COMPUTATION OF THE PSEUDOINVERSE OF

A MATRIX OF UNKNOWN RANK

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Abstract

A program is described which computes the pseudoinverse, and other related quantities, of an m X n matrix A of unknown rank. The program obtains least square solutions to singular and/or inconsistent linear systems Ax = B, where $m \leq n$ or $m > n$ and the rank of A may be less than min(m,n).   A complete description of the program and its use is given, including computational experience on a variety of problems.

I.   Introduction.

A method for the computation of the pseudoinverse, and other related quantities, corresponding to an **m x n** matrix A of unknown rank **r,** has recently been described [5]. The method determines the pseudoinverse A^+ of A and a related matrix $A^\#$. The pseudoinverse has the property that given the linear system Ax = b, the solution $x_m = A^+b$ satisfies $\|Ax_m - b\| < \|Ax - b\|$ for all x, and $\|x_m\| \leq \|x\|$ for all x such that $\|Ax_m - b\| = \|Ax - b\|$. The minimum basic solution $x_b = A^\#b$ has the property that $\|Ax_b - b\| \leq \|Ax - b\|$ for all x, and x_b has at most **r** non-zero elements.

Thecomputational difficulty for this problem arises primarily because the rank **r** is not known. In particular, it may be difficult to assign the correct rank if one or more of the singular values of A are small but non-zero [3]. Several other recent papers [1], [2], [4], on the computation of the pseudoinverse have not considered this . important practical question.

The approach used here to handle this difficulty can be summarized as follows. The desired matrices $A^\#$ and A^+ are formed from a matrix **B,** which consists of linearly independent columns selected from A'.

We would like to determine B so that it spans the same space as A, in which case B will contain r columns. Suppose we have a matrix B_q with q linearly independent columns selected from A, (where $q < r$) and the corresponding approximation A_q^+ to A+. Adding another linearly independent column of A to B_4, giving B_{q+1}, should give an improved approximation A_{q+1}^+ to the pseudoinverse. However, due to roundoff error in the calculation it may turn out for an ill-conditioned system that the new approximation is actually worse in the sense that $\|AA_{q+1}^+ - I\| > \|AA_q^+ - I\|$. Such a test is made in the pseudoinverse determination with the result that the effective-rank of A (the number of columns in B) is the maximum possible consistent with minimizing the error $\|AA^+ - I\|$.

A closely related aspect of the method used here to compute the pseudoinverse is what might be called its "smoothing" property. In many practical situations one would like to obtain a solution to a linear system which is stable in the sense that small changes in the matrix elements do not cause large changes in the solution vector. In general, the solution $x = A^+b$, where A^+ is the true pseudoinverse, will not behave smoothly. In fact, the norm of x will increase without bound as a singular value of A approaches zero. This difficulty can be eliminated by imposing a predetermined upper bound on the norm of $(B'B)^{-1}$. This is accomplished by estimating the effect of adding a new column of A to B_q and only adding this new column to B_q if it does not cause any element of A_{q+1}^+ to exceed the bound. Details of this selection procedure and the manner in which it depends on the choice of the bound SUPER is discussed in the next section.

In Section 3 the use of the **Algol** program, written to perform this algorithm, is described and suggested values for the input parameters are given. The program use is illustrated by means of a sample problem. A large number of problems have been solved using this program. Several different kinds of tests have been performed:

a) Very ill-conditioned matrices like the segments of the **Hilbert** matrix [6] have given a clear example of the smoothing property of the method.

b) Random rectangular matrices of random sizes have been generated and the pseudoinverse have been computed. The sizes were allowed to vary between 1 and 25. In all the cases the results were satisfactory.

c) Same as in b) but with random ranks. In every case the rank was correctly determined by the program.

d) Random matrices of specified size covering a range of values of m and n were run in order to obtain time estimates for different size problems.

e) A number of least-square problems, i.e., with $m \gg n$ and only one right-hand side.

f) A variety of matrices for which an independent check on the accuracy of the solution was available.

Tests b) through f) showed that in reasonable problems in which the rank is well determined the program will work very well, while a) has shown that in very ill-conditioned cases the smoothing property of the method is effective.

These test results are discussed more fully in Section 4. The notation used in [5] will also be followed here.

Details of storage requirements are given in Section 5.

A copy of the program appears in the Appendix.

II.   Program Description.

The method used to compute $A^{\#}$ and A^{+} from B is essentially that given in Section 2 of [5]. For convenience we will repeat the key relations here. The pseudoinverse of the m x r matrix B of rank r is given by

$$(2.1) \qquad\qquad B^{+} = (B'B)^{-1} B'$$

The non-zero rows of the n x m matrix $A^{\#}$ then consists of the corresponding rows of B^{+}. An r x n matrix of rank r is also obtained from B^{+} according to

$$(2.2) \qquad\qquad C = B+A$$

Note that, if B contains all the independent columns of A, then A = BC. Finally, A^{+} is obtained from C and B^{+} by

$$(2.3) \qquad\qquad A^{+} = C'(CC')^{-1} B^{+}$$

The determination of B is based on the algorithm of Section 3 in [5], using the more sophisticated selection procedure described below.

The program consists essentially of two parts. One part has all the input-output and the other is a PROCEDURE called PSEUDOINVER which may also be used separately as a part of other programs'. The program solves the matricial problem,

$$(2.4) \qquad\qquad A X = R H S$$

where RHS is a matrix containing several right hand sides.

a)   The first part of PSEUDOINVER normalizes the matrix A by scaling each column so that its Euclidean norm is equal to one. The normalization constants are saved in order to get back to the original problem.

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The search for independent columns of A is then made to determine the matrix B, according to the formulas described in section 3 of [5]. At this stage, the condition for a vector to be accepted as independent of the ones already included in the basis is that α, the square of the norm of the projection on the orthogonal **subspace** to that basis, be less than a quantity ORTP, which is an input parameter. Later we will discuss the appropriate choice of ORTP and the other parameters appearing in the program.

As the columns chosen in this fashion might not necessarily be the first columns of A, a record is kept of the column number of the accepted vectors,

After all the columns have been inspected two situations can arise; either all the n columns of A have been accepted or some have been rejected. In the first case we have finished and the computations indicated at the beginning of this section are performed to get A+, $A^{\#}$, X_b and X_m. Other computed quantities are the residuals, $NXM = \|AX_m - RHS\|$ $NXB = \|AX_b - RHS\|$ corresponding to X_m and X_b, and EST $= \|BC - A\|$.

EST would be zero if the computation were performed exactly; in general EST will be very small for well-conditioned matrices and will increase with the ill-conditioning or if an almost dependent column is added to **B.**

If only $q < n$ columns of A are selected for inclusion in B then the basis thus constructed is called B_q and the second part of PSEUDOINVER is called.

b) The projections on the **subspace** orthogonal to B_q are computed for all the rejected columns. The Euclidean norm of each projection is computed,

$$(2.5) \qquad \alpha_j = \left\| (I - B_q B_q^+) \, a_j \right\|^2$$

and the column corresponding to the **maximum** α_j (the most independent one) is stored in SAV.

c) A test is now performed which is based upon an estimation of the norm that $(B'_{q+1} B_{q+1})^{-1}$ would have if we were to include SAV in the basis,, The norm used is $\|A\| = \max\limits_i \sum\limits_{j=1}^{n} |a_{ij}|$ and the estimate is derived from the f ormula,

$$(2.6) \qquad (B'_{q+1} B_{q+1})^{-1} = \left(\begin{array}{c|c} (B'_q B_q)^{-1} & 0 \\ \hline 0 & 0 \end{array} \right) + \alpha_{q+1}^{-1} \begin{pmatrix} u_q \\ \hline -1 \end{pmatrix} (u'_q \mid -1)$$

where $u_q = B_q^+ \, \text{SAV}$ and α_{q+1} is the square of the norm of the projection of SAV on the orthogonal subspace to that spanned by B_q. Then

$$\left\| (B'_{q+1} B_{q+1})^{-1} \right\| \leq \text{ESTIM} = \left\| (B'_q B_q)^{-1} \right\| + \alpha_{q+1}^{-1} \left(1 + \sqrt{q} \right)$$

If ESTIM is larger than SUPER (an input parameter) then SAV is rejected and B_q is taken as the final B .

This test avoids large elements in the pseudoinverse and gives the smoothing property discussed in the introduction.

d) If the test in c) is passed then the PROCEDURE GARBG, which computes all the matrices and quantities mentioned at the beginning of this section, is called and a second test is made. GARBG is used again, now with the basis B_q plus the column SAV . The test consists in comparing the values of $\|AX_m - \text{RHS}\|$, $\|AX_b - \text{RHS}\|$ and $\|BC - A\|$ obtained with one basis, with the corresponding ones obtained with the incremented basis. If all these values for B_{q+1} are smaller than for B_q then SAV is definitely

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accepted. After shifting all the useful quantities, part (b) is repeated for the new basis B_{q+1} and so on, until either an exit is provided for one of the tests or the columns of A are exhausted. All the scalar products are performed in double precision. The block diagram in Fig. I shows the most essential parts of the program.

It is worth noting that this strategy has been dictated by the problem itself and achieves the best numerical pseudoinverse possible using the method of [5] and taking into account the numerical roundoff error of the computer being used. This strategy takes advantage of the step by step algorithm for determining B, and constructs an independent basis, the degree of independence being determined by the parameter ORTP. By picking the most independent vector among the remaining ones, and checking to see if this decreases the residuals (by taking this vector in the basis) we are answering in a direct manner the two questions: how many columns of A do we need to minimize the residual? and, among all the possible sets of independent columns,which set gives the best representation of the pseudoinverse?

Fig. 1 - BLOCK DIAGRAM

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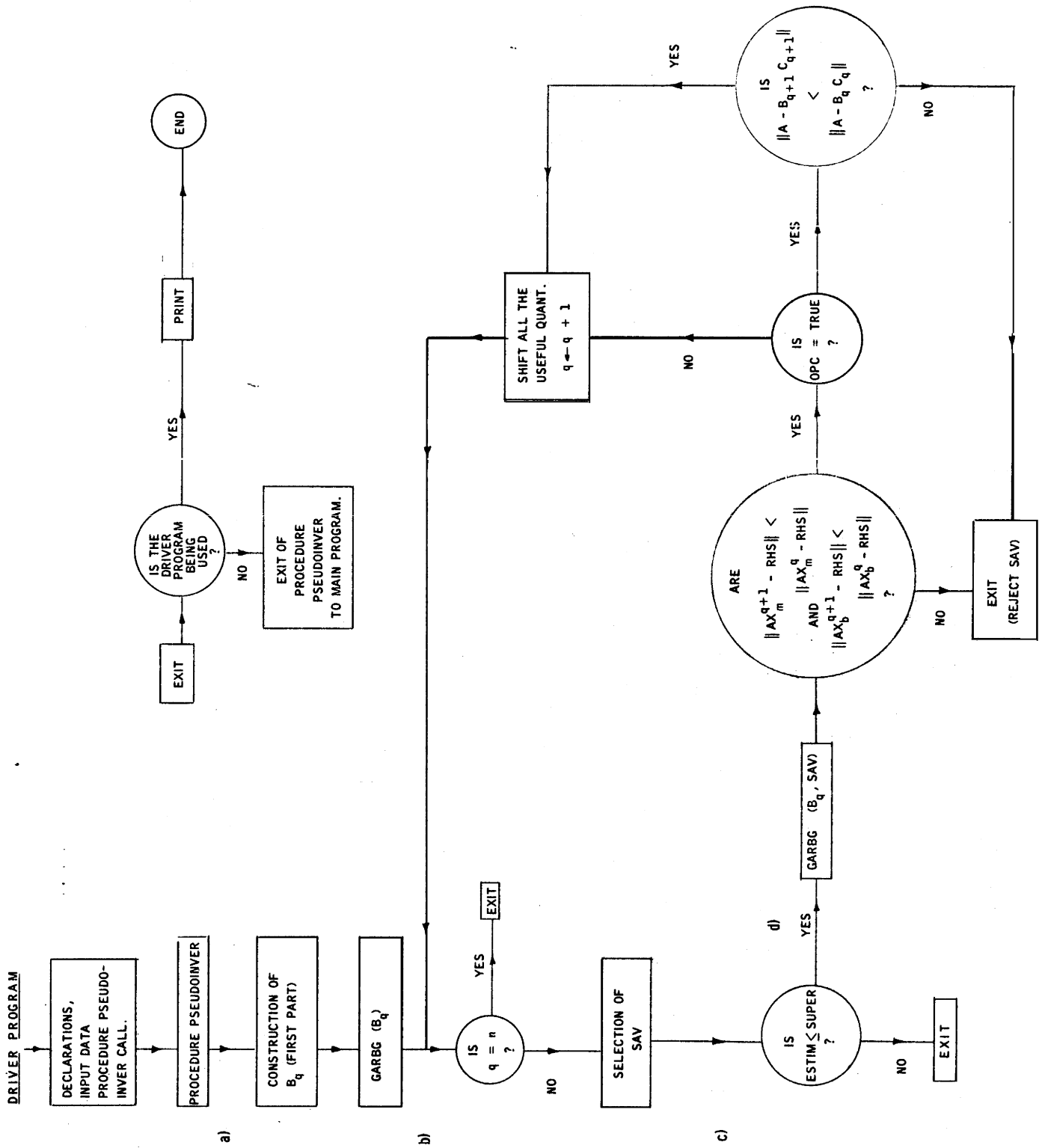
III.   Program Use.

As described in the previous section, several parameters are needed besides the matrix A . Now-we will explain the use and possibilities of these parameters.

Input.

M (integer) number of rows in A .

N (integer) -number of columns in A .

T (integer) number of right-hand sides.

OPC (Boolean) If OPC is equal to 1 then the program will compute the matrices A^+ and $A^\#$, and the right-hand side RHS = I(m x m) will be automatically provided.

Moreover, OPC decides if in the test described in Section 2,d) the quality of the representation (A = BC) is controlled. That test is done only if OPC = TRUE . If OPC is equal to FALSE then RHS an M x T matrix has to be provided and the program will compute $X_B = A^\#.RHS$, and $X_M = A^+.RHS$, matrices that will be printed out instead of A^+ and $A^\#$.

SUPER (real)   It is the SUPER of Section 2,c). If an upper bound for the elements of A^+ is known then SUPER can be set to this bound to take advantage of the smoothing property of this method; otherwise it is suggested that 10^{14} be used. It should be noted that, in general, a larger value than 10^{14}

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will increase computing time by throwing unnecessary decisions into the test of Section 2,d). On the other hand, much smaller values may completely eliminate from further consideration some columns which could be used to decrease the error,

ORTP (real)   This parameter was described in Section 2, a). Small values for ORTP (around 10^{-4}) in general will accelerate the process because the first part (construction of a basis of strongly independent column) is the fastest and as many columns as possible should be accepted there. Nevertheless, there are at least two cases in which a more careful choice of ORTP may be important. If higher precision in the answers is desired (at the cost of increased computing time), then a larger value of ORTP should be used, say 0.05. This will allow the second part of the program to choose "better" columns.

The other delicate case occurs when the matrix is very ill-conditioned and the rank is therefore not well defined. Here the use of a relatively'large ORTP is important. Again values around 0.05 are recommended.

Summarizing, in a reasonable, well behaved problem a recommended set of parameters is:

$$SUPER = 10^{14} \ , \ ORTP = 10^{-4} \ .$$

If the representation becomes very bad $(\|A - BC\|$ too large), the problem is not well behaved and more burden should be passed on to the second and safer test by increasing SUPER and decreasing ORTP .

If the user has information that certain variables are more significant than others, this information can be used by ordering the matrix A so that the columns of A corresponding to these variables appear first. This will insure that these columns are considered first for inclusion in the basis B .

If the complete program is used, then only the numerical data have to be punched,, This is done in the following way.

As all the read statements are in the FREE FIELD form, available in the EXTENDED ALGOL for the B5000 at Stanford, no special format is necessary. Numbers can be punched in any format, needing just one space in between to separate them.

1st. card:	M	N	T	O P C	SUPER	O R T P
for instance	10	10	10	1	@14	0.001

Next cards will contain the matrix A punched by rows. As each READ asks for a whole row, care must be taken not to mix different rows in the same card.

Finally, if 0 P C = 0 the right-hand sides (RHS) have to be provided and are read by columns. Each new column must be started on a new card, so that there will be at least T cards required for the RHS .

If the PROCEDURE is used separately, then all these quantities are input parameters (with the same names as above).

A complete sample input is given by,

$$
A \begin{cases}
\begin{array}{ccccccc}
3 & 3 & 1 & 0 & @14 & 0.001 \\
1.3 & 2 & & -5" & & \\
4 & 1 & & 0 & & \\
-1 & -1.3 & & @3 & &
\end{array}
\end{cases}
$$

$$
\text{RHS} \begin{cases}
\begin{array}{ccc}
1 & 2 & 2.1
\end{array}
\end{cases}
$$

<u>output</u>

All the matrices printed out by the program will have the following format:

Eight columns per line, each number in floating point with 6 significant digits. If the matrix is more than eight columns wide, then successive blocks will be printed in new pages. All the rows are printed together.

The output is described now in the order in which it will occur.

First the matrix A is printed out.

Then, if 0 P C = 0 , the right-hand sides are printed out.

Norm of (BC - A) .

The residuals $\|Ax_m^{(i)} - \text{RHS}^{(i)}\|$, $\|Ax_b^{(i)} - \text{RHS}^{(i)}\|$.

If 0 P C = 1 , then the matrix pseudoinverse is printed out with the format explained above; also in this case the non-zero rows of the matrix $A^{\#}$ are printed, each of them with a heading: ROW NUMBER.... .

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If O P C = O , then instead of these two last matrices, the minimum and basic solutions are printed out. As an example, we give the output for the problem:

$$\begin{pmatrix} 9 & 21 \\ 21 & 49 \end{pmatrix} X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

PSEUDOINVERSION OF THE MATRIX A . A IS 2 x 2 .

g. 00000 @ + 00   2.10000 @ + 01

2.10000 @ + 01   4.90000 @ + 01

[PAGE]

NORM OF (BC - A)   2.37582 @ - 12

RESIDUAL FOR XM

9.19145 @ - 01   3.93919 @ - 01

RESIDUAL FOR XB

9.19145 @ - 01   3.93919 @ - 01

[PAGE]

MATRIX APSEUDOINVERSE

2.67532 @ - 03   6.24314 @ - 03

6.24381 @ - 03   1.45794 @ - 02

[PAGE]

MATRIX ADAGGER

ROW NUMBER 1

1.72461 @ - 02   4.02332 @ - 02

END OF THE RUN

* * * * * * * * * * * * * * * * *

The PROCEDURE PSEUDOINVER.

The call for this PROCEDURE is,

PSEUDOINVER (M, N, TI, OPC, SUPER, ORTP, A, RHS, EST, NXM, NXB,

APSEUDO, ADAGER, COF, XM, XB);

The first 8 parameters are input parameters and they have been des-
cribed before. The only detail needed is: A(double real array
[O:M, O:N]), RHS [O:M, O:TI];

OUTPUT PARAMETERS:

EST (real) Contains $\|BC - A\|$.

NXM, NXB (single real arrays [O:TI]) . They contain the residuals
$\|AX_m - RHS\|$ and $\|AX_b - RHS\|$ respectively.

APSEUDO, ADAGER (Double real array [O:N, O:M])

They contain the pseudoinverse of A and the matrix $A^{\times}$.

COF (Single integer array [O:N]) .

If COF[I] = 0 then both, the Ith row of $A^{\#}$ and X_b are dif-
ferent from zero, otherwise they are zero and that means the program has
decided that the corresponding columns in A were linearly dependent
with respect to-the current basis.

XM, XB (double real arrays, [O:N, O:TI])

They contain the minimum and basic solutions.

IV.   Test Problems.

a)   Square segments of the Hilbert matrix have been tried, sizes

varying between 3 and 10.

For $5 \leq n \leq 10$ the rank found in each case was 4 .   The norm

of the pseudoinverses remained below 10^3 while for the true

pseudoinverse (the inverse in these cases) the norms ranged be-

tween 10^5 for n=5 and 10^{13} for n=10 . The norm, $\|A - BC\|$

was around 10^{-5} for all cases.

As is well known, the ill-conditioning of the Hilbert matrix seg-

ments increase with their dimension.   However, because of the

smoothing property of the method a bounded and reasonably accurate

representation for the pseudoinverse was always obtained.

b)   Eighteen random matrices with random dimensions varying between

1 and 25 were generated and pseudoinverted.   The norm $\|A - BC\|$

was always below 10^{-9} and the ranks were always found to be

equal to min(m,n).

c)   Given three random integers m, n and r in the interval $[1, 25]$

a routine generated two random matrices, L (mxr) and

R $(r \times n)$. Multiplying them we obtained a matrix A (m x n) with

rank at most equal to $r^{(1)}$. With 20 matrices generated in this

way, the results were similar to b) . In every case the rank r

was correctly determined. For most of these cases the rank r was

less than min(m,n), and of course was unknown for the program.

(1) This test was suggested by Professor Gene H. Golub.

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d) For each pair of values (m,n) several random matrices were generated and pseudoinverted. Average values of $\|A{-}BC\|$ for these matrices with m=10,20,30 and n=10,20,30 are shown in Table I. For the same problems, average computation time on the Burroughs B5000 at Stanford Computation Center are shown in Table II.

In all these matrices the rank was the maximum possible, i.e., rank = min (m,n) and it was properly determined by the program.

TABLE I

m \ n	10	20	30
10	$5 \times 10{-}9$	9×10^{-9}	4.5×10^{-8}
20	3×10^{-10}	4×10^{-8}	$2.9 \times 10{-}7$
30	5.8×10^{-10}	6.9×10^{-9}	2.6×10^{-7}

$\|A - BC\|$

TABLE I I

m \ n	10	20	30
10	7.6	13.9	20.7
20	16.5	49	91
30	27.3	92.5	180.6

Comp. time in seconds

For a 40 x 40 matrix the answers were:

t = 413 sec. $\|A - BC\| = 2.7 \times 10^{-7}$

rank = 40 .

e) A common problem in many branches of applied sciences is the least squares fit, and is therefore one of the most important applications for this program. A related feature of the program is that, by ordering the variables, the user will be able to test their independence and eventually to decide if his model is appropriate to the phenomena being investigated. This is done by ordering

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the matrix A so that the first coefficients correspond to the more important variables. The program will attempt, to use these columns first to form the basis B . The necessity for such an ordering is clear from the fact that if we have n columns in A and the **subspace** spanned by these columns has dimension (n-p) then we can construct with these columns as many as $\binom{n}{n-p}$ linearly independent sets.

In Figures II and III are shown the results obtained by running the program with least squares type problems. Again the elements of **the matrices** were generated randomly. Fig. II shows computation times on the B5000 for different values of m and n=5, 10. Fig. III shows the norm $\|A - BC\|$ for the same problems.

f) Matrices with exact known inverses were tried obtaining good results and accuracy. Of course this program should not be used to invert a matrix which is known to be nonsingular and **well-conditioned**, because it will be around four times slower than an efficient matrix inverter. The program has also been used to obtain the pseudoinverse of singular and almost singular matrices steming from the discretization of integral equations of the first kind, and problems in pattern recognition.

FIGURE II

X n = 10

O n = 5

FIGURE III

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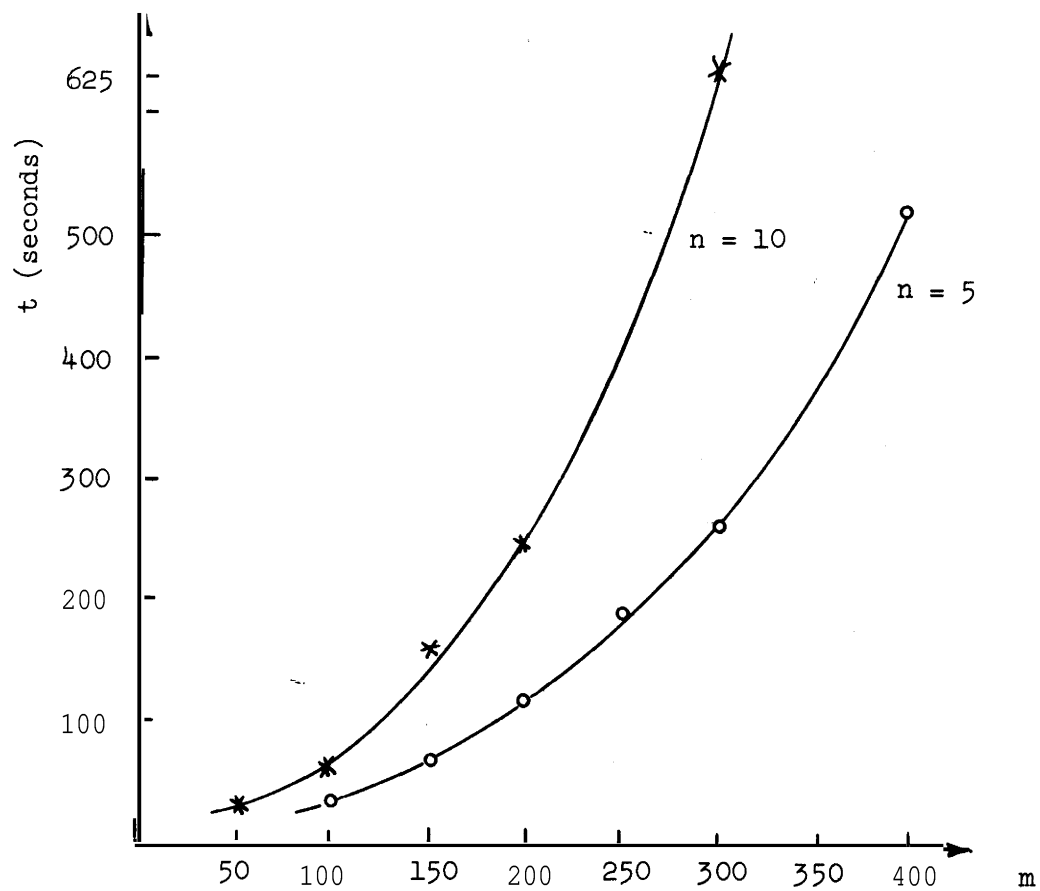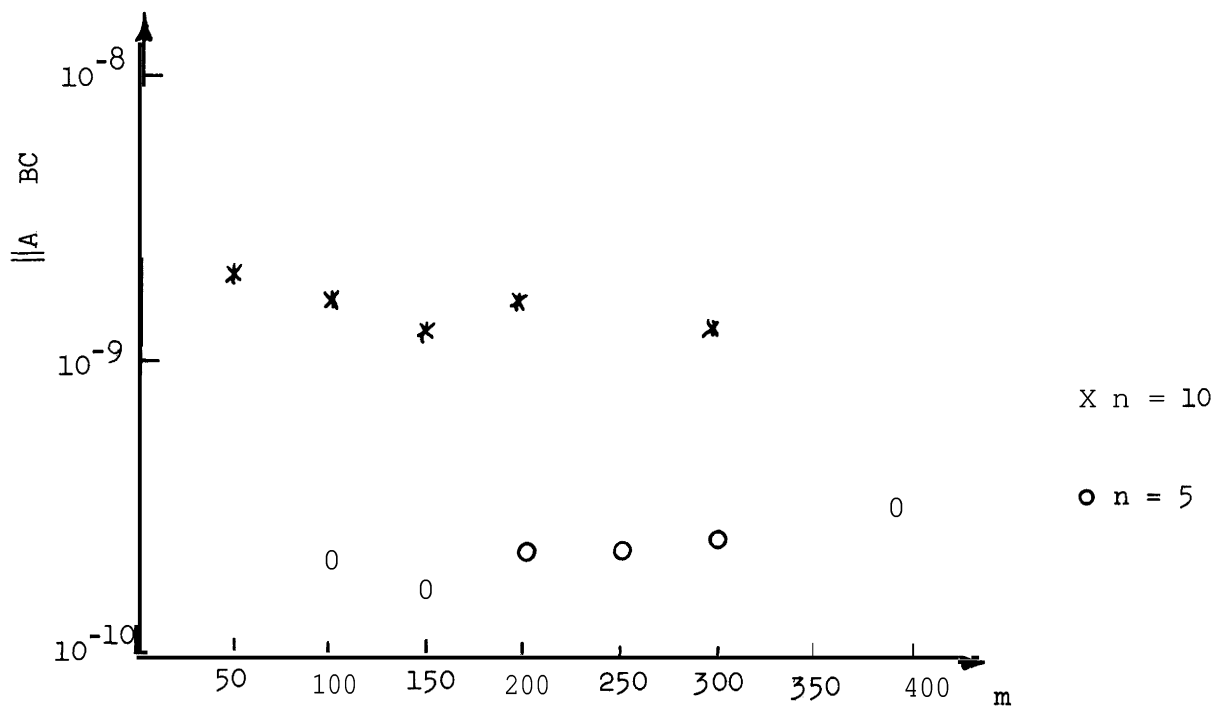
v.   <u>Storage Requirements.</u>

As all the array declarations are dynamical, the amount of
storage depends on several parameters. If M, N, T are as before, and
R is the final rank (number of accepted columns) then an estimate for
the storage used in the PROCEDURE PSEUDOINVER is,

$$\text{Storage} \cong N^2 + 5MN + MR + 2NR + 2NT + \max (R^2, MN, MT)$$

the last term is present because in the PROCEDURE GARBG we have several
independent blocks, and the storage corresponding to certain arrays is
not simultaneously used.

If, as usual, R is not known, then it can be replaced by
$\min(n,m)$.   If the complete program is used then additional storage is
needed,

$$\text{Addit. storage} \cong 3MN + MT + 2NT$$

Computer time rapidly increases when abusive use of the drum is made.
From the experience obtained with the test problems, it is suggested
that the total storage be less than 40,000 words.

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APPENDIX

AN EXTENDED **ALGOL** PROGRAM TO COMPUTE **THE**

PSEUDOINVERSE OF AN M x N REAL MATRIX

AND **OTHER** **RELATED** QUANTITIES.

```
BEGIN    COMMENT    PSEUDOINVERSE  OF  A  M×N  MATRIX  OF  UNKNOWN  RANK;
      INTEGER   M,N,TI,I,J ; REAL  SUPER,EST,TPO,ORTP ;
      BOOLEAN   OPC ; LABEL  OVER,FIU;
   COMMENT     DRIVER  PROGRAM  FIRST  PARAMETERS  ARE  READ  IN,
               AND  USED  FOR  FURTHER  DYNAMICAL  ARRAY  DECLARATIONS;
   OVER:   READ  (M,N,TI,OPC,SUPER,ORTP) [FIU];
   BEGIN    INTEGER  ARRAY  COF[0:N];
      ARRAY   A[0:M,0:N],RHS[0:M,0:TI],XM,XB[0:N,0:TI],NXM,NXB[0:TI],
              APSEUDO,ADAGER[0:N,0:M] ;  LABEL   NOPR,NOPR1 ;
      FORMAT    PRMAT(( 8E14.5)///),
         TIT1("PSEUDOINVERSION  OF  THE  MATRIX  A"///"A  IS",I3,"×",I3/),
         TIT7("NORM  OF  (BC-A)"//X3,E15.5/"RESIDUAL  FOR  XM"/(8E15.5/)),
         TIT8("MATRIX  A  PSEUDOINVERSE"/),TIT9("MATRIX  A  DAGGER"/),
         TIT10(X6,"XM"/),TIT20(//X6,"XB"/),TITL38(/"ROW  NUMBER",I3/),
         TIT77(/"RESIDUAL  FOR  XB  "/(8E15.5/)),SOL(8E15.6/),
         ENDE(//"END  OF  THE  RUN"//X15,"* * * * * * * * *"  /////);
      PROCEDURE  PRT(A,M,N) ;
         INTEGER  M,N; ARRAY  A[0,0];
   COMMENT     PRT  PRINT  OUT  THE  M×N  MATRIX  A;
      BEGIN    FORMAT  TITL46(/ 8E14.5)); INTEGER  SE,K,R;
            SE + N DIV 8    ; K + N MOD 8 ;
            FOR R+0  STEP  1  UNTIL  SE-I  DO
         BEGIN  FOR  I+1  STEP  1  UNTIL  M  DO
            WRITE  (TITL46,FOR  J+1  STEP  1  UNTIL  8  DO  A[I,8 ×R+J]);
            WRITE  ([PAGE])
          END ; IF  K≠0  THEN  FOR  I+1  STEP  1  UNTIL  M  DO
                    WRITE(TITL46,FOR  J+1  STEP  1  UNTIL' K  DO  A[I,8 ×SE+J]);
                    WRITE  ([PAGE])
      END  PRT ;
      PROCEDURE    PSEUDOINVER  (M,N,TI,SUPER,OPC,A,RHS,ORTP,EST,NXM,
                                 NXB,APSEUDO,ADAGER,COF,XM,XB);
         INTEGER  M,N,TI ; INTEGER  ARRAY  COF[0]; BOOLEAN  OPC;
         REAL  SUPER,  EST,ORTP ;
         ARRAY  A,RHS,APSEUDO,ADAGER,XM,XB[0,0],NXM,NXB[0];
   COMMENT     PSEUDOINVER  COMPUTES  THE  PSEUDOINVERSE  OF  A  M×N  MATRIX  A,
         AND  OTHER  RELATED  QUANTITIES.  THE  ESPECIAL  WAY  OF  ROUNDING-OFF
         AFTER  DOUBLE  PRECISION  OPERATIONS  IS  DUE  TO
         MR.  PETER  RUSINGER  AND  PROF.  GENE  GOLUB;
      BEGIN   INTEGER  J,CONT,Q,K,T,BUENO,R,I,PE,MA; BOOLEAN  SUITCH;
            ARRAY  BQ,AN[0:M,0:N],INVQ[0:N,0:N],NRHS,UPI,DOPI[0:TI],
                   G,UQ,SAV[0:N],X1,X2[0:N,0:TI],BPS,API,ADI[0:N,0:M];
            REAL  CLUF,ALFAQ,BEQ,AL,SUPALF,B,NXB1,NXM1,AAA,BBB,CCC,
                  UPI1,DOPI1,ESTI,ASO,SUM,ES,PERTUB,ESTIM,TPO,MINIR;
         MA t  IF  M<N  THEN  N  ELSE  M ;
         BEGIN    ARRAY  TRUC,TU,VQ,TEMP[0:MA]; LABEL  SECND,RFIN,
                  NONES,LOP,TRES,CUATRO,MAIS,CAS,FINI,OTRA,FORCED;
   COMMENT     TRMAVC,MATRIMUL,MULTIVEC,ESC,TRANSP  AND  VECSUM  ARE
            PROCEDURES  PERFORMING  MATRIX  AND  VECTOR  OPERATIONS,
            SOME  OF  THEM  IN  DOUBLE  PRECISION;
         PROCEDURE  TRMAVC  (A,V,I,J,M,N);
                    ARRAY  A[0,0],V[0]; INTEGER  I,J,M,N ;
            BEGIN    INTEGER  K;
               IF  I=0  THEN  FOR  K+1  STEP  1  UNTIL  M  DO  V[K]+ A[K,J]
               ELSE    FOR  K+1  STEP  1  UNTIL,  N  DO  V[K]+ A[I,K]
            END  TRMAVC  ;
```

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```
PROCEDURE    MATRIMUL  (A,B,C,P,Q,R) ;
             ARRAY  A,B,C[0,0] ;INTEGER P,Q,R ;
   BEGIN    INTEGER  I,J,K ;REAL AC,BC;
      FOR I+1 STEP 1 UNTIL P DO FOR J+1 STEP 1 UNTIL R DO
      BEGIN  AC+BC+0;    FOR K+1 STEP 1 UNTIL Q DO
         DOUBLE(A[I,K],0,B[K,J],0,x,AC,BC,+,+,AC,BC);
         C[I,J] + AC+BC&AC[1:1:8]/5.49755813891011
      END
   END    MATRIMUL ;
PROCEDURE    MULTIVEC  (A,V1,V2,P,Q);
             ARRAY  A[0,0],V1,V2[0] ;INTEGER P,Q;
   BEGIN    INTEGER  I,J; REAL AC,BC;
      FOR I+1 STEP 1 UNTIL P 00
      BEGIN    AC+BC+0; FOR J+1 STEP 1 UNTIL Q DO
         DOUBLE (A[I,J],0,V1[J],0,x,AC,BC,+,+,AC,BC);
         V2[I]+ AC+BC&AC[1:1:8]/5.49755813891011
      END
   END    MULTIVEC ;
PROCEDURE  ESC(A,B,C,P);
             ARRAY  A,B[0] ;REAL C ;  INTEGER  P;
   BEGIN    INTEGER I ;
      C+0; FOR I+1 STEP 1 UNTIL P 00 C+A[I]x B[I]+C
   END    ESC ;
PROCEDURE    TRANSP  (A,B,P,Q) ;
             ARRAY  A,B [0,0] ;  INTEGER  P,Q ;
   BEGIN    INTEGER  I,J;
      FOR I+1 STEP 1 UNTIL P DO
      FOR J+1 STEP 1 UNTIL Q DO    B[J,I]    + A[I,J]
   END    TRANSP ;
PROCEDURE    VECSUM  (A,B,C,ALF,BET,N) ;
             ARRAY  A,B,C[0]; REAL ALF,BET ;INTEGER N;
   BEGIN    INTEGER I;
      FOR I+1 STEP 1 UNTIL N DO C[I]+A[I]xALF+B[I]x BET
   END    VECSUM ;
PROCEDURE    PSEUDO  (NQ,ALFQ,UQ,Q) ;
             ARRAY  NQ[0,0],UQ[0]; REAL ALFQ;  INTEGER  Q ;
COMMENT    GIVEN (BQ"BQ) INVERSE,  PSEUOO CONSTRUCT (B(Q+1)"B(Q+1))
       INVERSE;
   BEGIN    REAL  A; INTEGER  I,J ;
      ALFQ + 1/ALFQ;
      FDR I+1 STEP 1 UNTIL Q DO FOR J+1 STEP 1 UNTIL Q DO
         DOUBLE(ALFQ,0,UQ[I],0,x,UQ[J],0,x,NQ[I,J],0,+,+,
             NQ[I,J],A); Q+Q+1 ; UQ[Q]+ -1 ;
      FOR J+1 STEP 1 UNTIL Q DO
         NQ[J,Q]+ NQ[Q,J] + -ALFQ x UQ[J]
   END    PSEUDO ;
PROCEDURE  GARBG(M,N,T,TI,COF,BQ,G,RHS,BPS,A,NRHS,AAA,
             NXM,NXB,NXM1,NXB1,APSEUDO,ADAGER,XM,XB,EST));
             REAL  NXM1,NXB1,EST,AAA; INTEGER N,M,T,TI;
             INTEGER ARRAY COF[0];
             ARRAY XM,XB,APSEUDO,ADAGER,BQ,RHS,BPS,A[0,0],
             NXB,NXM,G,NRHS[0]};
COMMENT    GIVEN THE BASIS BQ;A PSEUDOINVERSE,  A DAGGER, XM, XB,
       NXM, NXB,  AND NORM OF (A-BQC) ARE COMPUTED)
   BEGIN    INTEGER  I,Q,J,R,MA ; REAL B,PER,ALFAQ ;
```

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```
                     ARRAY  C[0:T,0:N], BQ1[0:M,0:T],UQ[0:N] ;
               IF  M≤N   THEN  MA←N  ELSE  MA ←M ;
               BEGIN     ARRAY  AN[0:T,0:N] ;
               BEGIN ARRAY  INVQ1[0:T,0:T],TU,TEMP[0:MA];
COMMENT    A DAGGER IS CALCULATED;
                   FOR I←1 STEP 1  UNTIL N DO
                   BEGIN.    Q ← COF[I] ;
                       IF Q≠0 THEN FOR J←1 STEP 1 UNTIL M OO
                       BEGIN   BQ1[J,Q]  ←  BQ[J,Q]/G[I]  ;
                           ADAGER[I,J]← BPS[Q,J]← BPS[Q,J]× G[I];
                       END
                   END;
COMMENT    C AND C-PSEUDOINVERSE ARE CALCULATED;
                   MATRIMUL(BPS,A,C,T,M,N);
                   TRMAVC(C,TU,1,1,T,N); ESC(TU,TU,B,N);
                   INVQ1[1,1]← 1/B;
                   FOR Q← 2 STEP 1 UNTIL T DO
                   BEGIN  R← Q-1; TRMAVC(C,TU,Q,Q,T,N);
                       MULTIVEC(C,TU,TEMP,R,N);
                       MULTIVEC(INVQ1,TEMP,UQ,R,R);
                       FOR I←1 STEP 1 UNTIL N DO
                       BEGIN  PER← 0;
                           FOR J←1 STEP 1 UNTIL R OO
                           PER+  PER+C[J,I] × UQ[J]; TEMP[I] ← PER
                       END;
                       VECSUM(TU,TEMP,TEMP,1,-1,N);
                       ESC(TEMP,TEMP,ALFAQ,N);
                       PSEUDO(INVQ1,ALFAQ,UQ,R)
                   END ;     MATRIMUL(INVQ1,C,AN,T,T,N) ;
               END;
               BEGIN     ARRAY BQC[0:MA,0:N] ;
COMMENT    NORM OF (A-BC) IS COMPUTED;
                   TRANSP(AN,BQC,T,N) ;
                   MATRIMUL(BQC,BPS,APSEUDO,N,T,M);
                   MATRIMUL(BQ1,C,BQC,M,T,N); EST ← 0;
                   FOR I←1 STEP 1 UNTIL M DO
                   BEGIN   PER← 0;
                       FOR J+1 STEP 1 UNTIL N DO
                       PER+ PER+ABS(BQC[I,J]-A[I,J]);
                       IF PER≥ EST THEN EST ← PER
                   END
               END
           END;
           EST ← EST/AAA;
COMMENT    THE MINIMUM AND BASIC SOLUTIONS ARE COMPUTED;
               MATRIMUL(APSEUDO,RHS,XM,N,M,TI);
               MATRIMUL(ADAGER,RHS,XB,N,M,TI);
               BEGIN    ARRAY  AN[0:M,0:TI];
COMMENT    THE RESIDUALS FOR THESE SOLUTIONS ARE COMPUTED!
               MATRIMUL (A,XM,AN,M,N,TI);
               FOR J←1 STEP 1 UNTIL TI DO
               BEGIN  NXM1← 0;
                   FOR I←1 STEP 1 UNTIL M OO
                   NXM ←(AN[I,J]-RHS[I,J])*2+NXM1;
                   NXM1← SQRT(NXM1); NXM[J] ← NXM1/NRHS[J]
```

```
                    END;
                    MATRIMJL(A,XB,AN,M,N,TI);
                    FOR J+1 STEP 1 UNTIL TI DO
                    BEGIN  NXB1+0;
                        FOR I+1 STEP 1 UNTIL M DO
                        NXB1+(AN[I,J]-RHS[I,J])*2 + NXB1 ;
                        NXB1+ SQRT(NXB1); NXB[J]+ NXB1/NRHS[J]
                    END ;  NXB1 + NXM1 + 0;
                    FOR I+1 STEP 1 UNTIL TI DO
                    BEGIN  NXB1 + NXB1+ NXB[I]*2 ;
                        NXM1  + NXM1+NXM[I] *2
                    END ; NXM1+ SQRT(NXM1); NXB1+ SQRT(NXB1);
                END;
            END  GARBG ;
```

PROCEDURE   PSEUDOINVER BODY ;
```
            HINIR +8-20;    AAA+CCC+0 ;
```

COMMENT   NORM OF A AND NORM OF THE RHS;
```
            FOR J+1 STEP 1 UNTIL N DO FOR I+1 STEP 1 UNTIL M DO
            BEGIN  BBB+ ABS(A[I,J]); If BBB>AAA THEN AAA+BBB
            END ; FOR J+1 STEP 1 UNTIL TI Do
            BEGIN  CCC+0; FOR I+1 STEP 1 UNTIL M DO
                CCC" CCC+RHS[I,J]* 2 ;   CCC+ SQRT(CCC);
            --- NRHS[J]+ IF CCC>1 THEN CCC ELSE 1 ;
            END;   IF AAA<1 ThEn AAA+1 ;
            FOR I+1 STEP 1 UNTIL N DO
            BEGIN  COF[I]+ 0;
                FOR J+1 STEP 1 UNTIL M DO ADI[I,J]+ADAGER[I,J]+ 0;
            END ;
```

COMMENT   THE MATRIX A IS NORMALIZED AND STORED ON AN ,
          G CONTAINS THE NORMALIZING COEFFICIENTS ;
```
            FOR J+1  STEP 1 UNTIL N DO
            BEGIN   TRMAVC(A,TU,O,J,M,N);
                ESC(TU,TU,CLUF,M);
                G[J]+IF CLUF > 1.0 THEN 1.0/SQRT(CLUF) ELSE 1.0;
            END ; FOR I+1 STEP 1 UNTIL M DO
            BEGIN FOR J+1 STEP 1 UNTIL N DO
                AN[I,J] + A[I,J]× G[J]
            END ;
```

COMMENT THE CONSTRUCTION OF A BASIS OF STRONGLY
          INDEPENDENT VECTORS IS STARTED;
```
            CONT+1; SUPALF+ 0 ; Q+ K+ T+ 1;
            COF[1]+1; INVQ[1,1]+ 1; SUITCH+ FALSE ;
            FOR I+1 STEP 1 UNTIL M DO BQ[I,1]+ AN[I,1];
```

COMMENT   SEARCH FOR INDEPENDENT COLUMNS OF A, WHEN THE COLUMNS
           ARE EXHAUSTED AN EXIT IS PROVIDED To LABEL FINI, IN CONT
           A RECORD IS KEPT ON THE WAY IN WHICH COLUMS ARE ACCEPTED1

LOP :   If Q=N THEN Go TO FXNI ;
```
            T+K  ; Q+Q+1; K+K+1;
```

COMMENT   PROJECTION OF A COLUMN OF AN ON THE ORTHOGONAL
           SUBSPACE OF BQ;

CAS a   TRMAVC(AN,VQ,O,Q,M,N);
```
            FOR I+1 STEP 1 UNTIL T DO
            BEGIN  ALFAQ+ 0;
                FOR J+1 STEP 1 UNTIL M DO
                ALFAQ+ ALFAQ+BQ[J,I]×VQ[J]; TEMP[I]+ ALFAQ
```

```
                    END ;
                    MULTIVEC(INVQ,TEMP,UQ,T,T);
                    MULTIVEC(BQ,UQ,TEMP,M,T);
                    VECSUM(VQ,TEMP,TEMP,1,-1,M);
                    ESC(TEMP,TEMP,ALFAQ,M);
                    If SUITCH THEN GO TO TRES;
COMMENT        FIRST TEST FOR ACCEPTANCE AS AN INDEPENDENT COLUMN;
                    If ALFAQ ≤ ORTP THEN GO TO NONES;
                    If CONT=2 THEN CONT← 3 ;
COMMENT        CONSTRUCTION OF B(Q+1);
                    PSEUDO (INVQ,ALFAQ,UQ,T);
                    FOR I←1 STEP 1 UNTIL M DO BQ[I,K]← VQ[I];
                    COF [Q]← K ; GO TO LOP ;
NONES :        COF[Q] ← 0; K← K-1 ;
                    If CONT ≠ 3 THEN
                    BEGIN CONT←2 ; GO TO TRES ;
                    END ; GO TO LOP ;
COMMENT        If CONT≠2 THEN THE REJECTED COLUMNS ARE REVISED IN ORDER
                    TO TAKE THE MOST INDEPENDENT WITH RESPECT TO THE BASIS BQ,
                    CONT=1 MEANS THAT ALL THE COLUMNS HAVE BEEN TAKEN IN THE
                    FIRST SWEPT, CONT=3 YEANS THAT A COLUMN HAS BEEN REJECTED
                    AND LATER ON,ANOTHER HAS BEEN ACCEPTED;
DTRA :        Q← 2 ; SUPALF← 0;  BUENO← 0;
MAIS :        If Q=N+1 THEN GO TO CUATRD ;
                    IF COF[Q]≠ 0 THEN
                    BEGIN Q← Q+1 ; GO TO MAIS ;
                    END ; GO TO CAS ;
TRES :        IF ALFAQ ≥ SUPALF THEN
                    BEGIN SUPALF ← ALFAQ ;
                         FOR I←1 STEP 1 UNTIL M DO TRUC[I]← VQ[I];
                         FOR I←1 STEP 1 UNTIL T DO
                         SAV[I]← UQ[I] ; BUENO ← Q
                    END ; IF CONT=2 THEN GO TO LOP ;
                    If Q=N THEN GO TO CUATRO ;
                    Q← Q+1; GO TO MAIS;
COMMENT        If THE PROJECTION OF THE SELECTED COLUMN IS LESS THAN ●-20
                    THEN SAV IS REJECTED AND WE FINISH;
CUATRD : If SUPALF ≤ MINIR THEN GO TO RFIN ;
                    AL ←SUM ← 0;
COMMENT        THE NORM OF (B(Q+1)*B(Q+1)) INVERSE IS ESTIMATED,AND ITS
                    VALUE IS CONTROLLED;
                    FOR I←1 STEP 1 UNTIL T DO
                    BEGIN FOR J←1 STEP 1 UNTIL T DO
                         SUM ←SUM + ABS(INVQ[I,J]);
                         IF SUM>AL THEN AL←SUM
                    END ;
                    ESTIM ← AL+(SQRT(T)+1.0)/SUPALF ;
                    If ESTIM ≥ SUPER THEN GO TO RFIN ;
COMMENT        SAV HAS PASSED THE TESTS OF SECTION 2,C),NOW IS USED
                    TENTATIVELY IN B(Q+1) TO SEE If THE RESIDUALS DIMINISH;
FORCED : PSEUDO(INVQ,SUPALF,SAV,T);
                    COF[BUENO]← T ;
                    FOR I←1 STEP 1 UNTIL M DO BQ[I,T]← TRUC[I];
    COMMENT           WE CONSTRUCT NOW B-PSEUDO,C,A-PSEUDO AND ADAGGER;
FINI :        FOR I←1 STEP 1 UNTIL T DO FOR J←1 STEP 1 UNTIL M DO
```

```
            BEGIN  ES←0; FOR PE←1 STEP 1   UNTIL  T  DO
                ES←  ES  + INVQ[I,PE] × BQ[J,PE] ; BPS[I,J] ←ES
COMMENT      AS  WE  WANT  TO  COMPARE  RESULTS  FOR TWO  DIFFERENT  BASES,
             SUITCH  PROVIDES  A  WAY  TO  DECIDE  THE  CALL  OF  GARBG;
             END ;   IF  SUITCH  THEN  GO  TO  SECND ;
             GARBG  (M,N,T,TI,COF,BQ,G,RHS,BPS,A,NRHS,AAA,
                   NXM,NXB,NXM1,NXB1,APSEUDO,ADAGER,XM,XB,EST) ;
             If  T=N  THEN  GO  TO  RFIN ;   SUITCH  ← TRUE ;
             If  CONT = 2   THEN
             BEGIN    CONT ← 1;    GO  TO  CUATRO ;
             END ;    GO  TO  OTRA ;
SECND ;   GARBG  (M,N,T,TI,COF,BQ,G,RHS,BPS,A,NRHS,AAA,
                   UPI,DOPI,UPI1,DOPI1,API,ADI,X1,X2,ESTI);
COMMENT      NOW  THE  TEST  OF  SECTION  2,0)  IS  MADE;
             If  NXB1≥ DOPI1   AND   NXM1≥UPI1   THEN
             BEGIN    IF  NOT  OPC OR  EST≥ ESTI   THEN
COMMENT      If  SAV  IS  ACCEPTED  THEN  ALL  THE  USEFUL  QUANTITIES
             ARE  SHIFTED;
                BEGIN    NXM1←UPI1; NXB1← DOPII;  EST←ESTI ;
                    FOR I←1 STEP 1  UNTIL  N  DO
                    BEGIN  FOR  J←1   STEP  1  UNTIL  M  DO
                        BEGIN  APSEUDO[I,J]← API[I,J] ;
                           ADAGER[I,J]←ADI[I,J]
                        END;   FOR  J←1  STEP  1 UNTIL  TI  DO
                        BEGIN  XM[I,J]←X1[I,J]; XB[I,J]← X2[I,J];
                             NXM[J]←UPI[J]; NXB[J]←DOPI[J];
                             END;
                    END ; IF T=N   THEN
                    BEGIN   BUENO←0 ; GO  TO  RFIN
                    END ;  GO  TO  OTRA
                END
            END ;
RFIN ;   COF[BUENO]← 0;
        END
    END    PSEUDOINVER ;
COMMENT    BODY OF  THE  DRIVER  PROGRAM,  THE  INPUT-OUTPUT  AND  THE  CALL
           Of  PSEUDOINVER  ARE  INCLUDED;
   FOR I←1 STEP 1  UNTIL  M  DO
   READ(FOR J←1 STEP  1  UNTIL  N  DO A[I,J]);
   WRITE( [PAGE]); WRITE(TIT1,M,N); PRT(A,M,N);
   IF  NOT  OPC  THEN  FOR  I←1 STEP  1  UNTIL  TI DO
   READ( FOR  J←1 STEP  1  UNTIL  M  DO RHS[J,I])
   ELSE  FOR I←1 STEP  1   UNTIL  M  DO FOR  J←1 STEP  1  UNTIL  M  DO
         RHS[I,J]← If  I=J  THEN  1 ELSE  0 ;
   PSEUDOINVER(M,N,TI,SUPER,OPC,A,RHS,ORTP,EST,NXM,
              NXB,APSEUDO,ADAGER,COF,XM,XB);
   WRITE(TIT7,EST,FOR I←1  STEP  1  UNTIL TI DO NXM[I]);
   WRITE(TIT77, FOR I←1  STEP  1  UNTIL  TX DO NXB[I]);
   If OPC  THEN
   BEGIN   WRITE([PAGE])  ; WRITE(TIT8);
       PRT(APSEUDO,N,M); WRITE (TIT9);
       FOR I←1 STEP  1  UNTIL  N  DO
       BEGIN IF COF[I]=0  THEN  GO  TO  NOPR ;
           WRITE (TITL38,I);
           WRITE (PRMAT,FOR J←1 STEP 1 UNTIL M DO ADAGER[I,J]) ;
```

```
NOPR:  END
      END          ELSE
                   BEGIN  WRITE([PAGE]); WRITE(TIT10) ;
                      FOR  I+1  STEP  1  UNTIL  N  DO
                      WRITE(SOL,FOR J+1  STEP  1  UNTIL  TI  DOXM[I,J]);
                      WRITE  (TIT20) ; FOR I+1  STEP  1  UNTIL  N  DO
                      BEGIN   IF  COF  [I]=0  THEN  GO  TO  NOPR1;
                         WRITE(TITL38,I );
                         WRITE(SOL,FOR J+1  STEP  1  UNTIL  TI  DO  XB[I,J]);
NOPR1 :               END;
COMMENT     JUST  BY  ADDING  NEW SETS  OF  DATAS  MORE  PROBLEMS  CAN  BE  RUN;
                   END ;  WRITE(ENDE); GO  TO  OVER ;

      END  ;
      FIU  ;
END .
```