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ELASTIC-PLASTIC ANALYSIS OF TRUSSES BY THE GRADIENT PROJECTION METHOD

BY

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Abstract

The gradient projection method has been applied to the problem of obtaining the elastic-plastic response of a perfectly plastic ideal truss with several degrees of redundancy to several independently varying sets of quasi-static loads. It is proved that the minimization of stress rate intensity subject to a set of yield inequalities is equivalent to the maximization process of the gradient projection method. This equivalence proof establishes the basis of the computational method. The technique is applied to the problem of investigating the possibilities of shake down and to limit analysis. A closed convex "safe load domain" is defined to represent the load carrying capacity characteristics of a truss subjected to various combinations of the several sets of loads.

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1. Introduction.

This paper is concerned with the elastic-plastic analysis of trusses by means of the gradient projection method of nonlinear programming which has been developed by one of the authors [4, 5]. It is a method of obtaining the global maximum of a nonlinear concave function in a convex region defined by linear or nonlinear constraints. In this paper, however, the application of the method is restricted to the case of linear constraints.

The geometrical representation of a state of stress given by Prager enables one to represent a change of state of stress in a truss as an infinitesimal vector in a convex yield polyhedron defined in a stress space. The minimum principle of Greenberg [1] is interpreted in this geometrical context. It is proved that the infinitesimal response vector obtained by use of the minimum principle is identical with that described by the maximization process of the gradient projection method, provided that the objective function is so chosen that the gradient vector always coincides with the direction determined by the increments of the load factors. The equivalence proof establishes the basis for the application of the computational method. The technique is illustrated by means of a simple example.

If the prescribed loading path is piecewise linear, the number of maximization steps is equal to the number of segments of the path, regardless of the degrees of redundancy of the truss., Because of this fact, we can readily investigate the possibilities of shakedown [3,6,7] if a loading cycle is prescribed or if it can be assumed that all the variations of the applied loads are bounded by a polyhedron or by a parallelopiped. A practical example is studied in detail to illustrate the technique. The results obtained from GP are shown to be identical with those which can be obtained graphically for this relatively simple case The load carrying capacity must be defined with respect to a particular combination of the applied loads. A set of all the load carrying capacities defines a closed convex "Safe Load Domain". This may be obtained by the GP procedure.

It is expected that this computational technique can be applied to frame structures. In this case an infinite number of yield strips are obtained corresponding to an infinite number of cross-sections of the members of a frame. Each yield strip represents the yield condition for a particular cross-section. The fact that most collapse modes of frames consist of a finite number of plastic hinges implies that even if continuously distributed loads are applied to a frame, the corresponding yield conditions can be represented by a polyhedron and not by a smooth convex hypersurface. Therefore we have again linear constraints and the gradient projection method can be used without modification.

2. Geometrical Representation of State of Stress.

The geometrical representation of a state of stress as given by Prager [2, 3] is introduced here in a generalized form to prepare for the ensuing -discussion.

An ideal truss is composed of n bars of elastic-perfectly-plastic materials. It is assumed that the velocities of settlement of the supports are-all zero. Let f be the number of reactions and h the number of joints. Suppose there are s sets of loads characterized by s independent load factors $\{\xi_i\}$ in the form

$$\{\xi_1 P_{1\ell}\}, \{\xi_2 P_{2\ell}\}, \ldots, \{\xi_s P_{s\ell}\} \quad (\ell = 1,2,\ldots,h). \tag{2.1}$$

acting upon the given ideal truss. Then the equations of equilibrium form an inhomogeneous system of linear algebraic equations of the form

$$\sum_{k=1}^{n+f} a_{\ell k} S_k = \sum_{i=1}^{s} \xi_i \, b_{i\ell} \qquad (\ell = 1,2,\ldots,2h) \tag{2.2}$$

where $S_1, S_2,\ldots,S_n$ denote the internal forces in the bars, $S_{n+1},\ldots,S_{n+f}$, the reactions at supports and $b_{i\ell}$, $P_{i\ell}$ multiplied by a direction cosine, The coefficients $a_{\ell k}$ are essentially the direction cosines of the kth bar. Unknown quantities are $\{S_k\}$ $(k = 1,2,\ldots,n+f)$.

If $n+f-2h \equiv r > 0$, then the set $\{S_k\}$ is not uniquely determined by (2.2) and the truss has r degrees of redundancy. Let

$$\{S_{1j}\},\ \{S_{2j}\}\ldots.\{S_{sj}\}$$

by a set of s particular solutions of the s inhomogeneous systems the ith of which is obtained by replacing the right-hand side of (2.2) by the ith inhomogeneous term $\{b_{i\ell}\}$ and

$$\{R_{1j}\},\ \{R_{2j}\},\ldots,\{R_{rj}\},$$

a set of r linearly independent solutions of the homogeneous equation obtained from (2.2). Although (2.2) cannot have a unique solution, any solution of (2.2) must be contained in the general solution given by a linear combination of s linearly independent particular solutions $\{S_{ij}\}$ and r linearly independent solutions $\{R_{kj}\}$ of the homogeneous system obtained from (2.2).

$$S_j = \sum_{i=1}^{s} \xi_i \, S_{ij} + \sum_{k=1}^{r} \eta_k \, R_{kj} \qquad (2.3)$$

$$(j = 1,2,\ldots,n+f).$$

where η_k's are arbitrary parameters. The particular solution $\{S_{ij}\}$ can be obtained by imposing r additional conditions on deformation that the truss responds elastically under the particular set of loads $\{b_{i\ell}\}$ $(\ell = 1,2,\ldots,2h)$. Thus $\{S_{ij}\}$ represents an elastic state of stress. $\{R_{kj}\}$ is obtained by considering the statically determinate structure which can be produced by replacing r redundant forces by a set of known forces. $\{R_{kj}\}$ then represents a state of self-stress.

It is always possible [2] by forming linear combinations of $\{S_{ij}\}$ or $\{R_{kj}\}$ to construct an orthonormal set of (s+r) solutions in the sense

$$\frac{1}{2}\sum_{j=1}^{n} \alpha_j \, S_{kj} \, S_{ij} = \delta_{ki}$$

$$\frac{1}{2}\sum_{j=1}^{n} \alpha_j \, S_{ij} \, R_{\ell j} = 0 \qquad (2.4)$$

$$\frac{1}{2}\sum_{j=1}^{n} \alpha_j \, R_{\ell j} \, R_{mj} = \delta_{\ell m}$$

$$(k, \; i = 1,2,\ldots,s; \qquad \ell, \; m = 1,2,\ldots,r).$$

where $\alpha_j = \frac{l_j}{E_j A_j}$, ℓ_j denoting the length of bar j, A_j its cross-section and E_j its elastic modulus, and where δ_{ki} and $\delta_{\ell m}$ are Kronecker delta's, It is to be understood in the following that $\{S_{ij}\}$ and $\{R_{kj}\}$ denote elements of this orthonormalized set where j varies from 1 to n. The original load factors can then be expressed by linear combinations of the ξ_i's for this orthonormalized set.

The elastic strain energy corresponding to the set of internal forces $\{S_j\}$ given by (2.3) is defined by

$$\begin{aligned} \epsilon &= \frac{1}{2} \sum_{j=1}^{n} \alpha_j S_j^2 \\ &= \frac{1}{2} \sum_{j=1}^{n} \alpha_j \left[\sum_{i=1}^{s} \xi_i S_{ij} + \sum_{k=1}^{r} \eta_k R_{kj} \right]^2 \end{aligned} \tag{2.5}$$

By virtue of the orthonormality (2.4), (2.5) is reduced to

$$\epsilon = \sum_{i=1}^{s} \xi_i^2 + \sum_{k=1}^{r} \eta_k^2 \tag{2.6}$$

With this preparation, we can now make use of the concept of "s-tress space." Any solution $\{S_j\}$, and the corresponding strain energy, is completely determined by the set of parameters

$$(\xi_1, \xi_2, \ldots, \xi_s; \quad \eta_1, \ldots\ldots, \eta_r) \tag{2.7}$$

as shown by (2.3) and (2.6). If we consider an (s+r)-dimensional stress space whose Cartesian coordinates are $\xi_1, \xi_2, \ldots\ldots, \xi_s$, $\eta_1, \eta_2, \ldots\ldots, \eta_r$, then any state of stress of the truss can be represented by a point in this space. Any state of loading is then represented by the set $(\xi_1, \xi_2, \ldots\ldots, \xi_s)$ while any state of residual stress by $(\eta_1, \eta_2, \ldots\ldots, \eta_r)$.

Since all the bars are assumed to be composed of perfectly-plastic materials, the corresponding yield conditions must be satisfied. The condition that the stress in any bar should not exceed the yield limit can be written

$$c_j \leq \sum_{i=1}^{s} \xi_i \, S_{ij} + \sum_{k=1}^{r} \eta_k \, R_{kj} \leq t_j \tag{2.8}$$

$$(j = 1,2,\ldots,n)$$

where c_j and t_j denote the yield limits of bar j in compression and in tension, respectively. Each one of the n inequalities in (2.8) defines a strip between two hyperplanes. The set of all the inequalities define a convex polyhedron in (s+r)-dimensional stress space as the common region of all the yield strips. Therefore, only the set of points on or inside the yield polyhedron can represent actual states of stress.

3. Geometrical Interpretation of the Minimum Principle.

The minimum principle of Greenberg [1] is expressed in the geometrical terms according to Prager [3] to prepare for the later use in Section 4.

In order to obtain the response of a given truss to a particular loading program, (s-1) relations between s load factors must be prescribed resulting in a "loading path" in the s dimensional load factor subspace. Let a vector

$$\vec{d\xi} = (d\xi_1, d\xi_2, \ldots, d\xi_s)$$

define a set of infinitesimal changes of load factors from an instantaneous state of loading $(\xi_1, \xi_2, \ldots, \xi_s)$, $\vec{d\xi}$ being a tangent vector to the loading path. Corresponding to this change is an infinitesimal translational displacement of the r-dimensional subspace of equilibrium. The corresponding new state of stress must be represented by a point in this displaced subspace of equilibrium. If we consider a local coordinate axis ξ in the direction of $\vec{d\xi}$ at a stress point denoted by

$$\vec{x} = (\xi_1, \xi_2, \ldots, \xi_s; \quad \eta_1, \eta_2, \ldots, \eta_r),$$

and an (r+1)-dimensional cross-section of the yield polyhedron spanned by ξ, $\eta_1, \ldots, \eta_r$ axes, then any stress change due to $d\xi$ can be represented by a vector

$$\vec{dx} = (d\xi, \; d\eta_1, \ldots, d\eta_r).$$

where $d\eta_1$, $d\eta_2, \ldots, d\eta_r$ denote the variation in η_1, $\eta_2, \ldots, \eta_r$ corresponding to $d\xi$. The problem is then to determine $\vec{dx}$ which does not violate the yield conditions (2.8).

The quantity termed as stress rate intensity by Greenberg may be written as

$$\begin{aligned} \Delta\epsilon &= d\xi^2 + d\eta_1^2 + \ldots + d\eta_r^2 \\ &= \|\vec{dx}\|^2 \end{aligned} \tag{3.1}$$

Since the new stress point must lie in the displaced subspace of equilibrium (a hyperplane in E_{r+1}) and in the yield polyhedron, a vector $\vec{dx}$ is said to be admissible if the point $\vec{x} + \vec{dx}$ is in the displaced hyperplane of equilibrium bounded by the yield polyhedron. Admissible vectors form a family any one of which can represent an admissible stress change. The minimum principle can be stated as follows: For a given $\vec{d\xi}$, the actual stress change is given by the vector which minimizes the absolute value of $\vec{dx}$ among all the admissible vectors. In other words, we wish to obtain the shortest distance from an initial point to the intersection of the equilibrium hyperplane and the yield polyhedron.

4. Equivalence Theorem.

The use of the gradient projection (GP) method is based essentially on the fact that in GP the global maximum is sought by cutting across the interior of the convex region of definition, if possible. It is proved in this section that the stress change $\vec{dx}$ obtained by use of the minimum principle stated above exactly coincides with the vector determined by GP, provided that the objective function F is chosen so that the gradient vector is always in the direction of the tangent to a prescribed loading path.

A hyperplane of equilibrium is expressed by

$$H_e \; : \; \xi = \text{const.} \tag{4.1}$$

The afore-mentioned condition will be satisfied if F is defined by

$$F = \xi \tag{4.2}$$

in the local coordinate system. Then

$$\text{grad } F = \vec{g}_o = (1, 0, \ldots, 0) \tag{4.3}$$

and

$$\vec{d\xi} = d\xi \, \vec{g}_o \tag{4.4}$$

The following preliminary results are required for the equivalence proof. Given an m x n real matrix A we let the finite set $\{A_i\}$ represent all submatrices which can be formed with linearly independent columns of A. For each such A_i we can form the m x m projection matrix $P_i \equiv I - A_i(A_i' A_i)^{-1} A_i'$, which takes any m-dimensional vector into the space orthogonal to that spanned by the columns of A_i. We let $P = \{P_i, I\}$ be the finite set of all P_i and the m x m identity matrix.

<u>Lemma</u>

Given an m-dimensional vector g and the convex cone $A x > 0$, the gradient projection algorithm will form the projection matrix $P_\ell \in P$, such that

$$\|P_\ell\, g\| = \max_{P_i \in P} \left\{ \|P_i g\| \;\middle|\; A' P_i g \geq 0 \right\} \qquad (4.5)$$

The proof follows directly from equation (4.48) in reference [5], which shows that an appropriate basis change is made whenever such a change will increase the norm of the projected gradient, subject to the feasibility restriction.

The minimum principle [1] for the elastic-plastic truss can be stated as that of finding a vector x which satisfies the following quadratic programming problem

$$\min \{x'x \mid A'x > 0,\ g'x = 1\} \qquad (4.6)$$

The columns of A represent the active constraints of the yield polyhedron at the point considered, and g is a normal vector to the equilibrium hyperplane H_e. The desired stress change dx is then given by $dx = x d\xi$.

<u>Equivalence Theorem</u>

<u>If $P_\ell g$ is the solution of (4.5), then $x = \alpha P_\ell g$ is the solution of (4.6), where $a > 0$ is a scalar.</u>

Proof: We consider all the possible projection matrices $P_i \epsilon P$, and let $x_i = \alpha_i P_i g$. In order to satisfy $g'x_i = 1$, we require

$\alpha_i g'P_i g = \alpha_i \|P_i g\|^2 = 1$, or $\alpha_i = \|P_i g\|^{-2}$.

Then $x_i'x_i = \|P_i g\|^{-2}$, so that the value $i = \ell$ which maximizes $\|P_i g\|$ in (4.5) also gives the desired minimum in (4.6)

5. Three-bar truss.

A truss consisting of three bars shown in Figure 1 is subjected to a vertical varying load P. Let the internal forces transmitted by the bars 1 (or 3) and 2 be S_1 and S_2, and the tensile rigidity (ℓ_i/A_iE) be 1 and 2 for simplicity.

The equation of equilibrium is written as

$$S_1 + S_2 = P \tag{5.1}$$

and the compatibility equation

$$2S_2 = 2 \cdot 1 \cdot S_1 \tag{5.2}$$

.The solution of (4.1) and (4.2) is

$$S_1 = \frac{P}{2}, S_2 = \frac{P}{2} \tag{5-3}$$

the normalized set is

$$\{S_{11}, S_{12}\} = \{\frac{1}{\sqrt{2}}, \quad \frac{1}{\sqrt{2}}\} \tag{5.4}$$

The state of self-stress is shown in Figure 2. After normalization, we obtain

$$\{R_{11}, R_{12}, = \{\frac{1}{\sqrt{2}}, \quad \sqrt{2} \tag{5.5}$$

thus a typical state of stress can be written

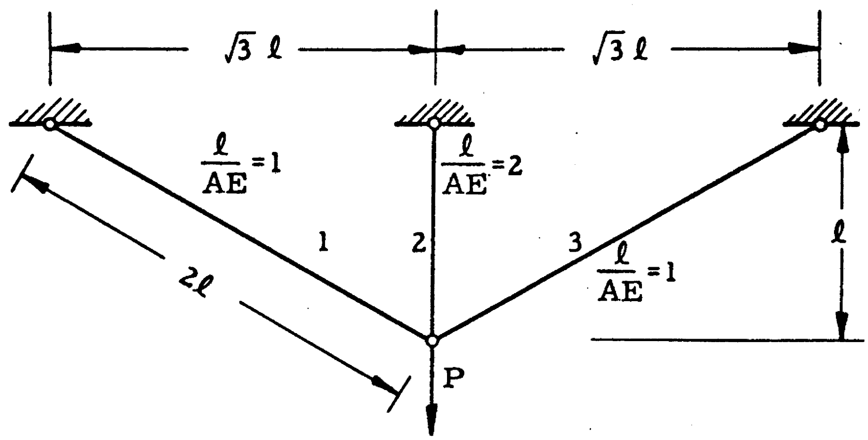


Figure 1. THREE-BAR TRUSS

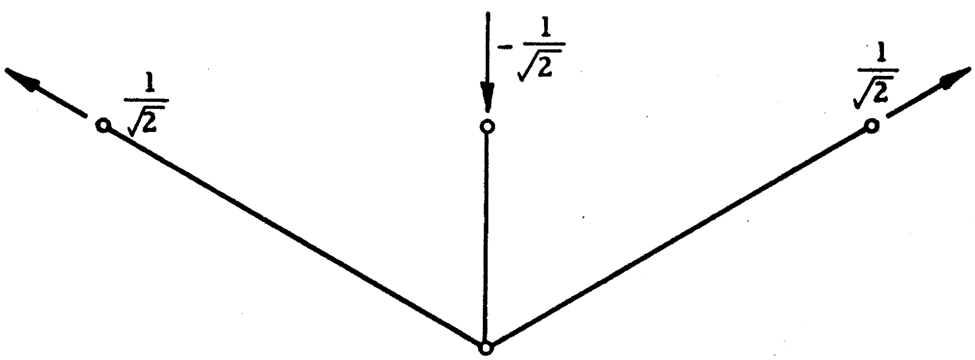


Figure 2. STATE OF SELF-STRESS

$$S_1 = \frac{\xi}{\sqrt{2}} + \frac{\eta}{\sqrt{2}}$$

$$S_2 = \frac{\xi}{\sqrt{2}} - \frac{\eta}{\sqrt{2}} \tag{5.6}$$

For the sake of simplicity, let the yield limits of the bars 1(3) and 2 be $\pm\sqrt{2}$ and $\pm\frac{1}{\sqrt{2}}$ respectively. Then the yield conditions are

$$\left.\begin{aligned} -2 < \xi + \eta \leq 2 \\ -1 \leq \xi - \eta \leq 1 \end{aligned}\right\} \tag{5.7}$$

The feasible region R is determined by the yield conditions (5.7) and the lines of equilibrium. Figure 3 shows the yield polygon (H_1, H_2, H_3, H_4), the upper and lower extreme values of the load factor ξ (H_5, H_6) and the correspondingly inward drawn unit vectors $\vec{n}_i$ (i =1,2,...,6). The objective function is $F = \xi$ whose contour lines are a family of equally-spaced parallel dotted lines which are the lines of equilibrium themselves.

Consider the loading and unloading process given by

$$\xi: \quad 0 \rightarrow 1.4 \rightarrow 0 \rightarrow -1.2 \rightarrow 0 \rightarrow 1.4 \tag{5.8}$$

The initial point is the origin with the gradient $|\vec{g}_o| = 1.0$. The largest **permitted** step length in the direction of $\vec{g}_o$ without leaving R is to H_1. The projection $P_1\vec{g}_1$ is shown in the Figure 3, where P_1 is the corresponding projection matrix to H_1. Since $P_1 g_1 = (\frac{1}{2}, \frac{1}{2})$ the new direction $\vec{z}_1 = P_1\vec{g}_1/|P_1\vec{g}_1|$ is $\vec{AB}$. The largest step length is to H_2 giving $\vec{g}_2 = \vec{g}_o$ as shown. At this point B, the projection of g_2 on H_5 becomes

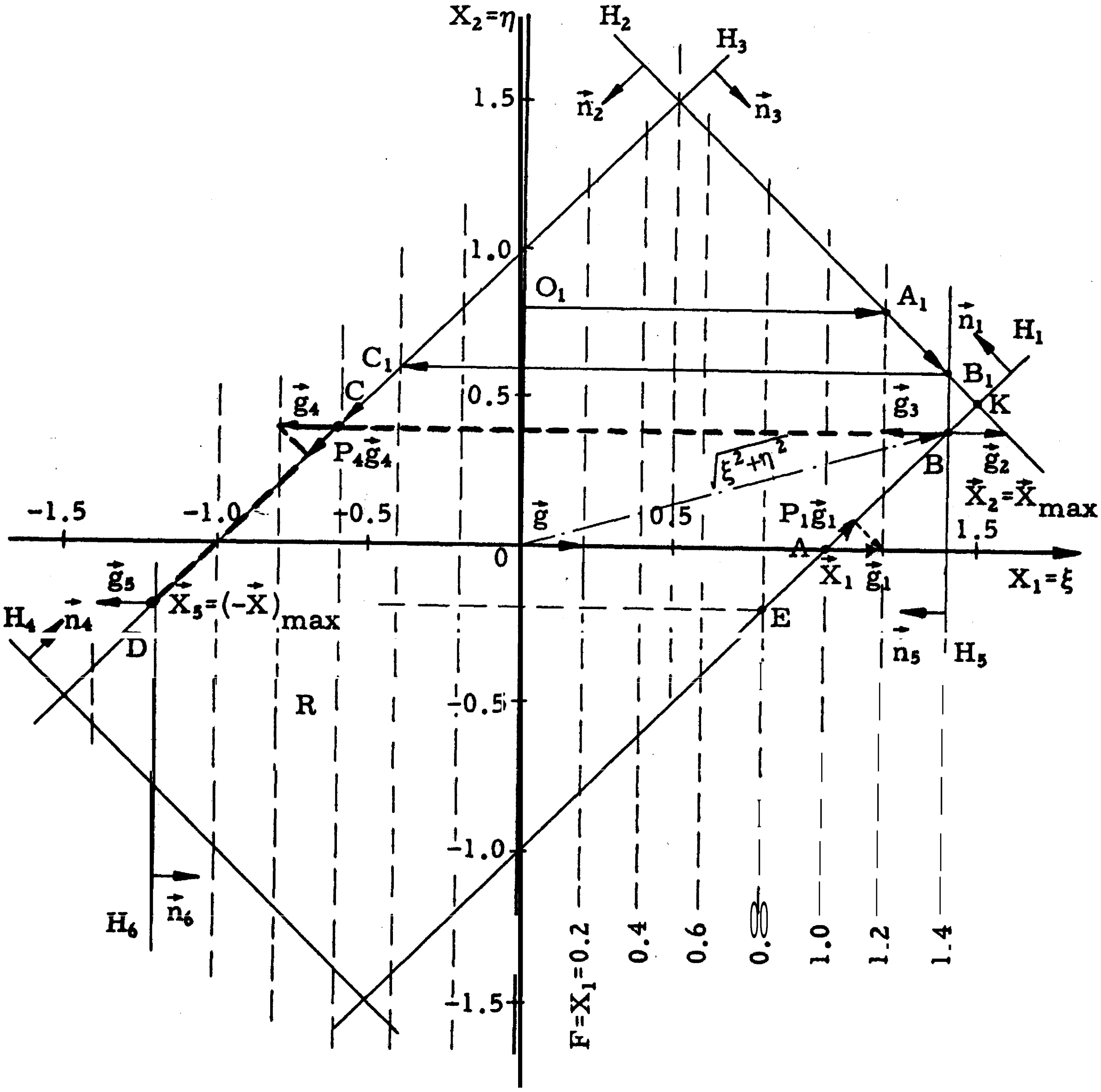


Figure 3. THE YIELD POLYGON

zero because H_5 is one of the contour lines. Thus at B the maximum of $F = \xi$ is achieved first. The computer describes exactly the path $\vec{OA}, \vec{AB}$ which represent the actual response of the truss. It should be noted that the computer stops at the point B where the maximum first achieved although there are an infinite number of points of maxima along H_5.

For the unloading process, ξ_i: 1.4 → 0 → −1.2, the program must be started with the new initial point B. The corresponding mathematical problem is stated as follows:

Maximize $F = -\xi$

subject to the constraints $\xi \geq -1.2$, $-\xi \geq -1.4$ and (5.7) with the new initial point B.

Tables 1 & 2 show the results obtained by GP. Theydemonstrate that the actual response of the truss to the prescribed variation of ξ is exactly traced by the computer within the round-off errors in the last digit.

Table 1. Response I. No shakedown.

$X_1(\xi)$	$X_2(\eta)$		Loading
0	0	0	+
1	0	A	Loading
1.40000000	o-39999997	B	
-0.60000002	0.39999997	C	
-1.20000000	-0.19999997	D	Unloading
0.80000003	-0.19999997	E	+
1.40000000	0.39999997	B	Loading

Table 2. Response II. Shakedown.

$X_1(\ )$	$X_2(\eta)$		Loading
0	0	0	+
1	0	A	Loading
1.40000000	0.39999997	B	
o-59999999	0.39999997	C	Unloading
1.40000000	0.39999997	B	Loading
o-59999999	o-39999997	C	Unloading

6. Shakedown.

In many practical cases, the precise variations of the loads applied to a structure are not known or are so complicated that it is difficult to prescribe them. For the purpose of designing structures, certain bounds on the working loads can be assumed which are based on statistical data. It is assumed here that the variable loads applied to the structure have a finite period.

The shakedown problem may be stated as follows: Consider a structure subjected to a set of periodically varying loads, whose bounds of variations are prescribed. We wish to determine whether the structure will shakedown to a state of self-stress after a finite number of cycles of loading and unloading process such that its response to all further cycles becomes purely elastic.

An example will be considered first. Figure 4(a) shows a simply redundant ideal truss of 8 bars with 5 joints subjected to a vertical load P and a horizontal load Q which vary independently. The elements $\{S_{1j}\}$, $\{S_{2j}\}$ and $\{R_j\}$ of the orthonormal stress set for the present example are shown in Figure 4(f) and in Table 3. The original varying loads P and Q may then be written as

$$P = \frac{5}{\sqrt{30}}\,\xi_2$$

$$Q = \sqrt{5}\,\frac{2}{5}\,\xi_1 + \frac{1}{\sqrt{30}}\,\xi_2$$

The yield conditions for the eight members may be expressed as

$$\left.\begin{array}{l}
-1 \leq -\frac{2}{3\sqrt{5}}\,\xi_1 - \frac{2}{\sqrt{30}}\,\xi_2 - \frac{\sqrt{2}}{3}\,\eta \leq 1 \\
-1 < \frac{4}{3\sqrt{5}}\,\xi_1 - \frac{1}{\sqrt{30}}\,\xi_2 - \frac{\sqrt{2}}{3}\,\eta \leq 1 \\
-2 < \frac{\sqrt{2}}{\sqrt{5}}\,\xi_1 - \frac{2\sqrt{2}}{\sqrt{30}}\,\xi_2 < 2 \\
-2 < \frac{1}{\sqrt{5}}\,\xi_1 + \frac{3}{\sqrt{30}}\,\xi_2 < 2 \\
-1 < -\frac{1}{3\sqrt{5}}\,\xi_1 + \frac{1}{\sqrt{30}}\,\xi_2 - \frac{\sqrt{2}}{3}\,\eta \leq 1 \\
-1 < \frac{\sqrt{2}}{3\sqrt{5}}\,\xi_1 + \frac{2\sqrt{2}}{\sqrt{30}}\,\xi_2 + \frac{2}{3}\,\eta \leq 1 \\
-1 \leq -\frac{\sqrt{2}}{3\sqrt{5}}\,\xi_1 - \frac{\sqrt{2}}{\sqrt{30}}\,\xi_2 + \frac{2}{3}\,\eta \leq 1 \\
-1 \leq -\frac{2}{3\sqrt{5}}\,\xi_1 + \frac{3}{\sqrt{30}}\,\xi_2 - \frac{\sqrt{2}}{3}\,\eta \leq 1
\end{array}\right\} \qquad (6.1)$$

where the yield limits of the bars are chosen as

$$t_j = |c_j| = 1 \qquad \text{for } j = 1,2,5,6,7,8,$$

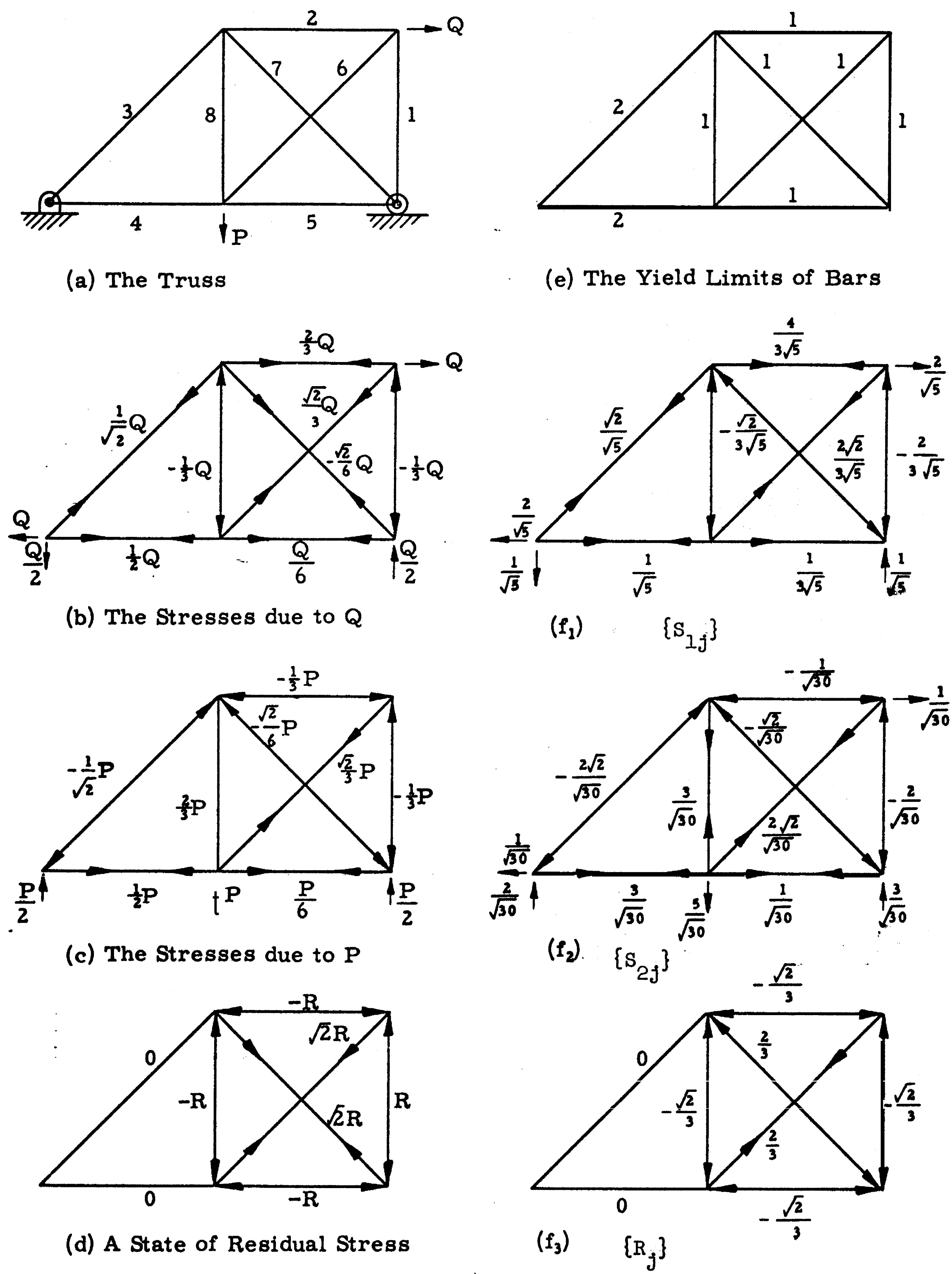
$$= 2 \qquad \text{for } j = 3,4.$$

This choice of t_j and c_j prevents the truss from collapsing due to yielding of the bars 3 and 4 in an incomplete mode. The yield polyhedron for this truss is shown in Figures 5 and 6. Suppose a complicated periodical loading path is entirely contained in a rectangular region given by

$$\left.\begin{array}{l}
-1.5 \leq \xi_1 \leq 0.5 \\
-1.2 \leq \xi_2 \leq 0.1
\end{array}\right\} \qquad (6.2)$$

Table 3. THE **ORTHONORMAL** SET OF STRESS SETS

j	ρ_j	S_{1j}	S_{2j}	R_j
1	1	$-\frac{2}{3\sqrt{5}}$	$-\frac{2}{\sqrt{30}}$	$-\frac{\sqrt{2}}{3}$
2	1	$\frac{4}{3\sqrt{5}}$	$\frac{1}{\sqrt{30}}$	$-\frac{\sqrt{2}}{3}$
3	2	$\sqrt{\frac{2}{5}}$	$-\frac{2}{\sqrt{30}}$	0
4	2	$\frac{1}{\sqrt{5}}$	$\frac{2}{\sqrt{30}}$	0
5	2	$\frac{1}{3\sqrt{5}}$	$\frac{1}{\sqrt{30}}$	$-\frac{\sqrt{2}}{3}$
6	1	$\frac{2\sqrt{2}}{3\sqrt{5}}$	$\frac{2\sqrt{2}}{\sqrt{30}}$	$\frac{2}{3}$
7	1	$-\frac{\sqrt{2}}{3\sqrt{5}}$	$-\frac{\sqrt{2}}{\sqrt{30}}$	$\frac{2}{3}$
8	1	$-\frac{2}{3\sqrt{5}}$	$\frac{3}{\sqrt{30}}$	$-\frac{\sqrt{2}}{3}$

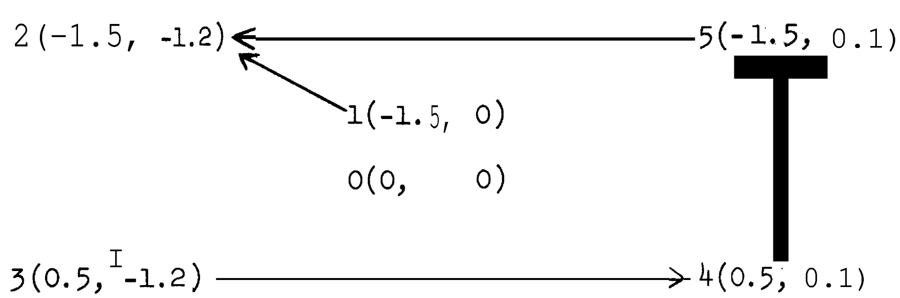


(a) The Truss
(e) The Yield Limits of Bars
(b) The Stresses due to Q
(f₁) {S1j}
(c) The Stresses due to P
(f₂) {S2j}
(d) A State of Residual Stress
(f₃) {Rj}
-R
√2R
-R
R
0

Figure 4

According to the general shakedown theorem, if there exists any state of self-stress which would enable the truss to respond.in a purely elastic manner to all further cycles of loading, then it will shakedown. In order to show that the truss will shakedown, it suffices therefore to find only one state of residual stress to which it might shake down. Any purely elastic response is characterized by the fact that the response curve is entirely on a plane parallel to ξ_1 ξ_2 **plane.** Then the problem may be conceived geometrically as that of imbedding the prescribed region of loading program into the yield polyhedron by a translational displacement normal to itself Only [3]. This leads us to investigate the possibility of imbedding the rectangular region defined by (6.2) into the yield polyhedron. The imbedding can be achieved if the fictitious response to the worst possible loading cycle,which consists of the circumference of the rectangle, shows that η becomes a constant eventually.

In the yield polyhedron shown in Figure 5, the response to the piecewise linear cycle: $0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 2$



is constructed as shown by the arrows

$$\vec{1};\ \vec{2}',\ \vec{2};\ \vec{3};\ \vec{4};\ \vec{5}',\ \vec{5};\ \vec{6}',\ \vec{6}$$

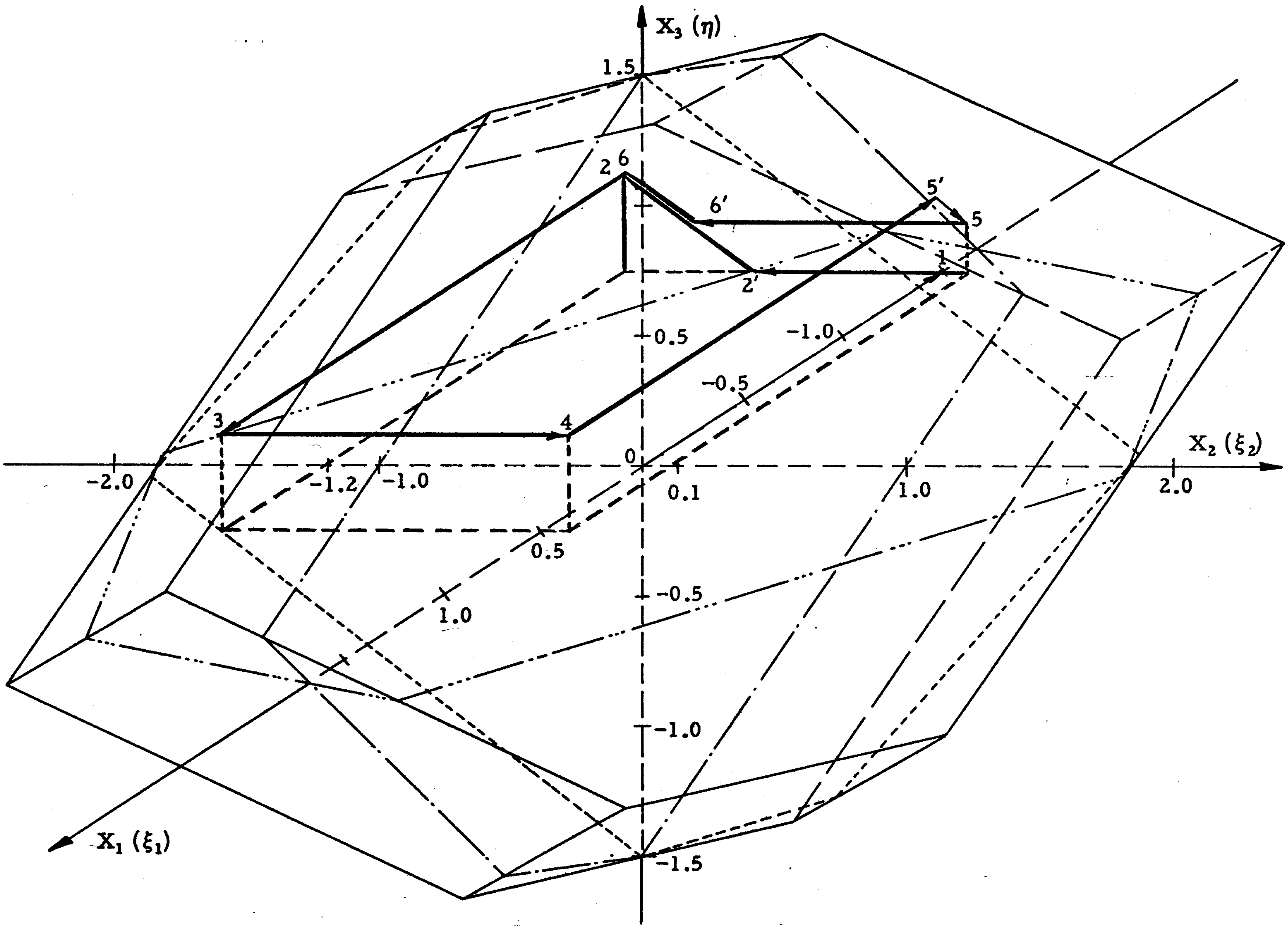


Figure 5. NON-SHAKEDOWN CYCLE

the corresponding **stepwise** formulation of the problem in terms of GP is given in Table 4. The result given in Table 5 shows the coincidence with that obtained graphically. The steps 5 and 6 require plastic deformation so that the truss will not shake down if all possible arbitrary loading cycles in the rectangular region must be taken into account.

In order to illustrate the case where shakedown actually occurs, the response to the small rectangular region

$$\begin{aligned} -1.3 \le \xi_1 \le 0.5 \\ -1.2 \le \xi_2 \le 0.1 \end{aligned} \qquad (6.3)$$

has been obtained as shown in Table 6. The result illustrated in Figure 6 shows that this smaller rectangular region is indeed imbedded in the polyhedron.

This procedure can easily be generalized. If a truss of r degrees of redundancy is subjected to s sets of loads characterized by s load factors whose bounds are prescribed by

$$L_i \le \xi_1 \le U_i \qquad (i = 1,2,\ldots,s) \qquad (6.4)$$

where L_i and U_i denote the lower and upper bounds on ξ_1, respectively, then any conceivable variation of the set $\{\xi_1\}$ is contained in the parallelopiped defined by (6.4). In this case we have r η-type parameters which will be denoted by $\eta_1,\ldots,\eta_r$. A yield polyhedron is then considered in an (s+r)-dimensional stress space. Shakedown will occur under any loading cycle contained in the parallelipiped if it can be imbedded in the yield polyhedron by translation normal to itself only. By virture of the convexity of the yield polyhedron, it is sufficient to consider a fictitious response of the truss to that loading cycle which passes through all the corners of the parallelopiped. If this response shows that all the η_k's become constant after a finite number of cycles, then the imbedding of the parallelopiped is indeed achieved and shakedown occurs, The loading cycle may be piecewise linear from one corner to another. Hence the GP program can be applied. On this basis it appears that the number of steps required to show the shakedown will be at least 2^s and at most 2×2^s. In the case of the example, s = 2. The number of steps N required should be

$$4 \leq N \leq 8$$

Six steps were necessary for the loading path chosen as above.

Table 4. FORMULATION FOR THE NON-SHAKEDOWN CYCLE

$\xi_1 \rightarrow x_1$. $\xi_2 \rightarrow x_2$, $\eta \rightarrow x_3$

Step	Maximize F =	Subject to: Bounds		Yield Ineq.	Path
1	$-x_1$	$x_1 > -1.5$	$-x_1 > -0.5$	(6.1) ~ (6.16)	$x_2 = 0$
2	$-x_2$	$x_2 \geq -1.2$	$-x_2 \geq -0.1$	"	$x_1 = -1.5$
3	x_1	$x_1 \geq -1.5$	$-x_1 \geq -0.5$	"	$x_2 = -1.2$
4	x_2	$x_2 \geq -1.2$	$-x_2 \geq -0.1$	"	$x_1 = 0.5$
5	$-x_1$	$x_1 \geq -1.5$	$-x_1 \geq -0.5$	"	$x_2 = 0.1$
6	x_2	$x_2 > -1.2$	$-x_2 > -0.1$	"	$x_1 = -1.5$

Table 5. THE RESPONSE TO THE NON-SHAKEDOWN CYCLE

Step	Maximum F	x_1 (ξ_1)	x_2 (ξ_2)	x_3 (η)
1	1.50000000	-1.5000000	0	0
2		-1.5000000	-0.71174686	0
2	1.20000000	-1.5000000	-1.20000000	0.37819923
3	0.50000000	0.50000000	-1.20000000	0.37819923
4	0. [illegible]	0.50000000	0. [illegible]	0.37819923
5		-1.3474396	[illegible]	0.37819923
5	1.50000000	-1.50000000	0. [illegible]	0.18522390
6		-1.50000000	-0.95086989	[illegible] 18522390
6	1.20000000	-1.50000000	-1.20000000	0.37819923

Table 6. FORMULATION FOR A SHAKEDOWN CYCLE

Step	Maximize F =	Subject to: Bounds		Yield Ineq.	Path
1	$-x_1$	$x_1 > -1.3$	$-x_1 > -0.5$	(6.1) ~ (6.16)	$x_2 = 0$
2	$-x_2$	$x_2 > -1.2$	$-x_2 > -0.1$	"	$x_1 = -1.5$
3	x_1	$x_1 > -1.3$	$-x_1 > -0.5$	"	$x_2 = -1.2$
4	x_2	$x_2 > -1.2$	$-x_2 > -0.1$	"	$x_1 = 0.5$
5	$-x_1$	$x_1 > -1.3$	$-x_1 > -0.5$	"	$x_2 = 0.1$
6	$-x_2$	$x_2 > -1.2$	$-x_2 > -0.1$	"	$x_1 = -1.5$

Table 7. THE RESPONSE TO THE SHAKEDOWN CYCLE

Step	Maximum F	x_1 (ξ_1)	x_2 (ξ_2)	x_3 (η)
1	1.30000000	-1.30000000	0	0
2	1.20000000	-1.30000000	-1.20000000	0.25170814
3	0.50000000	0.50000000	-1.20000000	0.25170811
4	0.09999999	0.50000000	0. [illegible]	0.25170814
5	1.30000000	-1.30000000	[illegible]	0.25170814
6	1.20000000	-1.30000000	-1.20000000	0.25170814

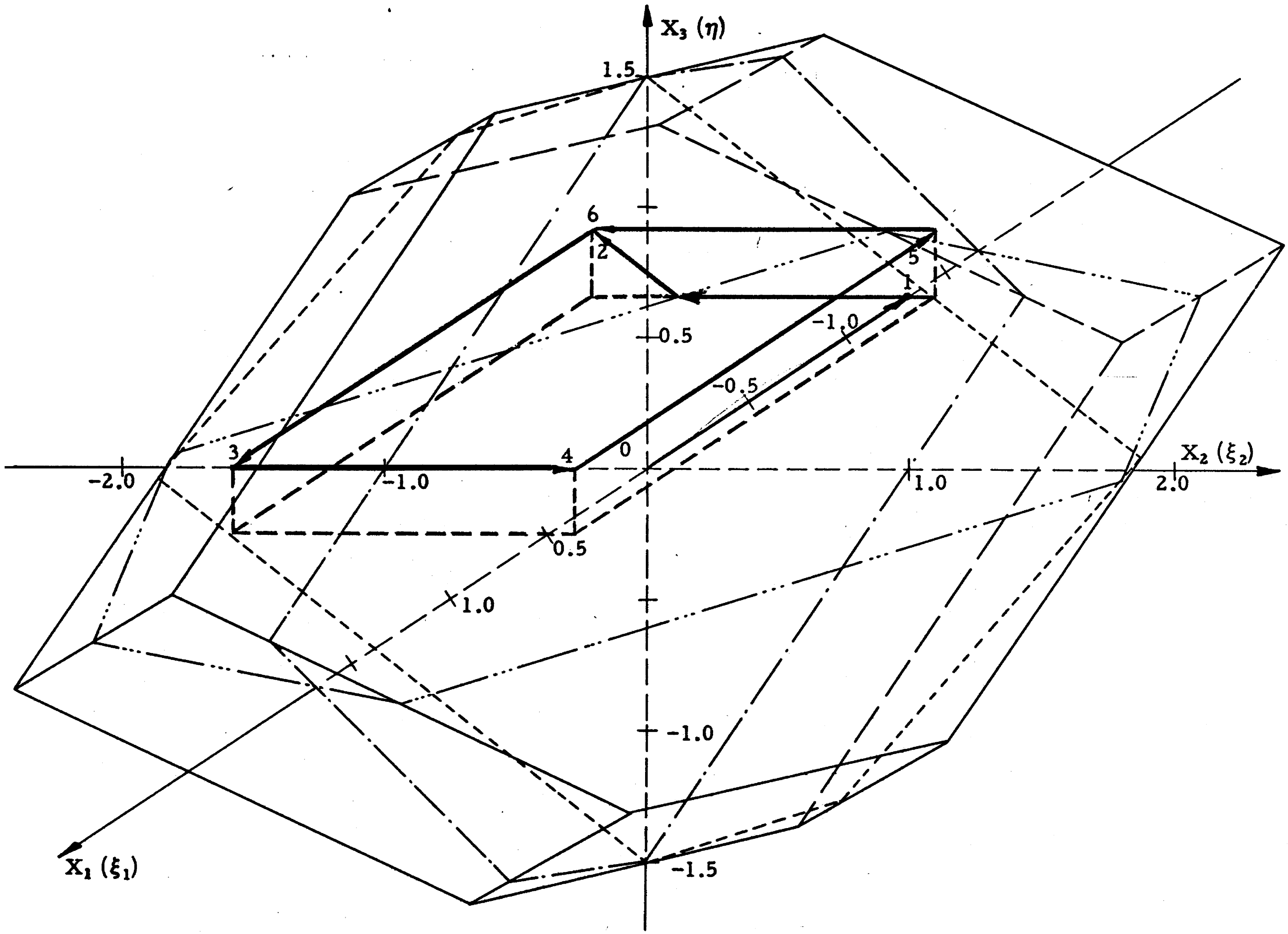


Figure 6. SHAKEDOWN

To investigate more extensively under what kind of loading cycles the given truss will shake down, it is necessary to know the shapes of the elastic subspaces defined by η = cons-t. i.e., cross-sections of the yield polyhedron parallel to $\xi_1 \xi_2$ plane. Responses of the truss are characterized by a family of an infinite number of elastic subspaces to which the truss could shake down. Since the original yield polyhedron is convex, these elastic subspaces are convex polygons. We will tentatively call any of these polygons a "shakedown polygon". Any loading cycle under which the given truss will shake down must therefore be contained in one of this family of an infinite number of shakedown polygons. For practical purposes, several shakedown polygons will be sufficient to reveal the shakedown characteristics of a truss. If we can draw them by some means, then it can be immediately inspected whether or not a given loading cycle or region can be imbedded into the yield polyhedron, or how it may be enlarged or should be shrunk, if the truss is to shake down.

In the present example it is not difficult to obtain these shakedown polygons graphically since there are only eight inequalities as given by (6.1). However, as the number of bars increases, the graphical solution becomes cumbersome. Furthermore, if the truss has r degrees of redundancy then a shakedown polygon is an intersection of the yield polyhedron (E_{r+2}) and r hyperplanes given by

$$\eta_1 = \text{const.}, \qquad \eta_2 = \text{const}, \ldots\ldots, \eta_r = \text{const.}$$

where the truss is subjected to two independently varying sets of loads.

These shakedown polygons can easily be obtained by use of the GP as follows. Since any shakedown polygon is convex, it is always possible to circumscribe it by a rectangle as shown in Figure 7. There are, in general, four points of contact with the rectangle or some of the sides of the polygon may coincide with those of the rectangle. Then it is obvious that the following five steps of maximization suffice to describe the polygon completely.

1:	max.	$F_1 = \xi_1$	with the initial point	0
$1 \to 2$:	max.	$F_2 = \xi_2$	"	, 1
$2 \to 3$:	max.	$F_3 = -\xi_1$	"	, 2
$3 \to 4$:	max.	$F_4 = -\xi_2$	"	, 3
$4 \to 1$:	max.	$F_5 = \xi_1$	"	, 4

As long as the truss is subjected only to two independently varying loads, $F_1, F_2, \ldots, F_5$ remain the same through all the shakedown polygons of the family. Only the right-hand sides of r equality constraints $\{\eta_i = \text{const.}\}$ are changed. The results from the GP program give all the vertices of the polygons. If the number of independent load factors is greater than two, then this technique cannot be used because it is very difficult to describe all the vertices of a complicated polyhedron by means of GP.

7. Load Carrying Capacity and Safe Load Domain.

If a structure is subjected to a set of loads characterized by only one load factor, then the corresponding load carrying capacity is uniquely defined. However, if it is subjected to s sets of loads characterized by s independently varying load factors $\{\xi_1\}$, then the set $\{\xi_i\}$ at collapse depends upon the prescribed loading path. In order to obtain all the sets $\{\xi_i\}$ at collapse, it suffices to consider a family of straight line paths defined by

$$\frac{\xi_1}{m_1} = \frac{52}{m_2} = \ldots\ldots\ldots = \frac{\xi_s}{m_s}$$

where $(m_1, m_2, \ldots, m_s)$ is a set of numbers which determine the ratios between ξ_i's. This family covers the s-dimensional load factor subspace completely. To every one of these paths there corresponds a set $(\xi_1, \xi_2, \ldots, \xi_s)$ at collapse. All these sets $(\xi_1, \xi_2, \ldots, \xi_s)$ form a closed hypersurface in the s-dimensional load factor subspace. Since this hypersurface can be regarded as a projection of the yield polyhedron into the s-dimensional load factor subspace, it must be a convex polyhedron in E_s. This will be called the "safe load domain". This domain is characterized by the property that any combination of the s sets of loads represented by a point interior to it does not cause collapse if the loads are monotonically increased from zero. This can easily by obtained by the GP method since some equality constraints have only to be added.

It should be noted that if we denote an infinite number of regions of shakedown polygons by $D_1, D_2, \ldots, D_m, \ldots\ldots$ then the union $D_1 \cup D_2 \cup \ldots \cup D_m \cup \ldots$ gives a safe load domain approximated from inside i.e., from safe side.

In the case of the example, since there are two load factors all the ratios ξ_1/ξ_2 must be considered. By virtue of the symmetry of the yield polyhedron, we have only to consider a family of straight line paths originating from the origin which cover a half $\xi_1 \xi_2$ plane completely. The safe load domain in this case is a polygon and practically several straight line paths suffice to draw the polygon. The result is shown in Figure 8.

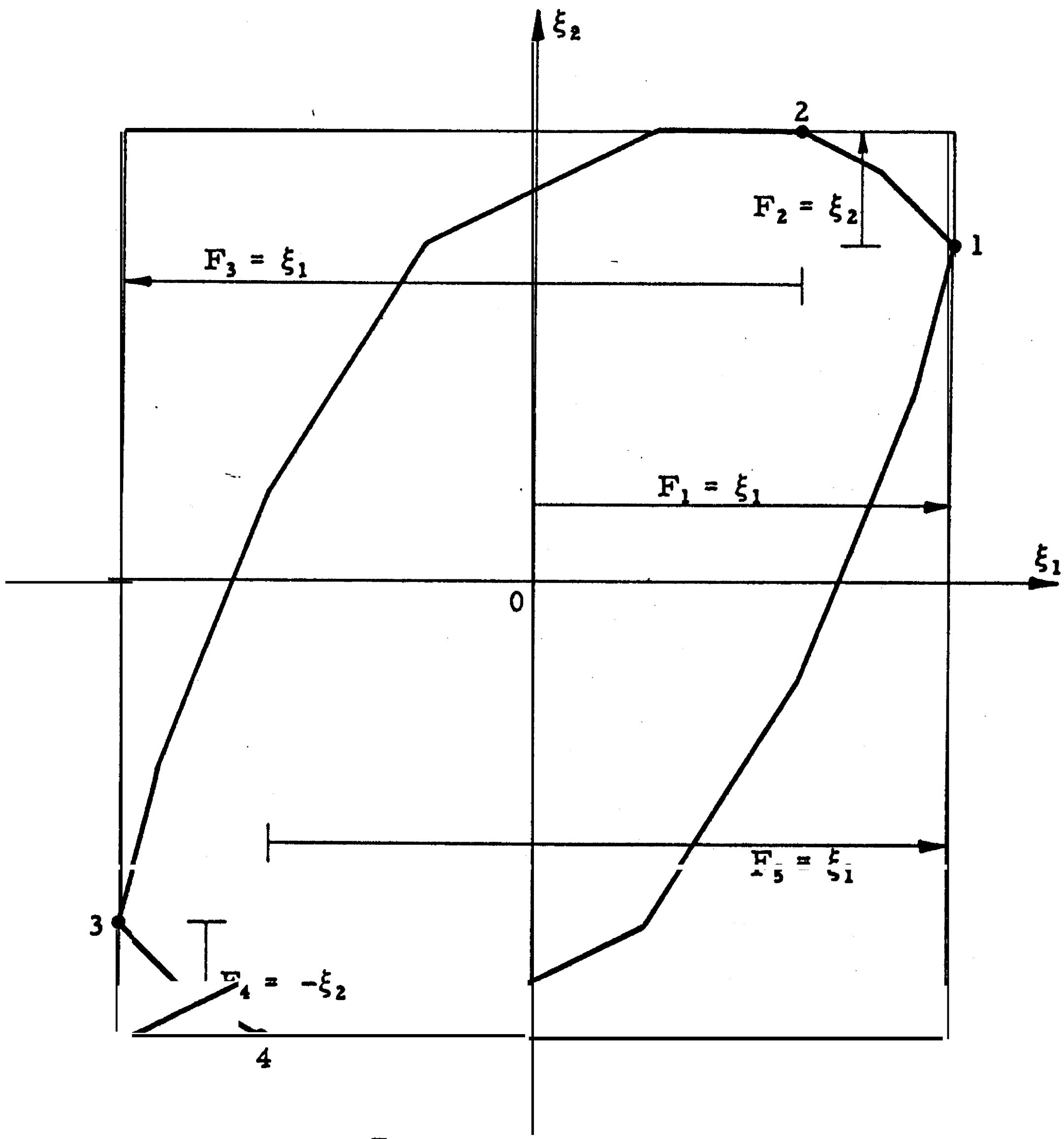


Figure 7. SHAKEDOWN POLYGON

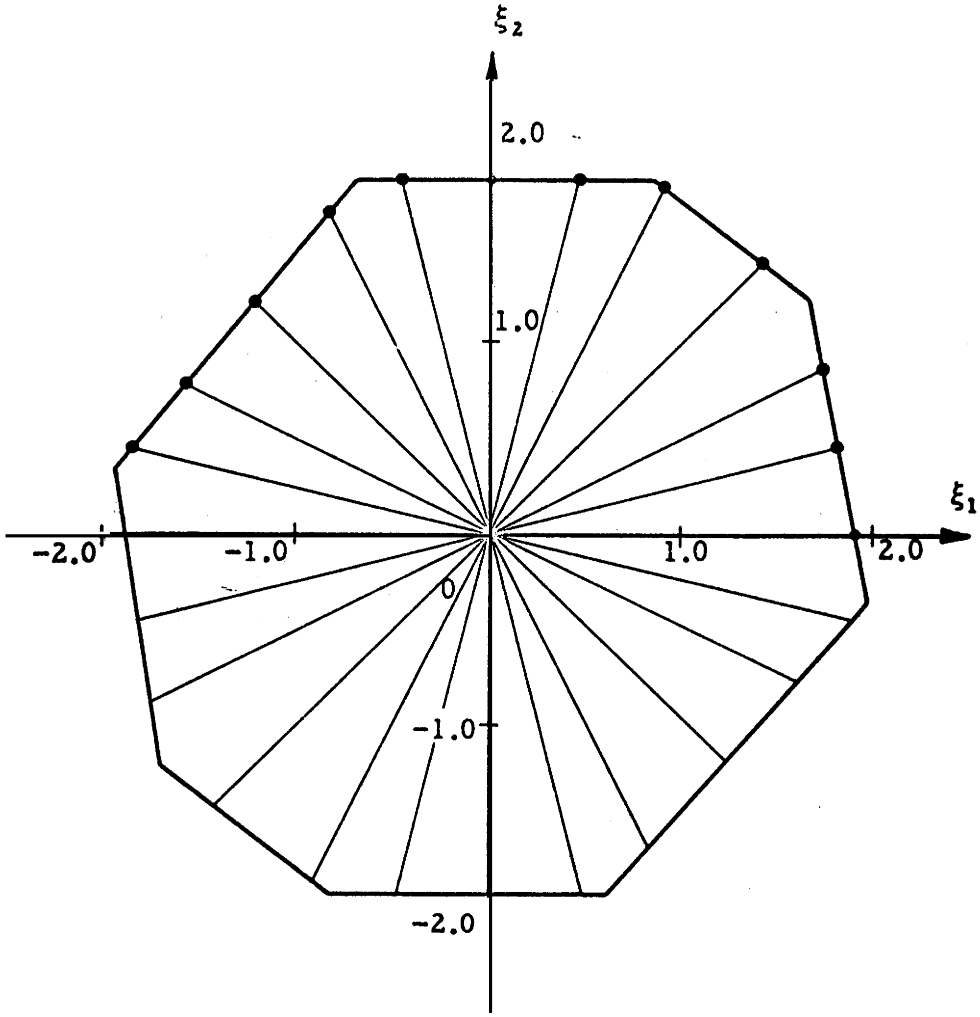


Figure 8. SAFE LOAD DOMAIN

8. Concluding Remarks.

An application of the gradient projection method of nonlinear programming to the elastic-plastic analysis of trusses has been shown to be straightforward and fruitful. It should be emphasized that as long as loading paths are piecewise linear, GP automatically gives integral results for every segment of the paths and the exact elastic-plastic responses of multiply redundant trusses to several independently varying loads can readily be obtained by GP.

It has also been shown that the gradient projection method is useful and powerful to investigate shakedown and load carrying capacities of trusses. Frame structures can be treated in the same manner as trusses without modification. In the case of a frame, it is expected that we have an infinite number of yield inequalities corresponding to an infinite number of cross-sections of its members. However, since most of the collapse modes of frames contain a finite number of plastic hinges, it appears that we should expect to obtain yield polyhedrons rather than smooth convex hypersurfaces. Hence those linear constraints make it possible to use the same technique as above.

References

1. Greenberg, H. J., "Complementary minimum principles for an elastic-plastic material", Q. Appl. Math. Vol. 7, pp. 85-95. 1949.

2. Prager, W., "Introduction to plasticity", Addison-Wesley, 1959.

3. Prager, W., "Stress analysis in the plastic range", Brown University Tech. Rept. C11-52. October 1959.

4. Rosen, J. B., 'The gradient projection method for nonlinear programming". I. Linear Constraints. J. Soc. Indust. Appl. Math. 8, pp. 181-217 (1960).

5. Rosen, J. B., "The gradient projection method for nonlinear programming. Part II. Nonlinear constraints". J. Soc. Indust. Appl. Math. 9, pp. 514-532(1961).

6. Neal, B. G., "The plastic methods of structural analysis", J. Wiley, 1956.

7. Koiter, W. T., 'General theorems for elastic-plastic solids'. Progress in Solid Mechanics. Vol. 1, Interscience Pub. Inc. 1960.