MATRIX THEOREMS FOR PARTIAL DIFFERENTIAL AND DIFFERENCE EQUATIONS

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We want to reexamine the Cauchy problem for systems with constant coefficients, together with the matrix questions which arise after a Fourier transformation, Our main results are in fact purely matrix-theoretic, so that after motivating those results in the following paragraphs, we hardly need to mention partial differential equations again. We do hope, however, that our ideas will prove to be useful locally in studying certain systems with variable coefficients; such an application will of course require a much fuller discussion of differential operators.

A simple example will illustrate the problem we solve here, Consider the system

$$(1) \qquad \frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \text{ at } t = 0 ,$$

which has the solution

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(2) $$u_1(t) = f_1 \ , \ u_2(t) = f_2 + t \frac{\partial f_1}{\partial x} .$$

Since we may choose $f_1 \in L_2(-\infty,\infty)$ such that $u_2 \notin L_2$, the system (1) fails to be well-posed over L_2. Nevertheless, on the closed subspace normed by

(3) $$\|u\|^2 = \int |u_1|^2 + \left|\frac{\partial u_1}{\partial x}\right|^2 + |u_2|^2$$

we no longer lose a derivative, and (1) becomes well-posed; in fact,

(4) $$\|u(t)\| \le \left(1 + t\sqrt{1 + \frac{t^2}{4}} + \frac{t^2}{2}\right)^{1/2} \|f\| \le e^{t/2} \|f\|$$

What we want is to associate with more general systems such a subspace, maximal in a certain sense, over which the problem is well-posed. Without the maximality requirement, this question has been treated independently by Birkhoff and others (see [1]).

After Fourier transformation, a linear differential (or pseudo-differential) system with constant coefficients looks like

(5) $$\frac{\partial \hat{u}}{\partial t} = P(\omega)\, \hat{u}, \quad \hat{u}(\omega,0) = \hat{f}(\omega)$$

where $\omega = (\omega_1, \ldots, \omega_d)$, $\hat{u} = (\hat{u}_1(\omega,t), \ldots, \hat{u}_m(\omega,t))$, and $m \times m$ matrix P is the symbol of the given differential operator. To stay within the framework of the Fourier transform we introduce the Hilbert spaces $L_2(H)$, normed by

$$\|u\|_H^2 = \int_{R^d} (H(\omega)\,\hat{u}(\omega),\ \hat{u}(\omega))d\omega \quad . \tag{6}$$

Here H is a measurable Hermitian matrix function, normalized by the requirement $H \geq I$, **i.e.,** $H - I$ shall be non-negative definite,

Let us call (5) <u>well-posed over</u> $L_2(H)$ provided that for some α,

$$\|u(t)\|_H \leq e^{\alpha t}\|f\|_H \tag{7}$$

for all $t > 0$ and all initial data **f.** Solving (5), this can be made more explicit:

$$e^{P^*(\omega)t}\ H(w)\ e^{P(\omega)t} \leq e^{2\alpha t}\ H(o) \quad . \tag{8}$$

Differentiating at $t = 0$, we come to a still simpler equivalent condition; for almost all ω,

$$H(o)\ P(w) + P^*(\omega)\ H(w) \leq 2\alpha H(\omega) \, . \tag{9}$$

(To recover (8), post-multiply by $\exp(P(\omega) - \alpha)t$, pre-multiply by its **adjoint,** and integrate.) In the example described by (1) and (3), for instance, the condition (9) becomes

$$\begin{pmatrix} 1+\omega^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ i\omega & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\omega \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1+\omega^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{10}$$

$$= \begin{pmatrix} 0 & -i\omega \\ i\omega & 0 \end{pmatrix} \leq \begin{pmatrix} 1+\omega^2 & 0 \\ 0 & 1 \end{pmatrix} \, .$$

Our definition (7) is stronger than the usual one, which permits a constant factor M on the right side. Nothing is changed, however, since if (5) is well-posed in this weaker sense with respect to $L_2(H_1)$, there is an equivalent norm H_2 such that (7) holds on $L_2(H_2) = L_2(H_1)$. This follows from Theorem III below, and in fact it is the chief result of the Kreiss theorems which our work extends,

It is no trouble to bound α from below. If some $P(\omega)$ has the eigenvalue λ with eigenvector v, we must have from (9) that

$$((HP + P^*H)v,v) \leq 2\alpha(Hv,v) ,$$

which yields

$$\text{(11)} \qquad \text{Re } \lambda \leq \alpha \quad .$$

Therefore a is not less than

$$\text{(12)} \qquad \sigma = \sup_{\omega, j} \text{Re } \lambda_j(P(\omega)) ,$$

and we must impose on the symbol P the <u>Petrowsky-Gärding condition</u> $\sigma < \infty$. Subtracting a constant multiple of the identity, we shall in fact suppose $\sigma < 0$. Now fixing $\alpha = 0$, there is no doubt that we can construct H(o) to satisfy (9). The delicate problem is to keep H as small as possible; this we achieve, up to a constant depending only on the order m, in Theorem III. The corresponding space $L_2(H)$ is consequently maximal; its norm is weaker than that of any $L_2(H')$ over which (5) is well-posed.

The theory of partial difference operators leads to a closely related matrix problem, In place of (5) we have

$$\hat{u}(\omega,t+k,k) = A_k(\omega)\ \hat{u}(\omega,t,k),\ \hat{u}(\omega,0,k) = \hat{f}(\omega)\ . \tag{13}$$

The $A_k(\omega)$ are called amplification matrices; we don't want to discuss such systems fully, but we need to explain that the time-step k ranges over some interval $0 < k < k_0$. The analogue of (9), equivalent to the condition (7) on $L_2(H_k)$, is simply

$$A_k^*(\omega)\ H_k(\omega)\ A_k(\omega) \leq e^{2\alpha k}\ H_k(\omega)\ . \tag{14}$$

Again there is a lower bound on α, namely

$$\sigma' = \sup_{\omega,\ j,k} \frac{\log |\lambda_j(A_k(\omega))|}{k}\ . \tag{15}$$

Therefore we impose on (13) the von Neumann condition $\sigma' < \infty$. By a simple manipulation, we may achieve $\sigma' < 0$ and fix a = 0 as before. Thus our two matrix problems can be very concisely stated: given suitable P and A, to construct two corresponding matrices H > I as 'small as possible so that

$$HP + P^*H \leq 0 \quad \text{and} \quad A^*HA < H\ ,$$

respectively, Since the second problem is perhaps the more familiar, and its solution leads to a solution of the first, it will be treated in full detail, We need the definitions

$|v| = (|v_1|^2 + \cdots + |v_m|^2)^{1/2}$, $|A| = \sup |Av|/|v|$, $\rho(A) = \max_{1 \leq j \leq m} |\lambda_j(A)|$.

Theorem I, For a suitable constant $K(m)$, depending only on the order m of the matrix A, each of the following statements implies the next:

i) $A^*HA < H$ for some $H \geq I$ with $(Hv,v)^{1/2} = C(v)$ for $|v| = 1$.

ii) $|SAS^{-1}| < 1$ for some S with $|S^{-1}| < 1$ and $|Sv| = C(v)$ for $|v| = 1$.

iii) $|A^n v| \leq C(v)$ for all $n > 0$ and $|v| = 1$.

iv) $|(zI-A)^{-1}v| \leq \frac{C(v)}{|z|-1}$ for all complex $|z| > 1$ and all $|v| = 1$.

v) $A^*HA \leq \left(\frac{1+\rho(A)}{2}\right)^2 H \leq H$ for some $H \geq I$ with $(Hv,v)^{1/2} \leq K(m)\ C(v)$ for all $|v| = 1$.

This theorem is very close to one originally proved by Kreiss [2], and studied subsequently by Morton [3] and Morton and Schechter [4]. Therefore we should clarify those respects in which it is new:

a) The previous estimates in v), established by induction on m, had a power $C^{p(m)}$ in place of C, with $p(m) \to \infty$ as $m \to \infty$.

b) We estimate the action of H on each vector v, where earlier there appeared only the single constant $C = \sup C(v)$. It follows that the H in v) is minimal in a stronger sense than just in norm: if $H' > I$ and $A^*H'A < H'$, then $H < K^2(m)\ H'$.

c) We construct the new H in v) explicitly, leading to the following additional information:

For some S with $S^*S = H$, $A' = SAS^{-1}$ is upper triangular, with $A'_{ij} = 0$ unless λ_i and λ_j are in the same cluster (see below), and $|A'_{ij}| \leq \frac{1}{2}(1 - \max(|\lambda_i|, |\lambda_j|))$ in this case,

A trivial modification of S reduces this constant $\frac{1}{2}$ to any other, say $1/2m$, so that the absolute row and column sums (the ℓ_∞ and ℓ_1 norms of A') may also be made less than $(1 + \rho(A))/2$.

It remains to determine the behavior of the best constant K(m). Our constant (which we don't compute) grows roughly like m!, while examples of McCarthy and Schwartz [5] show that it must grow at least as fast as some power of log m; this leaves a wide gap, It is not surprising that $K(m) \to \infty$ in view of the Foguel-Halmos counterexamples [6,7] to the Nagy conjecture.

2. In this section we establish the first three implications in Theorem I* These are easy steps, valid also for operators on Hilbert space.

With $H = S^*S$, the equivalence of i) and ii) follows from that of the inequalities

$$(A^*HAv,v) \leq (Hv,v) \quad \text{for all} \quad v$$

$$|SAv|^2 \leq |Sv|^2 \quad \text{for all} \quad v$$

$$|SAS^{-1}w|^2 < |w|^2 \quad \text{for all} \quad w .$$

In the applications, ii) corresponds to a change of variables and i) to a new norm, In one respect the use of H is to be preferred;

it may depend more smoothly on some relevant parameters than does an improperly chosen S. The positive square root $S = H^{1/2}$ is as smooth as H, but a diagonalizing S may not be, although the latter change of variables looks especially desirable. Mizohata [8] points out this difficulty when d = 2, arising from the multiple-connectedness of the circle; there is no difficulty in his context with H.

To show that ii) implies iii), we compute

$$(16) \qquad |A^n v| = |S^{-1}(SAS^{-1})^n Sv| < |S^{-1}||SAS^{-1}|^n |Sv| \leq C(v) .$$

Finally, given iii), we have for $|z| > 1$

$$(17) \qquad |(zI-A)^{-1}v| = \left| \sum_0^\infty \frac{A^n v}{z^{n+1}} \right| \leq \sum \frac{C(v)}{|z|^{n+1}} = \frac{C(v)}{|z|-1} .$$

3. Before coming to the final step in Theorem I, we warm up with a more special result of the same kind, which shows how the geometry of the eigenvalues enters the problem.

Theorem II. Suppose the resolvent condition iv) holds, and the eigenvalues of A satisfy

$$(18) \qquad \delta|\lambda_i - \lambda_j| \geq 1 - |\lambda_j| \quad \text{for all distinct } i,j .$$

Then $A^*HA \leq \rho^2(A)\, H < H$ for some $H > I$ with

$$(Hv,v)^{1/2} \leq m(2+4m\delta)(1+2\delta)^{2m-3}C(v)$$

for $|\nu| \neq 1$. Furthermore, there exists S such that $H = S^*S$ and SAS^{-1} is diagonal,

Proof. From iv) it is clear that no eigenvalue lies outside the unit circle, so $\rho(A) \leq 1$. Although (18) admits repeated eigenvalues of modulus one, suppose for the present that the eigenvalues are distinct, Then we construct the projections

$$(19) \qquad L_i = \prod_{j \neq i} \frac{A-h_j}{\lambda_i - \lambda_j}, \quad 1 \leq i \leq m .$$

Applying L_i to the eigenvectors $v_1, \ldots, v_m$, we find $L_i v_j = \delta_{ij} v_j$, so there are the standard identities

$$(20) \qquad L_i^2 = L_i, \quad L_i L_j = 0 \quad \text{for} \quad i \neq j$$

$$(21) \qquad \sum_1^m L_i = I, \quad \sum_1^m \lambda_i L_i = A .$$

Now define the Hermitian matrix H by

$$(22) \qquad H = m \sum_1^m L_i^* L_i .$$

From (20) and (21) we have

$$(23) \qquad A^*HA = \sum \bar{\lambda}_j L_j^* \, m \sum L_i^* L_i \sum \lambda_k L_k = m \sum |\lambda_i|^2 L_i^* L_i \leq \rho^2(A) H .$$

To prove $H \geq I$ we need only (21) and the Schwarz inequality:

$$(24) \qquad |v|^2 = |\sum L_i v|^2 \leq m \sum |L_i v|^2 = (Hv,v) \ .$$

From (22),

$$(25) \qquad (Hv,v) \leq m^2 \max |L_i v|^2 \ ,$$

and the crucial estimate is that of $|L_i v|$. We use the resolvent condition in the most natural way, by expanding

$$(26) \qquad L_i = \sum_{k=1}^{m} b_{ik}(z_k I - A)^{-1} \ .$$

We shall choose $z_k = 1/\bar{\lambda}_k$; if $|\lambda_k|$ is 0 or 1, then it is no longer true that $1 < |z_k| < \infty$, and a simple limiting argument is required in what follows, To compute the b_{ik}, apply (26) to the eigenvectors; for each i,

$$(27) \qquad \delta_{ij} = \sum_{k=1}^{m} b_{ik}(\bar{\lambda}_k^{-1} - \lambda_j)^{-1} \ , \quad 1 \leq j \leq m \ .$$

Solving this system, we get

$$(28) \qquad \frac{|b_{ii}|}{|z_i|-1} = (1+|\lambda_i|) \prod_{j \neq i} \frac{|1-\bar{\lambda}_j \lambda_i|^2}{|\lambda_j - \lambda_i|^2}$$

$$(29) \qquad \frac{|b_{ik}|}{|z_k|-1} = (1+|\lambda_k|)(1+|\lambda_i|) \frac{1-|\lambda_i|}{|\lambda_k-\lambda_i|} \frac{|1-\bar{\lambda}_k \lambda_i|}{|\lambda_k-\lambda_i|} \prod_{j \neq i,k} \frac{|1-\bar{\lambda}_j \lambda_i||1-\bar{\lambda}_k \lambda_j|}{|\lambda_j-\lambda_i||\lambda_j-\lambda_k|} \ ,$$

$$k \neq i \ .$$

For any distinct i and j,

(30) $$\left|\frac{1-\bar{\lambda}_j\lambda_i}{\lambda_j-\lambda_i}\right| = \left|\bar{\lambda}_j + (1+|\lambda_j|)\frac{1-|\lambda_j|}{\lambda_j-\lambda_{3-i}}\right| \leq 1 + 2\delta .$$

Putting the pieces together,

(31) $$|L_i v| \leq \sum |b_{ik}|\,|(z_k I-A)^{-1}v| \leq \sum \frac{|b_{ik}|\,C(v)}{|z_k|-1}$$

$$\leq [2(1+2\delta)^{2m-2} + (m-1)4\delta(1+2\delta)^{2m-3}]\,C(V) .$$

Simplifying the last term and using (25),

(32) $$(Hv,v)^{1/2} \leq m(2+4m\delta)(1+2\delta)^{2m-3}\,C(v) .$$

To complete the theorem, we introduce the left (row) eigenvectors r_k, so that

(33) $$Av_j = \lambda_j v_j , \quad r_k A = \lambda_k r_k .$$

Multiplying the first by r_k and the second by v_j, there is the familiar biorthogonality condition

(34) $$r_k v_j = (0) \text{ for } j \neq k .$$

Since v_j cannot be orthogonal also to r_j, we may fix the eigenvectors by the normalization

$$(35) \qquad m|v_j|^2 = 1 \quad \text{and} \quad r_j v_j = (1) \ , \quad 1 \le j \le m \ .$$

It follows that

$$(36) \qquad L_i = v_i r_i \ ,$$

since both sides, applied to v_j, give $\delta_{ij} v_j$.

Now let the rows of S be $r_1, \ldots, r_m$, so that SAS^{-1} is diagonal. By matrix multiplication

$$(37) \qquad S^*S = \begin{pmatrix} r_1^* & \cdot & \cdot & r_m^* \end{pmatrix} \begin{pmatrix} r_1 \\ \cdot \\ \cdot \\ r_m \end{pmatrix} = \sum_1^m r_i^* \, r_i \ .$$

Using (35) and (36), this is precisely

$$(38) \qquad \sum_1^m r_i^*(m v_i^* \, v_i) r_i = m \sum L_i^* \, L_i = H \ .$$

Finally, we have to return and admit eigenvalues λ_i of modulus one and multiplicity $M > 1$. From the resolvent condition iv), λ_i possesses M linearly independent corresponding eigenvectors; one puts A in Jordan form to compute the resolvent $(zI-A)^{-1}$, and then lets z approach λ_i. The eigenvectors may still be chosen to satisfy (34).

Let us number the eigenvalues so that $\lambda_1, \ldots, \lambda_N$ are distinct, and the rest are duplicates of these. Then instead of (19) we want

(39) $$L_i = \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{A-h_j}{\lambda_i - \lambda_j} , \quad i = 1,\ldots,N .$$

Simply replacing m by N in all the equations (20) to (32), the first part of the proof continues to hold. In place of (35) and (36), we have

(40) $$N|v_j|^2 = 1 \text{ and } r_j v_j = (1) , \quad 1 \leq j \leq m$$

(41) $$L_i = v_{i_1} r_{i_1} + \cdots + v_{i_M} r_{i_M} ,$$

where $\lambda_{i_1},\ldots,\lambda_{i_M}$ are the appearances of the eigenvalue λ_I. Then we may once more identify

(42) $$S^*S = \sum_1^m r_i^* r_i = \sum_1^m r_i^*(Nv_i^* v_i)r_i = N \sum_1^N L_i^* L_i = H .$$

Notice that when all $|\lambda_j| = 1$ we may take $\delta = 0$, so that $(Hv,v)^{1/2} \leq 2m\, C(v)$. Even this estimate is too large, since McCarthy [5] has shown that in this special case iii) implies v) with $K(m) \equiv 1$. A similar comment applies to Theorem 4, and is especially relevant for hyperbolic equations, in which Re $\lambda_j(P(\omega)) \equiv 0$ by definition,

4. To complete the proof of Theorem I, it remains to show that iv) implies v). From iv) we know the eigenvalues satisfy $|\lambda_j| \leq 1$; we shall put them into clusters as Morton [3] has done, Into the cluster

C_1 goes an eigenvalue, say λ_1, of largest modulus, together with all others that can be connected to λ_1 by a chain of eigenvalues, each link having length less than $(1-|\lambda_1|)/4m$. C_2 is formed in the same way from the remaining eigenvalues, and so on until every eigenvalue enters one of the clusters $C_1,\ldots,C_r$. Of course $r < m$; when $r = m$, our basic constructions coincide with those in Theorem I, Notice that an eigenvalue of modulus one and multiplicity M appears alone in M clusters.

Let us suppose that

$$\rho(A) < 1 \text{ and } \lambda_i \neq \lambda_j \quad \text{for } i \neq j \ , \tag{43}$$

and remove this hypothesis later by a continuity argument.

We want to associate with each cluster several matrices from which to construct H. Given the cluster C_α, let λ_α be the eigenvalue of largest modulus in C_α from which the cluster was formed. Recalling the projections L_i defined in (19), let

$$I_\alpha = \sum L_i \ , \quad A_\alpha = \sum \lambda_i L_i \ , \quad B_\alpha = \sum \frac{2(\lambda_i-\lambda_\alpha)}{1-|\lambda_\alpha|} L_i \ , \tag{44}$$

summing over the indices i such that $\lambda_i \in C_a$. Define

$$H_\alpha = I^*_\alpha I_\alpha + \sum_1^\infty (B^*_\alpha)^n (B_\alpha)^n \ . \tag{45}$$

From (20), I_α acts like the identity relative to C_a, and matrices associated with different clusters are orthogonal, In particular, we write down

$$(46) \qquad I_\alpha^2 = I_\alpha, \; I_\alpha B_\alpha = B_\alpha, \; I_\alpha A_\alpha = A_\alpha; \; I_\alpha A_\beta = B_\alpha A_\beta = H_\alpha A_\beta = 0, \quad \alpha \neq \beta \; .$$

From the definitions it follows that

$$(47) \qquad A_\alpha = \lambda_\alpha I_\alpha + \frac{1-|\lambda_\alpha|}{2} B_\alpha$$

$$(48) \qquad I^*_\alpha H_\alpha I_\alpha = H_\alpha, \; B^*_\alpha H_\alpha B_\alpha = H_a - I^*_\alpha I_\alpha \leq H_a \; .$$

Then from the appropriate triangle inequality

$$(49) \qquad A^*_\alpha H_\alpha A_\alpha \leq \left(|\lambda_\alpha| + \frac{1-|\lambda_\alpha|}{2} \right)^2 H_\alpha \leq \left(\frac{1+\rho(A)}{2} \right)^2 H_\alpha \; .$$

From (21) we see at once that

$$(50) \qquad \sum_1^r I_\alpha = I \, , \; \sum_1^r A_\alpha = A \; .$$

Now the matrix we want is just

$$(51) \qquad H = m \sum_1^r H_\alpha \; .$$

'Combining the last three equations with (46),

$$(52) \qquad A^*HA = \sum A^*_\alpha \, m \sum H_\beta \sum A_\gamma = m \sum A^*_\alpha H_\alpha A_\alpha \leq \left(\frac{1+\rho(A)}{2} \right)^2 H \; .$$

To see that $H \geq I$ we use the Schwarz inequality to compute

$$(53) \qquad |v|^2 = \left| \sum_1^r I_\alpha v \right|^2 \leq r \sum |I_\alpha v|^2 \leq m \sum (H_\alpha v, v) = (Hv, v) \; .$$

The essential problem is to bound

$$(54) \qquad (Hv, v) = m \sum_{\alpha} \left(|I_\alpha v|^2 + \sum_{1}^{\infty} |B_\alpha^n v|^2 \right) .$$

There are two means of carrying out this estimate. Conceptually, the simplest possible approach is to expand I_α and B_α^n as sums of resolvents, just as L_i was expanded in (26), and then apply iv). Unfortunately, the choice of the z_k has to be more complicated than it was there, and the consequent algebra is a sorry mess. Therefore we adopt a more economical alternative; with some minor refinements, the estimates we need can be lifted from those made by Morton [3]. We denote his equations by an added asterisk.

Morton's final result is

$$(55) \qquad \text{iv)} \implies |A^n v| \le K_1(m) \sup C(v) ,$$

but his proof works without requiring the supremum on the right side, by noticing the action on each v in (13*)-(16*) and (18*). Furthermore, his estimate of $A^n v$ is found precisely by bounding the contribution from each cluster; thus when n = 0, i.e., v = 0 in (18*),

$$(56) \qquad |I_\alpha v| \le K_2(m)\, C(v)$$

and also when n > 0,

$$(57) \qquad |A_\alpha^n v| \le K_2(m)\, C(v) .$$

Now we introduce one more matrix associated with C_α:

(58) $$D_\alpha = A_\alpha + \lambda_\alpha(I-I_\alpha) .$$

From the identities (46), we know

(59) $$D_\alpha^n = A_\alpha^n + \lambda_\alpha^n (I-I_\alpha) , \quad n > 0 .$$

According to (56) and (57),

(60) $$|D_\alpha^n v| \leq K_3(m) C(v) , \quad n \geq 0 .$$

Then the implication iii) $\Longrightarrow$ iv) gives

(61) $$|(zI-D_\alpha)^{-1}v| \leq \frac{K_3(m) C(v)}{|z|-1} , \quad |z| > 1 .$$

Manipulating with the definitions, we find

(62) $$(zI-B_\alpha)^{-1} = \frac{1-|\lambda_\alpha|}{2} (z_\alpha I-D_\alpha)^{-1} , \quad z_\alpha = \lambda_\alpha + \frac{1-|\lambda_\alpha|}{2} z .$$

Let z lie on the circle Z_α of radius 1 about the point $4\lambda_\alpha/|\lambda_\alpha|$ (or 4, if $\lambda_\alpha = 0$). The minimum of $|z_\alpha|$ on this circle occurs when z is closest to the origin, and an easy computation gives

(63) $$|z_\alpha| - 1 \geq \frac{1-|\lambda_\alpha|}{2} , \quad z \text{ on } Z_\alpha .$$

Thus it follows from (61) - (63) that

(64) $$|(zI-B_\alpha)^{-1}v| \leq K_3(m)\ C(v)\quad , \quad z \text{ on } Z_\alpha .$$

From (44), the eigenvalues μ_i of B_α are

$$\mu_i = \frac{2(\lambda_i-\lambda_\alpha)}{1-|\lambda_\alpha|}\ ,\ \lambda_i \in C_\alpha\ ;\ \mu_i = 0\ ,\ \lambda_i \notin C_\alpha\ .$$

Since each $\lambda_i \in C_\alpha$ is connected to λ_α by a chain with fewer than m links,

(65) $$|\lambda_i-\lambda_\alpha| \leq m\ \frac{1-|\lambda_\alpha|}{4m} = \frac{1-|\lambda_\alpha|}{4}\ .$$

Thus for all i,

(66) $$|\mu_i| \leq 1/2\ .$$

Using only (64) and (66), we will obtain the required bound (70); this result may have some independent interest. Looking a second time at Morton's argument, we put all the μ_i into one cluster, so his $\chi \equiv 1$. Denoting by D^p a divided difference formed at some p + 1 of the points μ_i, (11*) becomes

(67) $$|D^p(z^n)| \leq n^p(\tfrac{1}{2})^{n-p}\ .$$

Carrying out the contour integration (14*) over Z_α and applying (64), (16*) simplifies for $q<m$ to

(68) $$|D^q(P)| \leq K_4(m)\ C(v)\ .$$

Here $P(z) = (zI-B_\alpha)^{-1} \prod (z-\mu_i)$ is a matrix polynomial of degree less than m. As in (4*), $B_\alpha^n v$ is just the divided difference of order m-1 of the product $z^n P(z)v$ formed at the μ_i. Constructing a Leibnitz rule, this divided difference is the sum of 2^{m-1} products, each bounded by

$$(69) \qquad |D^p(z^n)D^{m-p-1}(P(z)v)| \le n^{m-1} (\tfrac{1}{2})^n K_5(m)\, C(v) \;.$$

Consequently

$$(70) \qquad |B_\alpha^n v| \le n^{m-1}(\tfrac{1}{2})^n K_6(m)\, C(v) \;.$$

Substituting (70) and (56) into (54), the infinite series converges to give the final estimate

$$(71) \qquad (Hv,v)^{1/2} \le K(m)\, C(v) \;.$$

We still have to eliminate the hypothesis (43). It is easy to choose M (after triangularizing A, for example) so that

$$A_\epsilon = (1-\epsilon)A + \epsilon^2 M$$

satisfies (43) as $\epsilon \to 0_+$. Then for $|v| = 1$ it follows from iv) that

$$(72) \qquad (|zI-(1-\epsilon)A)^{-1}v| \le \frac{C(v)}{|z|-(1-\epsilon)} \le \min\left(\frac{C(v)}{|z|-1}, \frac{C}{\epsilon}\right)$$

for $|z| > 1$, where $C = \sup c(v)$* (The uniform boundedness theorem applied to iv) assures that $C(v)$ can be chosen so that $C < \infty$.)

Therefore

$$(73)\qquad |(zI-A_\epsilon)^{-1}v| = |\sum_0^\infty [\epsilon^2(zI-(1-\epsilon)A)^{-1}M]^n (zI-(1-\epsilon)A)^{-1}v|$$

$$\leq \frac{1}{1-\epsilon C|M|} \cdot \frac{C(v)}{|z|-1} .$$

Since (43) holds for A_ϵ, there is an $H_\epsilon > I$ with

$$(74)\qquad A_\epsilon^* H_\epsilon A_\epsilon \leq \left(\frac{1+\rho(A_\epsilon)}{2}\right)^2 H_\epsilon;\ (H_\epsilon v,v)^{1/2} \leq \frac{K(m)\ C(v)}{1-\epsilon C|M|} .$$

As $\epsilon \to 0$, some subsequence of H_ϵ converges by compactness to an $H \geq I$, and taking the limit in (74) gives v).

5. In this section, we establish the italicized statement about **S** which follows Theorem I. Again we start by assuming (43), and we recall the left eigenvectors r_k defined in (33). Suppose we now number the eigenvalues in the order that they fall into clusters, and let C_1 contain $\lambda_1,\ldots,\lambda_q$. We want to prove that $H_1 = S_1^* S_1$, where the first q rows of S_1 are linear combinations of $r_1,\ldots,r_q$ and the other **m-q** rows are zero. From the definition (45),

$$(75)\qquad H_1 v_k = 0 \quad \text{for} \quad k > q\ ,\quad \text{rank}\ (H1) = q\ .$$

Writing $H_1^{1/2}$ for the positive semi-definite square root,

$$(76)\qquad |H_1^{1/2} v_k|^2 = (H_1 v_k, v_k) = 0 \quad \text{for } k > q\ .$$

By (34), $r_1,\ldots,r_q$ span the orthogonal complement of the space generated by $v_{q+1},\ldots,v_m$. Therefore each row of $H_1^{1/2}$ is a combination of $r_1,\ldots,r_q$. Let V be the space spanned by the columns of $H_1^{1/2}$. We construct orthonormal bases $u_1,\ldots,u_q$ and $u_{q+1},\ldots,u_m$ for V and $V^{\perp}$. Taking the u_i as the rows of a unitary matrix U_1, we have shown that $S_1 = U_1 H_1^{1/2}$ has the required properties; of course

$$S_1^* S_1 = H_1^{1/2} U_1^* U_1 H_1^{1/2} = H_1 \text{ "}$$

For every C_α, we construct in the same way an S_a satisfying $H_\alpha = S_a^* S_a$; row j of S_α is non-zero if and only if $\lambda_j \in C_\alpha$. Then defining $\bar{S} = m^{1/2} \sum S_\alpha$, and recalling the multiplication rule (37), we have $\bar{S}^* \bar{S} = H$.

Let $\bar{A} = \bar{S} A \bar{S}^{-1}$. Since the first row of $\bar{S}$ is by construction a combination of $r_1,\ldots,r_q$, and $r_k A = \lambda_k r_k$, the same is true of the first row of $\bar{S}A$. This must coincide with the first row of $\bar{A}\,\bar{S}$, which is a combination with weights $\bar{4}_j$ of the rows of $\bar{S}$. Again by construction, the rows of $\bar{S}$ after row q are combinations of $r_{q+1},\ldots,r_m$. Using the linear independence of the r_k and also of the rows of $\bar{S}$, we conclude that $\bar{4}_j = 0$ for $j > q$. In the same way, $\bar{A}_{ij} = 0$ whenever λ_i and λ_j are in different clusters. Therefore

$$\bar{A} = \begin{pmatrix} A_1 & & O \\ & \ddots & \\ O & & A_r \end{pmatrix},$$

the square block $\bar{A}_\alpha$ on the diagonal corresponding to the cluster C_α. With a final unitary similarity $\bar{U}$ of the same block form, we triangularize each $\bar{A}_\alpha$ separately. Thus with $S = \bar{U} \cdot \bar{S}$, we have $H = S^*S$, and $A' = SAS^{-1}$ has the required (triangular, block diagonal) form.

We have still to estimate the off-diagonal entries of A.'. Denoting by a prime the result of applying the similarity S, we conclude from the reasoning of the previous paragraph that A'_α, I'_α, B'_α and $L'_i(\lambda_i \in C_\alpha)$ all have zero entries outside block α. Since I'_α is the sum of the right number of mutually orthogonal projections L'_i, we know that I'_α is just the identity matrix in its block. Therefore by (47) the off-diagonal entries are introduced through B'_α. According to (48), $|B'_\alpha| \leq 1$, and the same must be true of all its entries. Then the off-diagonal entries of A'_α are bounded by $(1-|\lambda_\alpha|)/2 < (1-|\lambda_i|)/2$, $\lambda_i \in C_\alpha$.

Again we must circumvent (43). Recall that the sequence $A_\epsilon \to A$ led to a subsequence $H_\epsilon \to H$; for each H_ϵ we have seen how to construct S_ϵ, and taking a further subsequence, we get $S_\epsilon \to S$, where $S^*S = H$. Unless (43) is violated by a repeated eigenvalue of modulus one, the clusters for A_ϵ and A coincide for small ϵ. Therefore the limit matrix S gives an $A' = SAS^{-1}$ with the right properties. In case A has a repeated eigenvalue with $|\lambda_j| = 1$, we still know A' is upper triangular and $|A'| = 1$; but from this the off-diagonal entries in the rows containing λ_j must vanish, and once more A' is all right.

It is worth remarking that in v), H and S cannot be made continuous functions of A. The family

$$A_7 = \begin{pmatrix} e^{i\gamma} & |\gamma| \\ 0 & 1 \end{pmatrix} , \quad 7 \text{ real}$$

satisfies iv) with some $C(v)$ independent of 7. Since the eigenvalues of A_7 have modulus one, A_7 must be diagonal with respect to H_7 to satisfy $A^*_\gamma H_\gamma A_\gamma \leq H_7$. However, one of the eigenvectors of A_7 is discontinuous at $7 = 0$, from which one easily verifies that H_7 is too.

6. With the definitions

$$\tau(P) = \max \operatorname{Re} \lambda_j(P) , \quad \operatorname{Re} P = \frac{P+P^*}{2} \tag{78}$$

we can state the analogues of Theorems I and II for the exponential case.

Theorem III. For a suitable $K'(m)$ depending only on the order m of the matrix P, each of the following statements implies the next:

i') $HP + P^*H \leq 0$ for some $H > I$ with $(Hv,v)^{1/2} = c(v)$ for $|v| = 1$.

ii') $\operatorname{Re} SPS^{-1} \leq 0$ for some S with $|S^{-1}| \leq 1$ and $|Sv| = C(v)$ for $|v| = 1$.

iii') $|e^{Pt}v| \leq C(v)$ for all $t \geq 0$ and $|v| = 1$.

iv') $|(zI-P)^{-1}v| \leq \dfrac{C(v)}{\operatorname{Re} z}$ for $\operatorname{Re} z > 0$ and $|v| = 1$.

v') $HP + P^*H \leq \tau(P)H \leq 0$ for some $H > I$ with $(Hv,v)^{1/2} \leq K'(m)\, C(v)$ for $|v| = 1$.

Theorem IV. Suppose iv') holds, and the eigenvalues of P satisfy

$$(79) \qquad \delta|\lambda_i-\lambda_j| \geq -\mathrm{Re}\,\lambda_j \quad \text{for all distinct } i,j \,.$$

Then $HP + P^*H \leq 2\,\tau(P)\,H \leq 0$ for some $H > I$ with

$$(Hv,v)^{1/2} \leq m(2+4m\delta)(1+2\delta)^{2m-3}\,C(v)$$

for $|v| = 1$. Furthermore, there exists S such that $H = S^*S$ and SAS^{-1} is diagonal.

Of course Theorem IV goes almost exactly as Theorem II did; one makes the choice $z_k = -\bar{\lambda}_k$ in (26), as in the original paper by Kreiss [9], and recomputes (28), (29), and (30).

In Theorem III, the step iii') $\Longrightarrow$ iv') involves the Laplace transform in place of the power series in (17):

$$(80) \quad |(zI-P)^{-1}v| = \left|\int_0^\infty e^{-zt}e^{Pt}v\,dt\right| \leq C(v)\int_0 e^{-t\,\mathrm{Re}\,z}dt = C(v)/\mathrm{Re}\,z \,.$$

The cluster C_1' is now formed by starting with an eigenvalue λ_1 of largest real part (necessarily ≤ 0 by iv')) and connecting to it those eigenvalues which can be reached with links less than $-\mathrm{Re}\,\lambda_1/4m$. Then $C_2', \ldots, C_r'$ are formed in the same way, In analogy with (43) we may temporarily assume that

$$(81) \qquad \tau(P) < 0 \text{ and } \lambda_i \neq \lambda_j \text{ for } i \neq j \,,$$

and then remove this restriction as before. Now we can define

$$(82) \qquad I_\alpha = \sum L_i \,, \quad P_\alpha = \sum \lambda_i L_i \,, \quad G_a = \sum \frac{2(\lambda_\alpha - \lambda_i)}{\mathrm{Re}\,\lambda_\alpha} L_i \,,$$

summing over indices i such that $\lambda_i \in C'_\alpha$. Next we let

$$(83) \qquad H_\alpha = I^*_\alpha I_\alpha + \sum_1^\infty (G^*_\alpha)^n (G_\alpha)^n \,, \quad H = m \sum H_\alpha \,.$$

From the orthogonality of the L_i, it follows as usual that

$$(84) \qquad H_\alpha I_\alpha + I^*_\alpha H_\alpha = 2H_\alpha \,.$$

Obviously for $n > 0$

$$(85) \qquad (G^*_\alpha)^n (G_\alpha - I_\alpha)^* (G_\alpha - I_\alpha)(G_\alpha)^n \geq 0$$

or in other words,

$$(86) \qquad (G^*_\alpha)^n (G_\alpha)^{n+1} + (G^*_\alpha)^{n+1} (G_\alpha)^n \leq (G^*_\alpha)^{n+1} (G_\alpha)^{n+1} + (G^*_\alpha)^n (G_a)^n \,,$$

where the last term is to be interpreted as $I^*_\alpha I_\alpha$ when $n = 0$. Summing (86) from 0 to ∞,

$$(87) \qquad H_\alpha G_\alpha + G^*_\alpha H_\alpha \leq 2 H_\alpha \,.$$

From (82) we have

(88) $$P_\alpha = \lambda_\alpha I_\alpha - \frac{\operatorname{Re} \lambda_\alpha}{2} G_\alpha ,$$

so that (84) and (87) yield

(89) $$H_\alpha P_\alpha + P_\alpha^* H_\alpha \leq \operatorname{Re} \lambda_\alpha H_\alpha \leq \tau(P) H_\alpha .$$

Summing on α and using orthogonality,

(90) $$HP + P^*H \leq \tau(P) H .$$

The inequality $H \geq I$ is (53), and we have now to estimate (Hv, v). This time there are three possibilities. The first two — to expand I_α and G_α^n as sums of resolvents, or to repeat the argument of Theorem I with appropriate changes — would be safe but tedious. Therefore we shall try to derive the estimate from Theorem I itself, using only some essential remarks about its proof. In fact, we now give a complete proof of the last step in Theorem III <u>without</u> using the H defined explicitly in (83), and then identify the new $\bar{H}$ with that H.

For a given positive integer k, let $w = e^{z/k}$, so that $\operatorname{Re} z > 0 \iff |w| > 1$. Then as in (73)

(91) $$|(wI - e^{P/k})^{-1} v| = k |(zI - P + F_{k,z})^{-1} v|$$

$$\leq \frac{k|(zI-P)^{-1} v|}{1 - |(zI-P)^{-1}| \, |F_{k,z}|} \leq \frac{C(v)}{\operatorname{Re} z/k} \bullet \frac{1}{1 - \frac{,}{\operatorname{Re} z}} T$$

$$\leq \frac{1}{|w|-1} \; \frac{C(v)}{1 - \frac{C|F_{k,z}|}{\operatorname{Re} z}} ,$$

where we used $u < e^u - 1$ for real u. Estimating the perturbation $F_{k,z}$,

(92) $$|F_{k,z}| = k|(e^{z/k}-1-\frac{z}{k})I-(e^{P/k}-I-\frac{P}{k})| = O(\frac{1}{k})$$

as $k \to \infty$, uniformly for z in a compact set Z. If Re z > 0 in Z, we have

(93) $$C_k(v) = \sup_Z \frac{C(v)}{I - \frac{C|F_{k,z}|}{\text{Re } z}} \to C(v) \text{ as } k \to \infty .$$

We want to deduce from (91) that Morton's result (55) holds for $A=e^{P/k}$, in the strong form

(94) $$|e^{Pn/k}v| \leq K_1(m)\, C_k(v) \text{ for } n \geq 0 , |v| = 1 .$$

Then Theorem 1 provides an explicit $H_k \geq I$ such that

(95) $$(H_k v, v)^{1/2} \leq K(m)K_1(m)C_k(v)$$

(96) $$e^{P^*/k}H_k e^{P/k} \leq \left(\frac{1+\rho(e^{P/k})}{2}\right)^2 H_k = \left(\frac{1+e^{\tau(P)/k}}{2}\right)^2 H_k .$$

As $k \to \infty$, some subsequence H_{k_j} converges to a limit $\overline{H} \geq I$, with

(97) $$(\overline{H}v,v)^{1/2} \leq K(m)K_1(m)C(v) = K'(m)C(v) .$$

Expanding (96) in powers of k, subtracting H_k, multiplying by k, and taking the limit as $k_j \to \infty$, we get

$$(98) \qquad \overline{H}P + P^*\overline{H} \leq \tau(P)\overline{H} \quad .$$

All this is justified if, in applying Morton's argument to $e^{P/k}$, we actually need the estimate (91) only for z in a compact set Z in the right half-plane. It turns out that this is actually the case. Morton uses the resolvent condition in the contour integrations (14*), where $w = e^{z/k}$ lies on circles with

$$(99) \quad \text{radius} = \delta_a = 1 - e^{\operatorname{Re} \lambda_\alpha / k} \leq \frac{-\operatorname{Re} \lambda_\alpha}{k} , \quad \text{center} = (1+2\delta_\alpha)e^{i \operatorname{Im} \lambda_\alpha / k} \quad .$$

On this contour it is easy to bound z by Re λ_α and Im λ_α.

To make the identification $\overline{H} = H$, we want to match the clusters C'_α derived from P with the clusters C_α derived from $e^{P/k}$, k large, Clearly λ_α of maximum real part corresponds to $e^{\lambda_\alpha/k}$ of maximum modulus, and also the ratios which arise in forming clusters satisfy

$$(100) \qquad \frac{1 - |e^{\lambda_\alpha/k}|}{4m|e^{\lambda_i/k} - e^{\lambda_j/k}|} \to \frac{\operatorname{Re} \lambda_\alpha}{4m|\lambda_i - \lambda_j|} \quad \text{as } k \to \infty \quad .$$

Therefore $\lambda_i \in C'_\alpha$ if and only if $e^{\lambda_i/k} \in C_\alpha$, if we exclude eigenvalues of equal real part (which may make the choice of λ_α ambiguous) and also exclude the possibility that the limiting ratio in (100) is one, With these exceptions,

$$(101) \qquad B_\alpha \doteq \sum \frac{2\left|e^{\lambda_i/k} - e^{\lambda_\alpha/k}\right|}{1 - \left|e^{\lambda_\alpha/k}\right|} L_i \to G_\alpha = \sum \frac{2(\lambda_\alpha - \lambda_i)}{\text{Re } \lambda_\alpha} L_i$$

and $\bar{H} = \lim H_k = H$. In the excluded cases, as in the case when (81) fails, the proper estimate for (Hv,v) follows by a continuity argument.

Repeating the proof in Section 5, we can describe a further property of H:

For some S with $S^*S = H$, $P' = SPS^{-1}$ is upper triangular, with $P'_{ij} = 0$ unless λ_i and λ_j are in the same cluster $\mathcal{G}'_\alpha$ and $|P'_{ij}| \le \frac{1}{2} \min(-\text{Re } h_i, -\text{Re } \lambda_j)$.

There is one additional consequence of our method of proof which is significant in the applications to partial differential equations:

The conclusions in v) and v') may be changed to

$$A^*HA \le \left(\frac{2-\theta+\theta\rho(A)}{2}\right)^2 H \quad \text{and} \quad HP + P^*H \le \theta\tau(P)H ,$$

where $0 \le \theta < 2$ and the constants K and K' depend on θ as well as m.

It follows that our space $L_2(H)$, over which (5) is to be well-posed, does not depend on the constant multiple of the identity which was subtracted in order to make $\sigma < 0$. In other words, the minimal renorming families $H(\omega)$ used to achieve (7) are equivalent for any two choices $\alpha > \sigma$.

7. We want finally to extend Theorem I to apply to matrices such that $\rho(A) = 1$ but A^n is unbounded; this occurs if and only if some eigenvalue of modulus one has a non-simple elementary divisor, and consequently too few corresponding eigenvectors. The standard example is

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

It is easy to see that all the conditions i) – v) fail for A_1, no matter how large $C(v)$ is chosen; in particular, $|A_1^n|$ grows like n and the resolvent has a double pole at $z = 1$. The following result shows that such a relationship is typical.

Theorem V. There exist constants $\alpha(s)$ and $\beta(s)$ depending on $s > 0$, such that with $A_\epsilon = \epsilon A$ and the constant K(m) as in Theorem I, each of the following statements implies the next:

i") For $\frac{1}{2} < \epsilon < 1$, $A_\epsilon^* H_\epsilon A_\epsilon < He$ for some $H_\epsilon \geq I$ with $(H_\epsilon v, v)^{1/2} < C(v)/(1-\epsilon)^s$ for $|v| = 1$.

ii") For $\frac{1}{2} < \epsilon < 1$, $|S_\epsilon A_\epsilon S_\epsilon^{-1}| < 1$ for some S_ϵ with $|S_\epsilon^{-1}| < 1$ and $|S_\epsilon v| < C(v)/(1-\epsilon)^s$ for $|v| = 1$.

iii") $|A^n v| \leq \alpha(s)(n+1)^s C(v)$ for $n \geq o$ and $|v| = 1$.

iv") $$|(zI-A)^{-1}| < \frac{\alpha(s)\beta(s)\,|z|^s C(v)}{(|z|-1)^{s+1}}$$ for $|z| > 1$ and $|v| = 1$.

v") For $\frac{1}{2} < \epsilon < 1$, there exists $H_\epsilon > I$ such that

$$A_\epsilon^* H_\epsilon A_\epsilon < H_\epsilon \text{ and } (H_\epsilon v,v)^{1/2} \leq \frac{\alpha(s)\beta(s)K(m)C(v)}{(1-\epsilon)^s} \text{ for } |v| = 1.$$

Proof, The first two conditions are equivalent as before with $H_\epsilon = S^*_\epsilon S_\epsilon$. Given ii″), we have for $|v| = 1$

(103) $$|A^n_\epsilon v| \le C(v)/(1-\epsilon)^s, \quad \tfrac{1}{2} < \epsilon < 1$$

(104) $$|A^n v| \le C(v)/\epsilon^n(1-\epsilon)^s \le \alpha(s)(n+1)^s C(v), \quad n>0,$$

by maximizing the denominator with respect to ϵ. It follows that iv') holds; for $|z| > 1$,

(105) $$|(zI-A)^{-1}v| = \left|\sum_0^\infty \frac{A^n v}{z^{n+1}}\right| \le \alpha(s)\ C(v) \sum_0^\infty \frac{(n+1)^s}{|z|^{n+1}}$$

$$\le \frac{\alpha(s)\beta(s)C(v)|z|^s}{(|z|-1)^{s+1}} .$$

In order to apply Theorem I, we compute

(106) $$|(zI-A_\epsilon)^{-1}v| = \left|\frac{1}{\epsilon}\left(\frac{z}{\epsilon} I-A\right)^{-1}v\right|$$

$$\le \frac{\alpha(s)\beta(s)C(v)|z/\epsilon|^s}{\epsilon(|z/\epsilon|-1)^s} = \frac{\alpha(s)\beta(s)C(v)|z|^s}{(|z|-\epsilon)^{s+1}}$$

$$\le \frac{\alpha(s)\beta(s)C(v)}{|z|-1}\left(\frac{|z|}{|z|-\epsilon}\right)^s \le \frac{\alpha(s)\beta(s)C(v)}{(|z|-1)(1-\epsilon)^s} .$$

Now the last step in Theorem I yields v″).

We leave to the reader the exponential analogue of Theorem V, which arises naturally in the attempt to take $\alpha = \sigma$ in (7). When equality is impossible to achieve, as it is in our example (1), a sequence of norms H_ϵ with α decreasing to σ retains more information about the true growth of $e^{P(\omega)t}$ than any single norm--with respect to which $\alpha > \sigma$.

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