i_2, \dots i_h$ for which

$$i_1 < i_2 < i_3 < \dots < i_h \quad . \tag{2.9}$$

In words, $a_{n,h}$ is the h-th elementary symmetric function of the (-n)-th powers of the zeros we want to compute.

3. Computation of the zeros.

Let us now connect the considerations of the two foregoing sections. We form the products $\alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_h}$ according to (2.9), and we call them, increasingly ordered, $\gamma_1, \gamma_2,$ 739 . . Then

$a_{n,h}$ becomes s_n,

$a_1\ a_2 \dots \alpha_h$ becomes γ_1 = 7;

see (2.8) and (2.2), respectively., Hence (1.7) yields

$$\left(\frac{1}{a_{n,h}}\right)^{1/n} < \left(\frac{1}{a_{2n,h}}\right)^{1/2n} < \alpha_1 \alpha_2 \dots \alpha_h < \left(\frac{a_{2n,h}}{a_{4n,h}}\right)^{1/2n} < \left(\frac{a_{n,h}}{a_{2n,h}}\right)^{1/n} \tag{3.1}$$

provided that $n > p$, see (2.6). Thus, setting $n = 2^{m-1}$, we have a scheme to compute $\alpha_1 \alpha_2 \cdots \alpha_h$, and a scheme to compute α_h follows immediately. In fact, (3.1) yields

$$\alpha_h > \left(\frac{1}{a_{n,h}}\right)^{1/n} \frac{1}{\alpha_1 \alpha_2 \ldots \alpha_{h-1}} .$$

Then (3.1) yields also an upper bound for $\alpha_1 \alpha_2 \ldots \alpha_{h-1}$ and so a lower bound for α_h in terms of the a_k . An upper bound for α_h of the same nature is found similarly, and these bounds come closer to α_h when we pass from n to $2n$.

The reader may convince himself that the foregoing applies "essentially" also to the case of polynomials although, strictly speaking, this case was excluded from our reasoning by the (otherwise convenient) assumption that the series (2.3) is infinite,

The essential point in the foregoing is to observe the particular advantages the Graeffe process offers when it is applied to the particular class of entire functions here considered. Let me add that the zeros of a function of this not uninteresting class can be computed by still other techniques, Thus, in Section 1 we considered only those s_n as given for which n is a power of 2 . If we consider s_n as given for all n (or for all n from a certain one onward) we may base our computations instead of on (1.7) on the inequalities

$$\left(\frac{1}{s_n}\right)^{1/n} < \left(\frac{1}{s_{n+1}}\right)^{1/(n+1)} < \gamma < \frac{s_{n+1}}{s_{n+2}} < \frac{s_n}{s_{n+1}} \quad ; \qquad (3.2)$$

see 3. Moreover, the Hankel determinants

$$\begin{vmatrix} s_n & s_{n+1} & \cdots & s_{n+h-1} \\ s_{n+1} & s_{n+2} & \cdots & s_{n+h} \\ \cdot & \cdot & \cdot & \cdot \\ s_{n+h-1} & s_{n+h} & \cdots & s_{n+2h-2} \end{vmatrix}$$

considered by Hadamard can be used for computing the first h <u>different</u> zeros, see 4.

4. <u>Example,</u>

To try out the technique considered numerically, we use the Bessel function

$$F(z) = J_o(2\sqrt{z}) \ ,$$

also used in 3. Its smallest zero is $7_o \doteq 1.44576$. We use precisely the same number of terms in the power series for F(z) as in 3:

$$F(z) = 1 - z + \frac{z^2}{4} - \frac{z^3}{36} + \frac{z^4}{576} - \ \cdots \quad .$$

Then

$$F(z)F(-z) = 1 - \frac{z^2}{2} + \frac{z^4}{96} - \cdots$$

and

$$F(z)F(iz)F(-z)F(-iz) = 1 - \frac{11}{48} z^4 + \cdots .$$

Hence

$$a_{1,1} = 1 \ ; a_{2,1} = \frac{1}{2} \ ; \ a_{4,1} = \frac{11}{48} \quad .$$

Then

$$\frac{1}{a_{1,1}} = 1 \qquad \frac{a_{1,1}}{a_{2,1}} = 2 \quad ;$$

$$\left(\frac{1}{a_{2,1}}\right)^{1/2} = \sqrt{2} \doteq 1.41421 \quad ; \qquad \left(\frac{a_{2,1}}{a_{4,1}}\right)^{1/2} = \left(\frac{24}{11}\right)^{1/2} \doteq 1.47710 \ ;$$

$$\left(\frac{1}{a_{4,1}}\right)^{1/4} = \left(\frac{48}{11}\right)^{1/4} \doteq 1.44531 \quad .$$

Thus, using more decimals, we get the bounds

$$1 < 1.414213 < 1.445313 < 7 < 1.477098 < 2 \ .$$

For the first zero $27^{1/2} \doteq 2.404826$ of J_o , we get the corresponding bounds

$$2 < 2.37841 < 2.404424 < 2\gamma^{1/2} < 2.430719 < 2.82843 \ .$$

Thus our best bounds

$$2.404424 < 2\gamma^{1/2} < 2.430719$$

may be compared with the corresponding bounds

$$2.4006 < 27^{1/2} < 2.4121$$

from 3. We see that our present lower bound is much better, while the upper bound is much worse. (The upper bound in 3 results from $7 < s_3/s_4 = 16/11$ -- but here we have avoided using s_3 .)

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