A multi-item economic lot-sizing problem is considered wherein, as a consequence of a joint ordering or production setup cost, the ordering policies for individual items are interdependent.

The problem is to determine an optimal ordering plan in which the sum of the costs of carrying inventories and the costs of ordering are minimized and in which the known demands for each item in each time period are satisfied.

Two algorithms are presented for solving such problems. The first is a direct algorithm which yields periodic solutions and applies to problems in which demand occurs uniformly over time. The second is a dynamic programming algorithm which yields optimal solutions, whether periodic or aperiodic, and which applies to dynamic problems as well as to problems with constant demands.

A multi-item economic lot-sizing problem

by J. F. Pierce

In many inventory management systems, situations arise in which ordering time and ordering quantity of an item are interrelated with those for other items. In some of these cases, consideration of the interrelationship among items is mandatory. This can happen for a group of items, for instance, if there is a budgetary limit on the total dollar value of inventory that may be carried for the group, or if a limit exists on warehouse capacity available for storing items in the group.[1] In other cases, consideration of the interrelationship is optional, the interrelationship being considered only when the added complexity in decision-making is outweighed by the added benefits expected to be gained.

In a manufacturing context, for example, economies may accrue through joint replenishment as a consequence of savings in facility setup and changeover time. Or, in jointly ordering a number of items from a vendor, economies may also accrue as a consequence of the preparation, processing, and expediting of fewer orders and the receiving of fewer shipments.

Considered here are joint replenishment problems with this optional type of interrelationship. In these multi-item economic lot sizing problems, the objective is to minimize total replenishment and inventory carrying costs for all items.

common approaches

To solve such problems one common approach is to determine independently the lot size for each item as if no interrelationship among items exists, and then to "coordinate" the replenishment of items by aligning the reordering periods to minimize ordering costs. For specificity let us consider a problem involving two production items, A and B, having the annual demands Y, unit inventory carrying costs C_c, and production setup costs C_s, as shown in Table 1.

For each item considered independently, the lot size Q which minimizes the annual sum of the inventory carrying costs and the production setup costs, $(Q/2)C_c + (Y/Q)C_s$, for that item can be determined by the classical Wilson formula:

$$Q = \sqrt{\frac{2YC_s}{C_c}} \tag{1}$$

For the example, these lot sizes, together with the interorder time or production cycle time, $b = (12)Q/Y$, are as shown in the last two columns of the table. If these two items are planned independently, the total annual cost for the two items is $Z = 32{,}400 + 3600 = \$36{,}000$.

Assume, however, that items A and B represent different models of the same basic product and that economies in production setup time result whenever the items are produced in sequence. For example, to make the production line ready for the product class, in general, requires fixed cost $F = \$300$, and that the additional time to set up or change over the line for item A is $S_A = \$600$ and for item B is $S_B = \$150$. Thus, if A is produced alone, the total make-ready cost is, as before, $C_s = F + S_A = \$900$; if B is produced alone, $C_s = F + S_B = \$450$; but if both A and B are produced in sequence, the total make-ready and changeover cost is $F + S_A + S_B = \$1050$. Now by properly coordinating the production schedules of the two items (while still maintaining the lot sizes and cycle times in Table 1), it is possible to reduce total make-ready costs and hence total cost. For example, by producing item A in months 1, 3, 5, 7, 9, and 11, producing item B in months 1, 4, 7, and 10, and sequencing together the production in months 1 and 7, annual make-ready costs can be reduced by $[(900 + 450) - (1050)] = \$600$, yielding a reduced total cost of $Z = \$35{,}400$.

Further coordination in ordering the items can be achieved by modifying the lot sizes and cycle times of selected items as they were originally determined by Equation 1. One approach used in practice is to decrease the cycle time (and, accordingly, the lot size) of all items with a low reorder frequency, making it equal to (a multiple of) that of the highest frequency item. In this way, it is hoped to reduce the total number of production setups, and perhaps thereby to reduce total cost. For example, in the illustration we have $b_A = 2$ and $b_B = 3$ which, when coordinated, require eight setups per year. If we increase the frequency of item B to six so that $b_B = 2$, all runs can then be sequenced with runs of item A, thereby reducing the effective setup costs for B as well as the average inventory of item B but increasing the number of setups from four to six. The net effect of this decrease of $b_B = 3$ to $b_B = 2$, as can be verified, is to reduce total cost from \$35,400 to \$34,500.

Table 1 Data for illustrative example

Item	*Annual Demand* Y	*Unit Inventory Carrying Cost* C_c	*Production Setup Cost* C_s	*Optimal Lot Size* Q	*Optimal Production Cycle (Months)* b	*Annual Cost*
A	16 200	$12	$2 700	2 700	2	32 400
B	1 200	$12	$ 450	300	3	3 600

Table 2 Data for illustrative coordinated replenishment plans

	Annual Demand		*Changeover Cost*		*Production Cycle*		*Total Annual Cost*
	Y_A	Y_B	S_A	S_B	b_A	b_B	Z
Case I	2 700	7 200	150	2 400	2	3	26 400
					2	2	27 000
					3	3	26 250
Case II	16 200	7 200	2 400	2 400	2	3	53 400
					2	2	54 000
					3	3	55 500

Coordination approaches to joint replenishment of this type, which involve modification of the cycle time for selected items, are attractive in practice for several reasons. From a decision-making viewpoint, the added complexity and problem-solving effort which results by interrelating the items in this fashion is quite minimal. And, moreover, from an implementation and operational viewpoint, these approaches yield solutions that retain for each item the periodic property of constant lot size and constant interorder or production cycle time, a property sometimes desirable in practice in that it greatly facilitates other activities such as production scheduling and control, physical inventory planning and control, and so on.

The shortcomings of these approaches, however, are that they are unreliable. Even among solutions possessing this periodic property, they do not always yield a solution with minimum total cost; in fact, they sometimes yield solutions which are more costly than those in which all cycle times are simply left at their original values. For illustration, Table 2 shows the data for two examples obtained by slightly modifying the earlier example. As before, F = $300 and C_c = $12. In both cases the values determined independently according to Equation 1 are $b_A = 2$ and $b_B = 3$. As seen in the table, in neither case does a minimum cost solution result by increasing the frequency of item B: in Case II it is preferable to leave the frequencies and lot sizes of both items at their values originally determined independently according to Equation 1; in Case I it is actually better to reduce the frequency of item A and coordinate it with the replenishment of the lower frequency item. And more generally, discovery of a minimum cost solution may be more complicated than this illustration might suggest, for it is not always possible to determine a minimum cost periodic solution by simply adjusting a single item at a time. Sometimes combinations of items need to be altered concomitantly, a consideration which becomes potentially more troublesome as the number of interrelated items increases.

new algorithms

In the following sections, two different algorithms are presented for determining minimum cost solutions to multi-item problems of this type. Both algorithms consider a planning horizon of a fixed duration; for longer horizons it is assumed that the replenishment plan to be determined is simply replicated throughout the longer horizon as required. The planning horizon is then divided into a convenient[2] number of discrete time periods N of equal length. Solutions are sought in which all demands are met exactly, and for which total replenishment and inventory carrying costs are minimum.

The first algorithm is a direct algorithm for determining joint periodic solutions in which the lot size and cycle time for each item remain constant throughout the planning horizon. Under the assumption that each lot is to cover demand for an integral number of time periods in the horizon and that inventory balances are to be zero at the end of the planning horizon, the algorithm yields optimal joint replenishment solutions. Experience on an IBM 7094 computer with this algorithm as coded in FORTRAN has indicated that for $N = 12$, for example, problems with M items can be solved in approximately $M/300$ seconds for $M \leq 1500$. Although not usually of importance for problems with such short solution times, this algorithm has the desirable property that, since search proceeds from one feasible solution to a better feasible solution, the procedure can be terminated prior to its ultimate completion with a usable, although perhaps not optimal, solution.

The second algorithm is a dynamic programming algorithm. Up to this point attention has been focused on periodic solutions in which the lot size and interorder or cycle time for each item remains constant throughout the planning horizon. Upon foregoing this property it may be possible to achieve a lower total replenishment and inventory cost through use of an aperiodic solution. The dynamic programming algorithm to be presented yields a minimum cost solution to the joint replenishment problem whether the optimal solution be periodic or aperiodic. With this same algorithm, it is also possible to solve the dynamic problem in which demand varies from period to period. As would be expected, these more general solutions require considerably more problem-solving effort. In practice, the size of problem for which this approach is practical may be limited since, for N periods and M items the total number of states to be evaluated is

$$\sum_{t=1}^{N} t^M .$$

A direct algorithm for periodic solutions

presentation of problem

For a planning horizon of N periods, the total replenishment and inventory carrying cost for the multi-item joint replenishment problem to be considered in the present section may be expressed as:

$$Z(b_1, b_2, \cdots, b_k, \cdots, b_M) = \sum_{k=1}^{M} (Nd_k i_k b_k/2 + s_k N/b_k) + F \cdot n(b_1, b_2, \cdots, b_k, \cdots, b_M) \qquad (2)$$

where b_k is the number of periods between procurement or production of item k; Nd_k is the total demand for item k assumed to occur at a constant rate throughout the planning horizon; i_k is the unit inventory carrying cost for item k for one time period; F is the general setup or order writing-expediting-receiving cost for the placement of a replenishment order in a period; s_k is the additional setup, changeover or ordering cost incurred each time item k is reordered; $n(b_1, b_2, \cdots, b_k, \cdots, b_M)$, is the number of periods in N in which one or more items are ordered; and M is the number of items being considered.[3] In this formulation a set of interorder times $(b_1, b_2, \cdots, b_k, \cdots, b_M)$, or equivalently, a set of order frequencies $(f_1, f_2, \cdots, f_k, \cdots, f_M)$, $f_k = N/b_k$, is to be determined for which the total cost is minimum (Equation 2). The order quantity for item k is $Q_k = d_k b_k$.

As has been mentioned, it is assumed in the present case that the lot size Q_k is to remain the same in all reorder periods and that no inventory is to exist at the end of the N periods. Together with the assumption that the number of interorder periods b_k is to be integral, the value of b_k becomes limited to one of the possible integers $\alpha_1, \alpha_2, \cdots, \alpha_i, \cdots, \alpha_r$ wherein both α_i and N/α_i are integral, $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r$. For instance, for $N = 12$ the possible values of b_k are $\alpha = 1, 2, 3, 4, 6,$ and 12. Whenever N is a prime number, the possible values of b_k become $\alpha = 1$ and N. With these assumptions, the solution $(b_1, b_2, \cdots, b_M)$ as determined by the following algorithm is optimal.

initial solution

In principle, when there are r different allowable frequencies or interorder periods α_i in period N, an optimal solution can be determined by enumerating all of the r^M possible combinations $(b_1, b_2, \cdots, b_M)$ and selecting one for which total cost is minimum. In light of the structure of Equation 2, however, a solution guaranteed to be optimal can be found with considerably less search. We begin with an initial solution $B^* = (b_1{}^*, b_2{}^*, \cdots, b_k{}^*, \cdots, b_M{}^*)$,

Table 3 $\bar{n}(\delta_1, \delta_2, \ldots, \delta_r)$ N = 4

$\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 4$

δ_1	δ_2	δ_3	$\bar{n}$
1	X	X	4
0	1	X	2
0	0	1	1

Table 4 $\bar{n}(\delta_1, \delta_2, \ldots, \delta_r)$ N = 6

$\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \alpha_4 = 6$

δ_1	δ_2	δ_3	δ_4	$\bar{n}$
1	X	X	X	6
0	1	1	X	4
0	1	0	X	3
0	0	1	X	2
0	0	0	1	1

Table 5 $\bar{n}(\delta_1, \delta_2, \ldots, \delta_r)$ N = 8

$\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 4, \alpha_4 = 8$

δ_1	δ_2	δ_3	δ_4	$\bar{n}$
1	X	X	X	8
0	1	X	X	4
0	0	1	X	2
0	0	0	1	1

Table 6 $\bar{n}(\delta_1, \delta_2, \ldots, \delta_r)$ N = 12

$\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3,$
$\alpha_4 = 4, \alpha_5 = 6, \alpha_6 = 12$

δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	$\bar{n}$
1	X	X	X	X	X	12
0	1	1	X	X	X	8
0	1	0	X	X	X	6
0	0	1	1	X	X	6
0	0	1	0	X	X	4
0	0	0	1	1	X	4
0	0	0	1	0	X	3
0	0	0	0	1	X	2
0	0	0	0	0	1	1

where b_k^* is the integer α_i, $i = 1, 2, \cdots, r$ for which $q_k(b_k)$ is minimum:

$$q_k(b_k) = Nd_k i_k b_k/2 + s_k\, N/b_k$$

(When $F = 0$ and the M items are considered independently, N/b_k^* is the optimal order frequency for item k as determined by the classical Wilson formula). The total cost of the initial solution is then

$$Z^* = Z(b_1^*, \cdots, b_M^*) = \sum_{k=1}^{M} q_k(b_k^*) + F\cdot n(b_1^*, \cdots, b_M^*)$$

where $n(b_1^*, \cdots, b_M^*)$ is the minimum number of periods in which one or more orders must be placed so as to permit the realization of all frequencies N/b_k^*.

Given this initial solution B^*, subsequent interest resides only in solutions B for which $Z(b_1, b_2, \cdots, b_k, \cdots, b_M) < Z^*$. Since the quantity $q_k(b_k^*)$ is a minimum for each item k, a less costly solution $B' = (b_1' \cdots, b_M')$, when one exists, must necessarily be one in which the total number of periods having one or more orders is reduced, i.e., $n(b_1', \cdots, b_M') < n(b_1^*, \cdots, b_M^*)$. To guarantee an optimal solution, all such possibilities must be investigated, at least implicitly. Let us consider first the determination of the values $n(b_1, b_2, \cdots, b_M)$.

For any solution $B = (b_1, b_2, \cdots, b_M)$, the minimum value $n(b_1, b_2, \cdots, b_M)$ can be easily computed with the aid of a predetermined table of minimum values $\bar{n}(\delta_1, \delta_2, \cdots, \delta_r)$ which depends only on the total number of periods N and the set of allowable interorder times $\alpha_1, \alpha_2, \cdots, \alpha_r$. The variables $\delta_1, \delta_2, \cdots, \delta_r$ are determined from B as:

$$\delta_i = \begin{cases} 1 & \text{if } b_k = \alpha_i \text{ for any } k \\ 0 & \text{otherwise} \end{cases} \qquad \begin{matrix} k = 1, 2, \cdots, M \\ i = 1, 2, \cdots, r \end{matrix}$$

When $\delta_i = 1$, orders must be so placed during the N periods that an interorder time of α_i periods is realizable. For each of the 2^r possible combinations $(\delta_1, \delta_2, \cdots, \delta_r)$, the minimum value $\bar{n}(\delta_1, \delta_2, \cdots, \delta_r)$ can be determined and then tabled. Since, in practice, a number of combinations often have the same value $\bar{n}(\delta_1, \delta_2, \cdots, \delta_r)$, considerably fewer than 2^r entries need be explicitly tabled.

As illustrations, Tables 3 through 8 give the values of $\bar{n}(\delta_1, \delta_2, \cdots, \delta_r)$ for $N = 4$, 6, 8, 12, 16, and 24 respectively. An entry of X in a table indicates that the value of δ_i may be either zero or one. To explain the derivation of these entries, let us consider the combinations $(0, 0, 0, 1, \delta_5, \delta_6)$ for $N = 12$ in Table 6. With $\delta_4 = 1$, orders must be placed in a manner which permits an interorder interval of $\alpha_4 = 4$ periods. Orders are therefore placed in periods 1, 5, and 9. This ordering schedule then satisfies the case $\delta_5, = \delta_6 = 0$. Also, it satisfies the case $\delta_5 = 0$, $\delta_6 = 1$ since an interorder interval of $\alpha_6 = 12$ is realizable with an order in period 1. We therefore have $\bar{n}(0, 0, 0, 1, 0, X) = 3$. When $\delta_5 = 1$, an interval of $\alpha_5 = 6$ must be realizable; this requires that orders be placed in periods 1 and 7. Coupled with $\delta_4 = 1$, the result is an ordering schedule that includes periods 1, 5, 7, and 9. Since this schedule satisfies both the cases, $\delta_6 = 0$ and $\delta_6 = 1$, we have the single entry $\bar{n}(0, 0, 0, 1, 1, X) = 4$. In like manner, each of the other entries in the tables is derived.

Let us now return to the requirement for considering (at least implicitly) all solutions B for which $n(b_1, b_2, \cdots, b_M) < n(b_1^*, b_2^*, \cdots, b_M^*)$. This requirement is fulfilled with the help of these tables of minimum values $\bar{n}(\delta_1, \delta_2, \cdots, \delta_r)$. More explicitly, the consideration of all such solutions is guaranteed, first, by considering (at least implicitly) all "states" $(\delta_1', \delta_2', \cdots, \delta_r')$ in the table for which $\bar{n}(\delta_1', \delta_2', \cdots, \delta_r') < \bar{n}(\delta_1^*, \delta_2^*, \cdots, \delta_r^*)$, where $(\delta_1^*, \delta_2^*, \cdots, \delta_r^*)$ is the state determined by the initial solution $B^* = (b_1^*, b_2^*, \cdots, b_M^*)$; and, second, by considering (at least implicitly) for each such state $(\delta_1', \delta_2', \cdots, \delta_r')$ all solutions B corresponding to that state. Condition one is fulfilled by ordering the states in lexicographically decreasing order and proceeding to systematically investigate all states lexicographically smaller than the initial state, $(\delta_1^*, \delta_2^*, \cdots, \delta_r^*)$ determined by the initial solution $(b_1^*, b_2^*, \cdots, b_M^*)$. A state $(\delta_1^a, \delta_2^a, \cdots, \delta_r^a)$ is defined to be lexicographically smaller than state $(\delta_1^b, \delta_2^b, \cdots, \delta_r^b)$ if $\delta_i^a = \delta_i^b = 0$, $i = 1, 2, \cdots, s - 1$ and either $\delta_s^b = 1$, $\delta_s^a = 0$, or $\delta_s^b = \delta_s^a = 1$ and $\bar{n}(\delta_1^a, \delta_2^a, \cdots, \delta_r^a) < \bar{n}(\delta_1^b, \delta_2^b, \cdots, \delta_r^b)$, for some s, $1 \leq s \leq r$. (The states in Tables 3 through 8, for instance, are tabled in decreasing order.) Condition two is satisfied by determining for each state explicitly considered a solution for which $Z(b_1, b_2, \cdots, b_M)$ is minimum among all solutions corresponding to that state.

Table 7 $\bar{n}(\delta_1, \delta_2, \ldots, \delta_r)$ N = 16

$\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 4, \alpha_4 = 8, \alpha_5 = 16$

δ_1	δ_2	δ_3	δ_4	δ_5	$\bar{n}$
1	X	X	X	X	16
0	1	X	X	X	8
0	0	1	X	X	4
0	0	0	1	X	2
0	0	0	0	1	1

bounding solution

In the following discussion, reference will frequently be made to a solution B for which, among all solutions constrained by the condition $b_k \geq \alpha_i$, the sum $\sum_{k=1}^{M} q_k(b_k)$ is minimum. We will term such a solution "a bounding solution for $b_k \geq \alpha_i$," and denote it by $B^i = (b_1^i, b_2^i, \cdots, b_M^i)$. The value of the sum itself, $\sum_{k=1}^{M} q_k(b_k^i)$, will be denoted as π_i. In addition, we will on occasion wish to distinguish between states in which, for $\delta_1 = \delta_2 = \ldots = \delta_{i-1} = 0$ and $\delta_i = 1$, $\bar{n}(\delta_1, \delta_2, \cdots, \delta_r) = N/\alpha_i$ and $\bar{n}(\delta_1, \delta_2, \cdots, \delta_r) > N/\alpha_i$. Such states will be termed prime and nonprime states respectively. Solutions B corresponding to each of these states will be similarly termed prime and nonprime. Finally, at each point throughout the search process, the best (minimum total cost) solution found thus far will be denoted as $B^0 = (b_1^0, b_2^0, \cdots, b_M^0)$ and its total cost as Z^0.

We start, then, with the initial solution $(b_1^*, b_2^*, \cdots, b_M^*)$ as the best thus far, $B^0 = (b_1^*, b_2^*, \cdots, b_M^*)$ and $Z^0 = \sum q_k(b_k^*) + F \cdot \bar{n}(\delta_1^*, \delta_2^*, \cdots, \delta_r^*)$, and proceed to investigate the states lexicographically smaller than $(\delta_1^*, \delta_2^*, \cdots, \delta_r^*)$ in search of a less costly solution. If s denotes the smallest index i in $(\delta_1^*, \delta_2^*, \cdots, \delta_i^*, \cdots, \delta_r^*)$ such that $\delta_i^* = 1$, then the initial solution is a bounding solution for $b_k \geq \alpha_s$, $(b_1^s, b_2^s, \cdots, b_M^s)$, with

Table 8 $\bar{n}(\delta_1, \delta_2, \ldots, \delta_r)$ N = 24

$\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \alpha_4 = 4, \alpha_5 = 6, \alpha_6 = 8, \alpha_7 = 12, \alpha_8 = 24$

δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8	$\bar{n}$
1	X	X	X	X	X	X	X	24
0	1	1	X	X	X	X	X	16
0	1	0	X	X	X	X	X	12
0	0	1	1	X	X	X	X	12
0	0	1	0	X	1	X	X	10
0	0	1	0	X	0	X	X	8
0	0	0	1	1	X	X	X	8
0	0	0	1	0	X	X	X	6
0	0	0	0	1	1	X	X	6
0	0	0	0	1	0	X	X	4
0	0	0	0	0	1	1	X	4
0	0	0	0	0	1	0	X	3
0	0	0	0	0	0	1	X	2
0	0	0	0	0	0	0	1	1

$\pi_s = \sum_{k=1}^{M} q_k(b_k^*)$. In general, this (or any) bounding solution $(b_1^s, b_2^s, \cdots, b_M^s)$, may be either prime or nonprime. Let us consider each of these possibilities in turn.

When the state determined by the bounding solution $(b_1^s, b_2^s, \cdots, b_M^s)$ is prime, then $\delta_s = 0$ for all lexicographically smaller states, as may be seen in Tables 3 through 8. In this case, all subsequent solutions requiring consideration are therefore constrained by the condition $b_k \geq \alpha_{s+1}$, $k = 1, 2, \cdots, M$. From among all such solutions, we first consider a solution for which the sum $\sum_{k=1}^{M} q_k(b_k)$ is minimum; this solution, by definition, is a bounding solution $B^{s+1} = (b_1^{s+1}, b_2^{s+1}, \cdots, b_M^{s+1})$ for $b_k \geq \alpha_{s+1}$. The sum π_{s+1} constitutes a lower bound on $\sum_k q_k(b_k)$ for all subsequent solutions requiring consideration.

Computationally a solution B^{s+1} can be readily derived from solution B^s. For in general, if $(b_1^*, b_2^*, \cdots, b_M^*)$ is the unconstrained solution for which $\sum_k q_k(b_k)$ is minimum, then a bounding solution $(b_1^q, b_2^q, \cdots, b_M^q)$ for which $b_k \geq \alpha_q$ and $\sum_k q_k(b_k)$ is minimum is:

$$b_k^q = \max\,[b_k^*, \alpha_q] \qquad k = 1, 2, \cdots, M \tag{3'}$$

This follows from the fact that for each k, $q_k(b_k)$ assumes a minimum at $b_k = b_k^*$ and is a non-decreasing function of b_k for $b_k > b_k^*$. From Equation 3′ it follows that, starting with the initial solution $(b_1^s, b_2^s, \cdots, b_M^s)$, subsequent bounding solutions may conveniently be determined successively with:

$$b_k^{s+j+1} = \max\,[b_k^{s+j}, \alpha_{s+j+1}]; \quad k = 1, 2, \cdots, M; \quad \alpha_{s+j+1} \leq \alpha_r \tag{3}$$

In the present case, then, we have for B^{s+1}, $b_k^{s+1} = \max\,[b_k^s, \alpha_{s+1}]$ for all k.

Determining from B^{s+1} the state $(\delta_1^{s+1}, \delta_2^{s+1}, \cdots, \delta_r^{s+1})$ and then from the table the value of $\bar{n}(\delta_1^{s+1}, \delta_2^{s+1}, \cdots, \delta_r^{s+1})$, the total cost $Z^{s+1} = \pi_{s+1} + F \cdot \bar{n}(\delta_1^{s+1}, \delta_2^{s+1}, \cdots, \delta_r^{s+1})$ is computed. If $Z^{s+1} < Z^0$ solution B^{s+1} is retained as the best found thus far, setting $B^0 = B^{s+1}$ and $Z^0 = Z^{s+1}$.

Having determined and evaluated the bounding solution B^{s+1}, we next proceed to the investigation of solutions corresponding to states lexicographically smaller than $(\delta_1^{s+1}, \delta_2^{s+1}, \cdots, \delta_r^{s+1})$. That no other solutions need be first investigated results from the following consideration:

If $(\delta_1^q, \delta_2^q, \cdots, \delta_r^q)$ is the prime state for $\delta_1 = \delta_2 = \cdots = \delta_{q-1} = 0$ and $\delta_q = 1$, and $(\delta_1^{q+1}, \delta_2^{q+1}, \cdots, \delta_r^{q+1})$ is the state determined by the bounding solution B^{q+1}, then $Z^{q+1} \leq Z(b_1', \cdots, b_M')$ for all solutions B' corresponding to states lexicographically smaller than $(\delta_1^q, \delta_2^q, \cdots, \delta_r^q)$ and lexicographically at least as large as $(\delta_1^{q+1}, \delta_2^{q+1}, \cdots, \delta_r^{q+1})$. (4)

This follows directly from the fact that for all solutions B' corresponding to these states $\sum_k q_k(b_k') \geq \pi_{q+1}$ and, $n(b_1', b_2', \cdots, b_M') \geq n(b_1^{q+1}, b_2^{q+1}, \cdots, b_M^{q+1})$ so that $\sum_k q_k(b_k') + F \cdot n(b_1', b_2', \cdots, b_M') \geq Z^{q+1}$.

In proceeding to the investigation of states lexicographically smaller than $(\delta_1^{s+1}, \delta_2^{s+1}, \cdots, \delta_r^{s+1})$, two possible cases arise, just as they did with the initial solution B^s: the bounding solution B^{s+1} is prime or it is non-prime. When B^{s+1} is prime, we proceed to the determination of the next bounding solution B^{s+2} in the same manner as described for B^{s+1}. Let us consider, then, the case when a bounding solution is non-prime.

In the event a bounding solution B^q is non-prime, it is necessary, in general, to individually consider each of the lexicographically smaller states for which $\delta_q = 1$, as listed in the appropriate table of states. For each such state $(\delta_1', \delta_2', \cdots, \delta_r')$, only a single solution B' need be determined for which the sum $\sum_k q_k(b_k)$ is minimum. In each case such a minimum solution can be derived from the bounding solution B^q by setting:

$$b_k' = \begin{cases} \alpha_j & \text{for } b_k^q = \alpha_i, \delta_i' = 0 \\ b_k^q & \text{otherwise} \end{cases} \tag{5}$$

where[4]

$$q_k(\alpha_j) = \min_{\{t/\delta_t'=1\}} q_k(\alpha_t) .$$

(In light of the convexity of the function $q_k(b_k)$, at most two values of t, $\{t/\delta_t' = 1\}$, need explicitly be considered: the largest value of t such that $t < i$; and the smallest value of t such that $t > i$, when one exists.) For the resulting solution B', the total cost $Z' = \sum_k q_k(b_k') + F \cdot \overline{n}(\delta_1', \delta_2', \cdots, \delta_r')$ is computed, and when its value is less than Z^0, B' is saved as the best solution thus far.[5] When a minimum solution has been determined and evaluated for each lexicographically smaller state with $\delta_q = 1$, the investigation then proceeds to the consideration of states with $\delta_q = 0$.

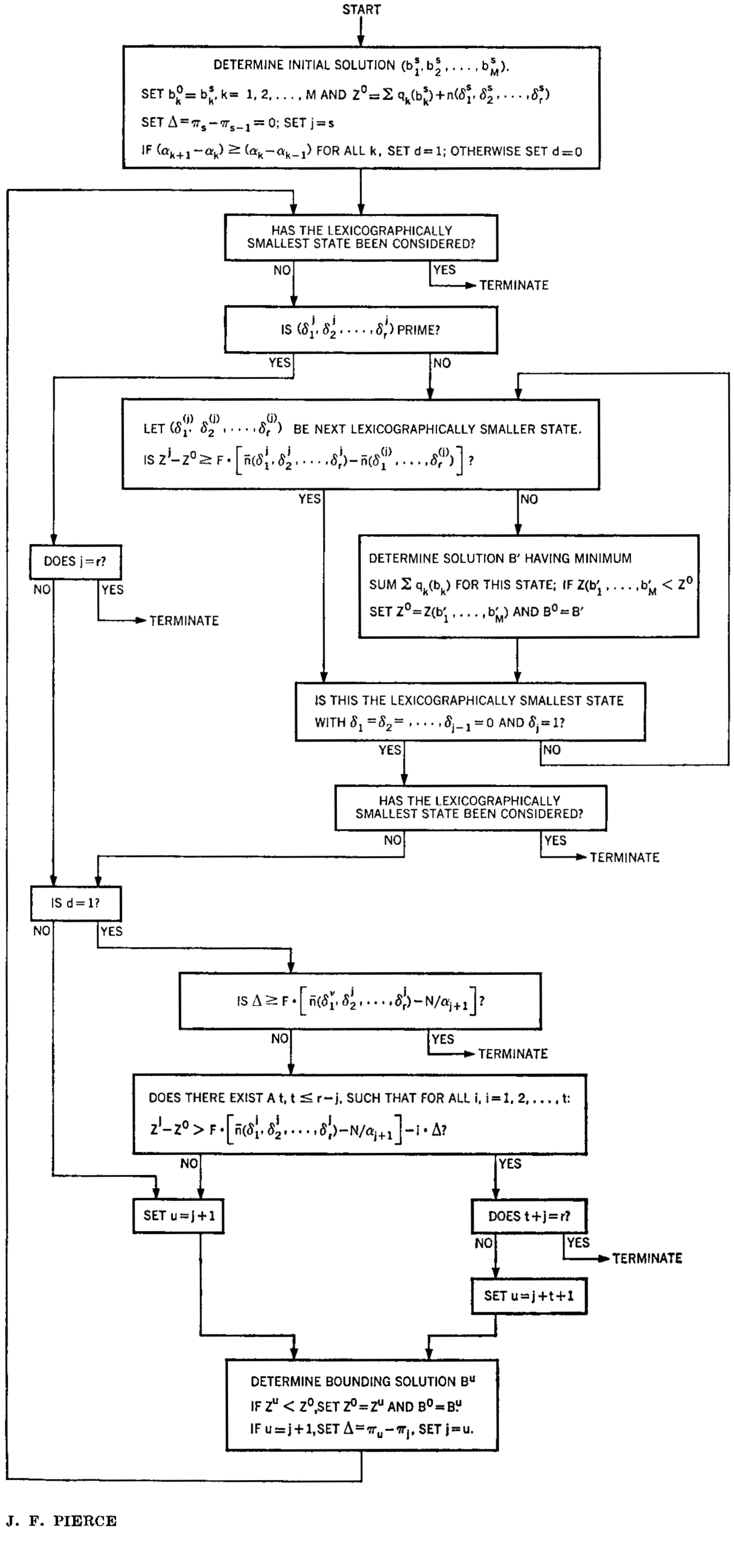
algorithm summary

In summary, the algorithm consists of determining an initial solution B^s and then proceeding to systematically investigate the states lexicographically smaller than $(\delta_1^s, \delta_2^s, \cdots, \delta_r^s)$ as given by the appropriate predetermined table. The main progression of the search begins with B^s and continues in succession to the $(r - s)$ bounding solutions $B^{s+1}, B^{s+2}, \cdots, B^r$, each solution being determined from its predecessor by Equation 3. For each bounding solution B^j, $s \leq j \leq r$, that is found to be nonprime, it is necessary to determine and evaluate a minimum solution B' for each state lexicographically smaller than B^j for which $\delta_j = 1$; each such solution B' is determined from B^j according to Equation 5. Search is complete when the prime state for $\delta_r = 1$ has been considered.

In some instances, it may be possible to further reduce the number of states which need be explicitly considered. Three such instances are as follows:

Let B^{j-1} and B^j, $s < j$, be any successive bounding solutions in any problem for which $(\alpha_{k+1} - \alpha_k) \geq (\alpha_k - \alpha_{k-1})$ for[6] all $k = 2, \cdots, M - 1$. If $\pi_j - \pi_{j-1} \geq F \cdot [\overline{n}(\delta_1^j, \delta_2^j, \cdots, \delta_r^j) - N/\alpha_{j+1}]$, then it is unnecessary to explicitly investigate states with $\delta_1 = \delta_2 = \cdots = \delta_j = 0$. (6)

Figure 1 General flowchart of search procedure



Let B^{j-1} and B^j, $s < j$, be any successive bounding solutions in any problem for which $(\alpha_{k+1} - \alpha_k) \geq (\alpha_k - \alpha_{k-1})$ for all $k = 2, 3, \cdots, M - 1$. If $Z^j - Z^0 + t(\pi_j - \pi_{j-1}) \geq F\cdot[\bar{n}(\delta_1{}^j, \delta_2{}^j, \cdots, \delta_r{}^j) - N/\alpha_{j+t}]$ for any t, it is unnecessary to explicitly investigate states with $\delta_1 = \delta_2 = \cdots = \delta_{j+t-1} = 0$ and $\delta_{j+t} = 1$. (7)

Proofs of Conditions 6 and 7 are given in the Appendix.

For any state $(\delta_1{}^{(j)}, \delta_2{}^{(j)}, \cdots, \delta_r{}^{(j)})$ with $\delta_1{}^{(j)} = \delta_2{}^{(j)} = \cdots = \delta_{j-1}{}^{(j)} = 0$ and $\delta_j{}^{(j)} = 1$ which is lexicographically smaller than the bounding state $(\delta_1{}^j, \delta_2{}^j, \cdots, \delta_r{}^j)$, if $Z^j - Z^0 \geq F\cdot[\bar{n}(\delta_1{}^j, \delta_2{}^j, \cdots, \delta_r{}^j) - \bar{n}(\delta_1{}^{(j)}, \delta_2{}^{(j)}, \cdots, \delta_r{}^{(j)})]$, the state need not be explicitly investigated. (8)

This follows from the fact that the value $\hat{Z}^{(j)} = Z^j - F\cdot[\bar{n}(\delta_1{}^j, \delta_2{}^j, \cdots, \delta_r{}^j) - \bar{n}(\delta_1{}^{(j)}, \delta_2{}^{(j)}, \cdots, \delta_r{}^{(j)})]$ constitutes a lower bound on the total cost of all solutions for state $(\delta_1{}^{(j)}, \delta_2{}^{(j)}, \cdots, \delta_r{}^{(j)})$, i.e., $\hat{Z}^{(j)} \leq Z^{(j)}$ for all $B^{(j)}$. Thus if $Z^0 \leq \hat{Z}^{(j)})$, there exists no solution $B^{(j)}$ which is preferred to B^0, and hence solutions $B^{(j)}$ need not be further considered. Condition 8 results upon substituting for $\hat{Z}^{(j)}$ in the condition $Z^0 \leq \hat{Z}^{(j)}$, and rearranging terms.

Incorporating these considerations into the general search process, the resulting procedure is as shown in Figure 1. To illustrate the algorithm, we solve a problem involving two items wherein it will be assumed that $N = 12$. With $d_1 = 35$, $i_1 = 4$, $d_2 = 150$, $i_2 = 5$, $s_1 = s_2 = 200$ and $F = 280$, the quantities $q_k(b_k) = N\cdot d_k\cdot i_k\cdot b_k/2 + s_k\, N/b_k$ for $k = 1, 2,$; $b_k = \alpha = 1, 2, 3, 4, 6$, and 12 are calculated as shown in Table 9.

From these values the initial solution is seen to be $b_1{}^0 = 2$ and $b_2{}^0 = 1$ with $g_1(b_1{}^0) = 2880$ and $g_2(b_2{}^0) = 6900$. The minimum sum is therefore $\pi_r = 2880 + 6900 = 9780$. With $\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = \delta_4 = \delta_5 = \delta_6 = 0$, reference to Table 6 gives $\bar{n}(\delta_1', \delta_2', \cdots, \delta_6') = 12$ periods. The total cost is thus $Z^0 = \pi_1 + F(12) = 9780 + 280(12) = 13{,}140$. Switch d is set to 1.

Starting with the second block in Figure 1, the path traced out is determined by the following responses at successive blocks: the smallest state has not been investigated; $(\delta_1', \delta_2', \cdots, \delta_6')$ is prime; $j \neq r$; $d = 1$; $\Delta = 0$; there exists no such t; and $u = j + 1 = 2$. Determining the bounding solution $(b_1{}^2, b_2{}^2)$, we have from $(b_1{}^1, b_2{}^1)$ that $b_1{}^2 = b_1{}^1$ and $b_2{}^2 = 2$; consequently $q_1(b_1{}^2) = 2880$, $q_2(b_2{}^2) = 10{,}200$ and $\pi_2 = 13{,}080$. Referring to Table 5, it is seen that

Table 9 Initial calculation for 2-item example

	$\alpha_1 = 1$	$\alpha_2 = 2$	$\alpha_3 = 3$	$\alpha_4 = 4$	$\alpha_5 = 6$	$\alpha_6 = 12$
$k = 1$	3 240	2 880	3 320	3 960	5 440	10 280
$k = 2$	6 900	10 200	14 300	18 600	27 400	54 200

Table 10 Data for 11-item example

Item k	Demand Nd_k	Inventory Cost Ni_k	Cost $q_k(b_k)$					
			$\alpha_1 = 1$	$\alpha_2 = 2$	$\alpha_3 = 3$	$\alpha_4 = 4$	$\alpha_5 = 6$	$\alpha_6 = 12$
1	80	.20	12.67	7.33	6.00	5.67	6.00	9.00
2	49	1.00	14.04	10.08	10.12	11.17	14.25	25.50
3	80	1.25	16.17	14.33	16.50	19.67	27.00	51.00
4	180	.20	13.50	9.00	8.50	9.00	11.00	19.00
5	100	1.00	16.17	14.33	16.50	19.67	27.00	51.00
6	320	1.25	28.67	39.33	54.00	69.67	102.00	201.00
7	36	.25	12.38	6.75	5.13	4.50	4.25	5.50
8	125	.20	13.04	8.08	7.12	7.17	8.25	13.50
9	64	1.00	14.67	11.33	12.00	13.67	18.00	33.00
10	180	1.25	21.38	24.75	32.13	40.50	58.25	113.50
11	16	.25	12.17	6.33	4.50	3.67	3.00	3.00

Table 11 Optimal ordering schedule for 11-item example

Period	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
	1				1				1			
	2		2		2		2		2		2	
	3		3		3		3		3		3	
	4		4		4		4		4		4	
	5		5		5		5		5		5	
Items	6		6		6		6		6		6	
Ordered	7						7					
	8				8				8			
	9		9		9		9		9		9	
	10		10		10		10		10		10	
	11						11					

$\bar{\pi}(0, 1, 0, 0, 0, 0) = 6$ so that $Z(b_1^2, b_2^2) = \pi_2 + F(6) = 13{,}080 + 280(6) = 14{,}760$. Because $Z(b_1^2, b_2^2) > Z^0$, the best solution thus far remains the initial solution. Since $u = j + 1$, we set $\Delta = \pi_2 - \pi_1 = 3300$, and finally, $j = 2$.

Returning to the second block at the top of the flowchart, we proceed to trace out the same path. The smallest state has not been investigated; $(\delta_1^2, \delta_2^2, \cdots, \delta_6^2)$ is prime; $j \neq r$ and $d = 1$. However, this time $\Delta = 3300 > F \cdot [\bar{\pi}(\delta_1^2, \delta_2^2, \cdots, \delta_6^2) - N/3] = 280\ [6 - 4] = 560$ so that search is terminated with the solution $b_1 = 2$ and $b_2 = 1$ being optimal.

Table 12 Optimal ordering schedule for constrained 11-item example $b_1 \leq 3; b_7 \leq 3; b_{11} \leq 3$

Period	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
	1		1		1		1		1		1	
	2		2		2		2		2		2	
	3		3		3		3		3		3	
	4		4		4		4		4		4	
Items	5		5		5		5		5		5	
Ordered	6		6		6		6		6		6	
	7		7		7		7		7		7	
	8				8				8			
	9		9		9		9		9		9	
	10		10		10		10		10		10	
	11		11		11		11		11		11	

For a somewhat larger illustration, we use the data for the 11 items of Naddor and Saltzman.[7] This data together with the value of $q_k(b_k)$ for $N = 12$ is shown in Table 10. For all items $s_k = 1.00$ and $F = 5.00$.

As can be verified by tracing through the procedure in Figure 1, an optimal solution for this problem is given by the schedule in Table 11 showing by period the items that are to be ordered in each period; the cost of this schedule is 173.25.

In some contexts, limits may be imposed on the acceptable range of interorder times b_k for specific items as, for instance, when items are perishable. Such limits are readily accommodated in the algorithm, the general effect being to reduce the amount of search that is required. As an illustration, we impose shelf-life constraints on items 1, 7, and 11, requiring that $b_k \leq 3$. With these constraints, an optimal solution is given by the order schedule of Table 12. As a consequence of these constraints, the minimum cost is increased from 173.25 to 180.75.

summary comment

In summary, the algorithm presented in this section proceeds directly to a feasible solution for the joint replenishment problem and then continues to search for successively better and better feasible solutions until ultimately one is discovered that is shown to be optimal among periodic solutions. Computationally the algorithm appears quite efficient, solving multi-item problems with several hundred items in seconds on an IBM 7094 computer.

In some problems, however, it may be desirable to explicitly consider the dynamic aspects of item demand. Or, in some, the added problem-solving effort expended in determining an optimal solution, periodic or aperiodic, may be warranted. We conclude with a dynamic programming approach to the solution of these problems.

A dynamic programming formulation

Assume that d_{kt} denotes the demand for item k in time period t which we assume is known for all items for each of the N time periods. The present problem is then to determine the quantity x_{kt} of each item k to procure for each time period t (assumed to be on hand at the beginning of period t) so as to fill all demands d_{kt} and so as to minimize the sum of the variable costs of carrying the items in inventory and the costs of ordering.

optimal solution

Letting I_{kt} denote the inventory of item k at the beginning of period t before the arrival of x_{kt}, we have for period t:

$$I_{kt} = I_{k0} + \sum_{j=1}^{t-1} x_{kj} - \sum_{j=1}^{t-1} d_{kj} \geq 0$$

where I_{k0} is the initial inventory on hand at the beginning of the planning period. As an extension of the single-item case reported by Wagner and Whitin,[8] the functional equations for the minimal cost policy in the multi-item case can be written as:

$$\begin{aligned}
f_t(I_{1t}, I_{2t}, \cdots, I_{Mt}) = \min_{\substack{x_{k,t} \geq 0,\\ I_{kt} + x_{kt} \geq d_{kt}\\ k = 1, \cdots, M}} \Bigg[& \sum_{k=1}^{M} i_k I_{kt} + \sum_{k=1}^{M} \delta(x_{kt}) \cdot s_k \\
& + F \cdot \delta^*(x_{1t}, x_{2t}, \cdots, x_{Mt}) \\
& + f_{t+1}(I_{1t} + x_{1t} - d_{1t}, I_{2t} + x_{2t} - d_{2t}, \cdots, I_{Mt} + x_{Mt} - d_{Mt}) \Bigg]
\end{aligned}$$

$$\text{for } t = 1, 2, \cdots, N - 1;$$

and

$$f_N(I_{1N}, I_{2N}, \cdots, I_{MN}) = \min_{\substack{x_{kN} \geq 0,\\ I_{kN} + x_{kN} = d_{kN}\\ k = 1, \cdots, M}} \left[\sum_{k=1}^{M} i_k I_{kN} + \sum_{k=1}^{M} \delta(x_{kN}) \cdot s_k + F \cdot \delta^*(x_{1N}, x_{2N}, \cdots, x_{MN}) \right] \tag{9}$$

where

$$\delta(x_{kt}) = \begin{cases} 0 & \text{if } x_{kt} = 0 \\ 1 & \text{otherwise} \end{cases}$$

and

$$\delta^*(x_{1t}, x_{2t}, \cdots, x_{Nt}) = \begin{cases} 0 & \text{if } x_{kt} = 0 \text{ for all } k \\ 1 & \text{otherwise} \end{cases}$$

The quantity $f_t(I_{1t}, I_{2t}, \cdots, I_{Mt})$ represents the minimum total cost to be incurred in periods $t, t+1, \cdots, N$ when period t begins with inventory balances $I_{1t}, I_{2t}, \cdots, I_{M-1,t}$, and I_{Mt} on hand, and when an optimal policy for periods $t, t+1, \cdots, N$ is employed.[9] Starting with f_N and evaluating recursively the functions $f_N, f_{N-1}, \cdots, f_2, f_1$, an optimal solution becomes determined when the combination of inventory balances $I_{11}, I_{21}, \cdots, I_{M1}$ has been considered.

reduced computation

When, in practice, starting inventories are netted successively against demands for period 1, period 2, and so on—so that in the problem formulation $I_{k0} = 0$ and all demands d_{kt} represent demands net of initial inventories—computation can be reduced considerably as a consequence of a theorem proved by Wagner and Whitin[8] for the single-item problem. This theorem states simply that in searching for an optimal solution it is unnecessary to consider for any item in any time period a case in which both inventory is brought forward into the period and an order is placed for the period, i.e., there exists an optimal solution in which $I_{kt} \cdot x_{kt} = 0$ for all k and all t. As an important consequence of this theorem, an optimal solution exists in which for all items k and all time periods t:

$$x_{kt} = 0, \text{ or } \sum_{j=t}^{r} d_{kj} \text{ for some } r, t \leq r \leq N \,.$$

Therefore, in period t only $(N - t + 2)$ values of I_{kt} need be considered for each item k. Hence, for M items there are $(N - t + 2)^M$ states to be evaluated at each stage t.

For illustration, we compute the optimal solution for the example of the previous section with four time periods. For item 1 we have $d_{11} = d_{12} = d_{13} = d_{14} = 35$; $i_1 = 4$, and $s_1 = 200$; for item 2, $d_{21} = d_{22} = d_{23} = d_{24} = 150$, $i_2 = 5$, $s_2 = 200$; and finally, we have $F = 280$. Table 13 shows the computational results. For each period t, we list $(N - t + 2)$ states as defined by the pair of inventory balances (I_{1t}, I_{2t}) on hand at the beginning of the period. For each such state in the problem, the table shows the optimal order quantities x_{1t} and x_{2t} for that period together with the minimum cost $f_t(I_{1t}, I_{2t})$ to be incurred for the remaining time periods when an optimal policy is employed.

To determine the entries in Table 13, we begin with period 4. Since, by Equation 1, $I_{k4} + x_{k4} = d_{k4}$ and, by the Wagner-Whitin theorem, $x_{kt} \cdot I_{kt} = 0$, there are two possibilities to be considered for each item k: $I_{k4} = 0$, $x_{k4} = d_{k4}$ and $I_{k4} = d_{k4}$, $x_{k4} = 0$. For two items, therefore, 2^2 states are to be evaluated according to: $f(I_{14}, I_{24}) = i_1 I_{14} + i_2 I_{24} + \delta(x_{14}) s_1 + \delta(x_{24}) s_2 + \delta^*(x_{14}, x_{24}) \cdot F$, the values of which are shown in Table 13. For instance, for $I_{14} = 0$ and $I_{24} = 150$ we have $f(0, 150) = 0 + 5(150) + 0 + 200 + 280 = 1230$.

Next, period 3 is considered. For each item k, the initial inventory balance I_{k3} should be zero, or sufficient to cover the demand for period 3, d_{k3}, or sufficient to cover the demand for both periods 3 and 4, $d_{k3} + d_{k4}$; therefore 3^2 states must be evaluated for period 3. For all states with $I_{k3} > 0$, we have immediately that $x_{k3} = 0$. If both $I_{13} > 0$ and $I_{23} > 0$, we then have simply $f_3(I_{13}, I_{23}) = i_1 I_{13} + i_2 I_{23} + f_4(I_{13} - d_{13}, I_{23} - d_{23})$. For instance, for $I_{13} = 35$

Table 13 Dynamic programming calculations for two products and four time periods

Period 1

f_1	I_{11}	I_{21}	X_{11}	X_{21}
2600	0	0	70	150
3150	0	150	70	0
4170	0	300	70	0
6080	0	450	105	0
8740	0	600	70	0
2600	35	0	0	150
2870	35	150	0	0
4170	35	300	0	0
5940	35	450	0	0
8540	35	600	0	0
2680	70	0	0	150
2950	70	150	0	0
3970	70	300	0	0
6020	70	450	0	0
8540	70	600	0	0
2960	105	0	0	150
3230	105	150	0	0
4250	105	300	0	0
6020	105	450	0	0
8820	105	600	0	0
3320	140	0	0	150
3590	140	150	0	0
4610	140	300	0	0
6380	140	450	0	0
8900	140	600	0	0

Period 2

f_2	I_{12}	I_{22}	X_{12}	X_{22}
1980	0	0	70*	150
2530	0	150	70*	0
3550	0	300	70	0
5400	0	450	105	0
1920	35	0	0	150
2190	35	150	0	0
3490	35	300	0	0
5260	35	450	0	0
2060	70	0	0	150
2330	70	150	0	0
3350	70	300	0	0
5400	70	450	0	0
2280	105	0	0	150
2550	105	150	0	0
3570	105	300	0	0
5340	105	450	0	0

* alternately, $X_{12} = 35$.

Period 3

f_3	I_{13}	I_{23}	X_{13}	X_{23}
1300	0	0	70	150
1850	0	150	70	0
2870	0	300	70	0
1300	35	0	0	150
1570	35	150	0	0
2870	35	300	0	0
1380	70	0	0	150
1650	70	150	0	0
2670	70	300	0	0

Period 4

f_4	I_{14}	I_{24}	X_{14}	X_{24}
680	0	0	35	150
1230	0	150	35	0
620	35	0	0	150
890	35	150	0	0

and $I_{23} = 150$, we have $f_3(35,150) = 4(35) + 5(150) + 680 = 1570$. When $I_{k3} = 0$ for one or both items, however, the order quantities x_{k3} must be chosen so that a minimal state is reached at period 4. For example, when $I_{13} = 0$ and $I_{23} = 150$, we have $f_3(0, 150) = 0 + 5(150) + 1(200) + 0 + 1(280) + \min [f_4(x_{13} - d_{13}, 0)] = 1230 + \min [f_4(0, 0), f_4(35, 0)] = 1850$ for $x_{13} = 70$. In like manner, each of the other states in period 3 are evaluated, followed by those in period 2 and finally by those in period 1.

From the completed table, it is seen that the cost of an optimal solution for the 4 periods when starting with no inventory is \$2600. To determine the optimal solution itself, we start with period 1 and proceed to each succeeding period in turn, determining at each stage the subsequent state from the present state and present order quantities.

The results are as follows:

$$
\begin{array}{llll}
I_{11} = 0 & I_{21} = 0 & x_{11} = 70 & x_{21} = 150 \\
I_{12} = 35 & I_{22} = 0 & x_{12} = 0 & x_{22} = 150 \\
I_{13} = 0 & I_{23} = 0 & x_{13} = 70 & x_{23} = 150 \\
I_{14} = 35 & I_{24} = 0 & x_{14} = 0 & x_{24} = 150
\end{array}
$$

Item 2 is therefore to be ordered every period, item 1 every two periods.[10]

The ordering of item 1 every other period and item 2 every period for an even number of periods N was also the optimal policy as determined with the direct algorithm. However, the total cost determined by the two methods differs, since in the dynamic programming algorithm there is no reflection in the total cost of the inventory cost incurred in period t for the quantities d_{kt}, $k = 1, 2, \cdots, M$. When these costs are represented as $i_k \cdot d_{kt}/2$ as in the static model, total cost is the same for both methods.

summary comment

In practice, the size of problems for which this dynamic programming formulation is useful may be rather limited since the total number of states S to be evaluated increases rapidly with both N and M: $S = 1 + \sum_{t=2}^{N} (N - t + 2)^M = \sum_{t=1}^{N} t^M$. For example, with a planning horizon of 12 periods there are, for two items, $S = 506$ states; upon increasing the number of items to 5, S soars to 381,876 states. In any case, however, the decision ultimately rests on a comparison of the expected benefits to be gained from a more globally optimum solution and the added problem-solving effort to be expended in obtaining it.

CITED REFERENCES AND FOOTNOTES

1. For analysis of problems of this type see, for instance, G. Hadley and T. M. Whitin, *Analysis of Inventory Systems*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1963), or F. Hanssmann, *Operation Research in Production and Inventory Control*, John Wiley & Sons, Inc., New York, 1962.
2. A number, for example, yielding a convenient review period for items. As N is increased for a given planning horizon, we increase the number of prospective opportunities for reordering and thereby increase the prospects for a solution with lower total costs. However, at the same time we also tend to increase the problem-solving effort required to solve the problem, so that a balance between the two must be struck.
3. A variation of this problem with a continuous planning horizon of infinite duration has been reported by E. Naddor and S. Saltzman, "Optimal reorder periods for an inventory system with variable costs of ordering," *Operations Research* 6, No. 5, 676–685 (September-October 1958). They present an approximate solution for the case wherein an order is to be placed regularly with the vendor or plant every t years and item k is to be requisitioned every $x_k t$, where x_k is some positive integer. By comparison, in the present algorithm each item k is requisitioned regularly every b_k periods, but the time between orders placed on the vendor need not be regular.
4. In the event $q_k(\alpha_j^*) = qk(\hat{\alpha}_j) = \min_{\{t/\delta_t^{j} = 1\}} q_k(\alpha_t)$ either of the values α_j or α_j^* may be chosen for b_k' since they yield identical cost $qk(\hat{\alpha}_j)$ and since in no way is the resulting solution B' itself used in generating subsequent states.
5. An exception to this procedure occurs when, to conform with state $(\delta_1', \delta_2', \cdots, \delta_r')$, all δ_i^q are transformed to zero by means of Equation 5 for which $\delta_i' = 0$, and there results a $\delta_k = 0$ for some k such that $\delta_k' = 1$. In every such case there exists another state in the table with $\delta_i'' = \delta_i'$ for all $i \neq k$ and $\delta_k'' = 0$, and with $\bar{n}(\delta_1'', \delta_2'', \cdots, \delta_r'') < \bar{n}(\delta_1', \delta_2', \cdots, \delta_r')$ (otherwise the present state would not be distinguished and the exception would not arise). Since upon ultimately transforming the δ_i^q as required to conform to the δ_i'' of this other state, the resulting values of the b_i will necessarily be the same as those just determined for state

$(\delta_1', \delta_2', \cdots, \delta_r')$, it follows that $Z(b_1'', b_2'', \cdots, b_M'') < Z(b_1', b_2', \cdots, b_M')$. Therefore whenever this exception occurs we simply locate the state $(\delta_1'', \delta_2'', \cdots, \delta_r'')$ in the appropriate table, evaluate $Z(b_1'', b_2', \cdots, b_M'')$, and proceed to the evaluation of the remaining states, giving no further consideration to state $(\delta_1', \delta_2', \cdots, \delta_r')$.

6. This condition is satisfied for many cases of practical interest, including, for example, each of those given in Tables 3 through 8. However the condition need not always be satisfied as is illustrated, for example, with the case $N = 323$ and $\alpha = 1, 17, 19$, and 323.
7. From Table IV of Reference 3. The problem for which they employ the data differs somewhat from that being considered here.
8. H. M. Wagner and T. M. Whitin, "Dynamic version of the economic lot size model," *Management Science* **5**, No. 1, 89–96 (October 1958).
9. In these cost expressions inventory carrying costs for period $(t - 1)$ for item k are represented as $i_k I_{kt}$, being based only on the balance on hand at the end of the period. This understates the carrying cost for the period by the cost incurred for the quantity d_{kt} supplied during the period. However, this cost is fixed, regardless of ordering policy, and hence need not be explicitly represented in the cost expression.
10. As can be seen from the table, had the number of periods N been odd e.g., 3, 5, . . ., item 2 would still be ordered each period; item 1 would be ordered every two periods except for one arbitrary period when only a quantity sufficient to cover the demand for a single period is ordered.

Appendix

Proof of Condition 6

Let B^{j-1} and B^j, $s < j$, be any successive bounding solutions in any problem for which $(\alpha_{k+1} - \alpha_k) \geq (\alpha_k - \alpha_{k-1})$ for all $k = 2, 3, \cdots, r - 1$. If $\pi_j - \pi_{j-1} \geq F[\bar{n}(\delta_1^j, \delta_2^j, \cdots, \delta_r^j) - N/\alpha_{j+1}]$, then it is unnecessary to explicitly investigate states with $\delta_1 = \delta_2 = \cdots = \delta_j = 0$.

For bounding solution B^j, the total cost is $Z^j = \pi_j + F \cdot \bar{n}(\delta_1^j, \delta_2^j, \cdots, \delta_r^j)$, and for any solution $B^* = (b_1^*, b_2^*, \cdots, b_M^*)$ with $b_k^* \geq \alpha_v$ for all k, any $v > j$, $Z^* = \sum_k q_k(b_k^*) + F \cdot \bar{n}(\delta_1^*, \delta_2^*, \cdots, \delta_r^*)$. Since $\sum_k q_k(b_k^*) \geq \pi_v$ and $\bar{n}(\delta_1^*, \delta_2^*, \cdots, \delta_r^*) \geq N/\alpha_v$ for all B^*, with $\delta_i^* = 0$ for all $i < v$ and $\delta_v^* = 1$, $\hat{Z}^* = \pi_v + N/\alpha_v$ constitutes a lower bound on Z^*. Since $Z^0 \leq Z^j$ and $\hat{Z}^* \leq Z^*$, if $Z^j \leq \hat{Z}^*$, then $Z^0 \leq Z^*$ and no solution B^* exists which is preferred to B^0 and therefore need be investigated. Upon substituting for Z^j and $\hat{Z}^*$, rearranging terms and expanding, the condition $Z^j \leq \hat{Z}^*$ can be written:

$$\begin{aligned}&(\pi_v - \pi_{v-1}) + (\pi_{v-1} - \pi_{v-2}) + \cdots + (\pi_{j+1} - \pi_j)\\ &\geq F(\bar{n}(\delta_1^j, \delta_2^j, \cdots, \delta_r^j) - N/\alpha_{j+1}) + F(N/\alpha_{j+1} - N/\alpha_{j+2})\\ &+ \cdots + F(N/\alpha_{v-1} - N/\alpha_v)\end{aligned} \tag{i}$$

To prove condition 6, it will be shown (a) that $(\pi_{k+1} - \pi_k) \geq (\pi_k - \pi_{k-1})$ for all k; (b) that $(\bar{n}(\delta_1^j, \delta_2^j, \cdots, \delta_r^j) - N/\alpha_{j+1}) \geq (N/\alpha_{j+1} - N/\alpha_{j+2})$; and (c) that $(N/\alpha_{k-1} - N/\alpha_k) \geq (N/\alpha_k - N/\alpha_{k+1})$. Then when $(\pi_j - \pi_{j-1}) \geq F \cdot \bar{n}(\delta_1^j, \delta_2^j, \cdots, \delta_r^j) - N/\alpha_{j+1}$, inequality (i) is satisfied so that $Z^0 \leq Z^*$, and hence B^* need not be investigated, as asserted in condition 6.

Beginning with (a) we first prove the following inequality:

If $(\alpha_{t+1} - \alpha_t) \geq (\alpha_t - \alpha_{t-1})$ for all $t = 2, 3, \cdots, r-1$
then $q_j(\alpha_{k+1}) - q_j(\alpha_k) \geq q_j(\alpha_k) - q_j(\alpha_{k-1})$ for all $k > s$ (ii)
where $q_j(\alpha_s) = \min_u \{q_j(\alpha_u)\}$, all $j = 1, 2, \cdots, M$

Suppose for some k and j, $q_j(\alpha_{k+1}) - q_j(\alpha_k) < q_j(\alpha_k) - q_j(\alpha_{k-1})$. Upon substituting $(Nd_ji_j\alpha/2 + s_j \cdot N/\alpha)$ for $q_j(\alpha)$ in each term and rearranging, this becomes:

$$[Nd_ji_j/2 - NS_j/\alpha_k\alpha_{k+1}][(\alpha_{k+1} - \alpha_k) - (\alpha_k - \alpha_{k-1})] \qquad \text{(iii)}$$
$$< NS_j[(\alpha_k - \alpha_{k-1})/\alpha_k][(\alpha_{k-1} - \alpha_{k+1})/\alpha_{k-1}\alpha_{k+1}]$$

If we denote by $\Delta^2 q_j(\alpha)$ the second difference of $q_j(\alpha)$ for any item j and cycle time α, then $\Delta^2 q_j(\alpha) = q_j(\alpha+2) - 2q_j(\alpha+1) + q_j(a) = 2NS_j/[\alpha(\alpha+1)(\alpha+2)]$ which is always positive. Therefore for all $k \geq s$, $q_j(\alpha_{k+1}) - q_j(\alpha_k) \geq 0$. Upon substituting $Nd_ji_j\alpha/2 + s_j \cdot N/\alpha$ for $q_j(\alpha)$ as above and rearranging, this inequality becomes

$$[Nd_ji_j/2 - NS_j/\alpha_k\alpha_{k+1}][\alpha_{k+1} - \alpha_k] \geq 0$$

from which it follows that $[Nd_ji_j/2 - NS_j/\alpha_k\alpha_{k+1}] \geq 0$ since $[\alpha_{k+1} - \alpha_k] > 0$. Together with the assumption that $(\alpha_{k+1} - \alpha_k) > (\alpha_k - \alpha_{k-1})$ this implies that the left hand side of (iii) is therefore never negative. On the other hand, since $\alpha_{t+1} > \alpha_t$ for any t, the right hand side of (iii) is always negative. Therefore (iii) can never hold for any k and j, contradicting the assertion $q_j(\alpha_{k+1}) - q_j(\alpha_k) < q_j(\alpha_k) - q_j(\alpha_{k-1})$ for some j and $k > s$, thereby establishing (ii). With the help of (ii) the following inequality can now be established:

If $(\alpha_{t+1} - \alpha_t) \geq (\alpha_t - \alpha_{t-1})$ for all $t = 2, 3, \cdots, r-1$,
then $q_j(b_j^{k+1}) - q_j(b_j^k) \geq q_j(b_j^k) - q_j(b_j^{k-1})$ for all (iv)
k and j.

Since $b_j^p = \max[\alpha_s, \alpha_p]$ where $q_j(\alpha_s) = \min_u \{q_j(\alpha_u)\}$, for all p, $b_j^{k+1} = b_j^k = b_j^{k-1}$ for all $k \leq s-1$, and strict equality results in (iv). For $k = s$, (iv) becomes $q_j(\alpha_{s+1}) - q_j(\alpha_s) \geq 0$, which will always be true in light of the positive second difference of $q_j(\alpha)$. And finally, for $k > s$, (iv) becomes $q_j(\alpha_{k+1}) - q_j(\alpha_k) \geq q_j(\alpha_k) - q_j(\alpha_{k-1})$ which is true by (ii), thus establishing (iv). We now establish (a), as was the initial objective:

If $(\alpha_{t+1} - \alpha_t) \geq (\alpha_t - \alpha_{t-1})$ for all $t = 2, 3, \cdots, r-1$,
then $(\pi_{k+1} - \pi_k) \geq (\pi_k - \pi_{k-1})$ for all k (a)

Substituting $\pi_w = \sum_{j=1}^{M} q_j(b_j^w)$ for $w = k-1, k$, and $k+1$, and rearranging terms, (a) can be written

$$\sum_{j=1}^{M} \{[q_j(b_j^{k+1}) - q_j(b_j^k) - q_j(b_j^k) - q_j(b_j^{k-1})]\} \geq 0$$

which is necessarily true since by (iv) every term is nonnegative. Next consider (c):

If $\alpha_{t+1} - \alpha_t) \geq (\alpha_t - \alpha_{t-1})$ for all $t = 2, 3, \cdots, r-1$,
then $(N/\alpha_{k-1} - N/\alpha_k) \geq (N/\alpha_k - N/\alpha_{k+1})$ for all k (c)

For every α_k in the set $\{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ there exists another element $\alpha_k' = N/\alpha_k$ which is also in the set. Since for all k; $\alpha_{k-1} < \alpha_k < \alpha_{k+1}$ implies $\alpha_{k-1}' > \alpha_k' > \alpha_{k+1}'$, and since for all elements $\alpha_{t-1} < \alpha_t < \alpha_{t+1}$ in the set $(\alpha_{t+1} - \alpha_t) \geq (\alpha_t - \alpha_{t-1})$, it follows that $(\alpha_{k-1}' - \alpha_k') \geq (\alpha_k' - \alpha_{k+1}')$, as asserted.

Finally, the proof of condition 6 is completed by establishing (b):

$$\begin{aligned}&\text{If } (\alpha_{t+1} - \alpha_t) \geq (\alpha_t - \alpha_{t-1}) \qquad \text{for all } t = 2, 3, \cdots, r-1,\\ &\text{then } (\bar{n}(\delta_1{}^j, \delta_2{}^j, \cdots, \delta_r{}^j) - N/\alpha_{j+1}) \geq (N/\alpha_{j+1} - N/\alpha_{j+2}) \qquad \text{(b)}\\ &\text{for all } j.\end{aligned}$$

Expanding the inequality and rearranging terms we get:

$$[\bar{n}(\delta_1{}^j, \delta_2{}^j, \cdots, \delta_r{}^j) - N/\alpha_j] + [(N/\alpha_j - N/\alpha_{j+1}) - (N/\alpha_{j+1} - N/\alpha_{j+2})] \geq 0$$

The first term is either zero or positive depending on whether B^j is prime or non-prime, and the second term is always nonnegative as a consequence of (c) above. Therefore the equality or inequality in (b) will always be fulfilled.

Proof of Condition 7

Let B^{j-1} and B^j, $s < j$, be any successive bounding solutions in any problem for which $(\alpha_{k+1} - \alpha_k) \geq (\alpha_k - \alpha_{k-1})$ for all k. If $Z^j - Z^0 + t[\pi_j - \pi_{j-1}] \geq F[\bar{n}(\delta_1{}^j, \delta_2{}^j, \cdots, \delta_r{}^j) - N/\alpha_{j+t}]$ for any t, it is unnecessary to explicitly investigate states with $\delta_1 = \delta_2 = \cdots = \delta_{j+t-1} = 0$ and $\delta_t = 1$.

Letting $v = j + t$, we have as with condition 6 that $\hat{Z}^* = \pi_v + N/\alpha_v$ constitutes a lower bound on the value of a solution B^* with $\delta_i{}^* = 0$ for $i < v$ and $\delta_v{}^* = 1$ so that if $\hat{Z}^* \geq Z^0$ there exists no such B^* which is preferred to B^0. Upon adding Z^j to both sides of the inequality $\hat{Z}^* \geq Z^0$ and rearranging terms we get:

$$Z^j - Z^0 + (\pi_{j+t} - \pi_j) \geq F[\bar{n}(\delta_1{}^j, \delta_2{}^j, \cdots, \delta_r{}^j) - N/\alpha_{j+t}] \qquad \text{(v)}$$

Expanding the term $(\pi_{j+t} - \pi_j) = (\pi_{j+t} - \pi_{j+t-1}) + \cdots + (\pi_{j+2} - \pi_{j+1}) + (\pi_{j+1} - \pi_j)$ and employing from (a) the fact that $(\pi_{i+1} - \pi_i) \geq (\pi_i - \pi_{i-1})$ for all i, we conclude that $(\pi_{j+t} - \pi_j) \geq t \cdot (\pi_j - \pi_{j-1})$. Therefore whenever the inequalities in condition 7 are satisfied, the inequality in (v) is satisfied, and hence no solutions B^* need be explicitly investigated, as was to be shown.