The reliability aspects of polymorphic systems are examined within the confines of a simple failure and repair model.

Emphasized are the derivation and use of easy-to-calculate approximations to the unavailabilities of system capacity levels.

On the reliability of polymorphic systems

by P. D. Welch

This discussion of the reliability aspects of "polymorphic" data processing systems stresses the two essential features of such systems—the load sharing over lower capacity units and the complete interconnection of functional levels. It is the author's intention that the discussion provide an approximate reliability analysis of a polymorphic system that will be more convenient for initial planning purposes than more exact analyses.

definitions

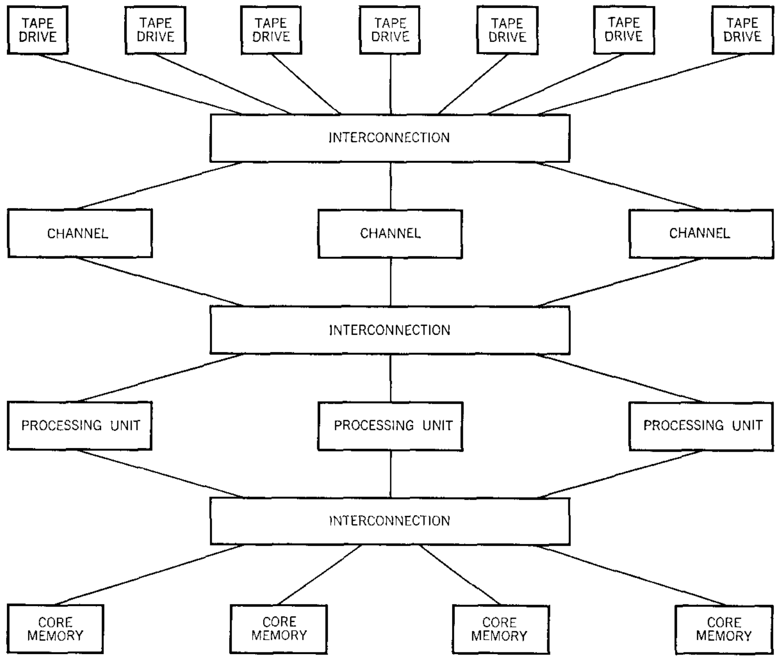
An example of a hypothetical polymorphic system is illustrated in Figure 1. The system is characterized by:

- A multiplicity of units at each functional level.
- The existence of all possible interconnections between the units at the various functional levels.

The organization ensures that the failure of any one unit removes only this unit from the system; all unfailed units are available for system use. For more detailed discussion of such systems, see References 1 through 5.

We are interested in the potential capacity characteristics of such systems when the units of which they are composed are subject to failure and repair.[6] In this analysis, we assume the simple "independent" failure and repair model. In this model there is no delay between the occurrence of a failure, its detection, and the beginning of the repair operation. In other words, there effectively exists a repair facility for each unit. Further, the times to failure are identically distributed, independent, random vari-

Figure 1 An example of a polymorphic configuration



ables as are the times to repair; and the times to repair are independent of the times to failure. For additional discussion see Reference 7.

As units fail, are removed and repaired, the system takes on different forms; hence the term "polymorphic." To assess the capacity characteristics of the system given this process, we must assign a system capacity to each possible system configuration. Suppose the system has N levels ($N = 4$ in the example of Figure 1) with n_i units at the ith level ($n_1 = 7$, $n_2 = 3$, $n_3 = 3$, $n_4 = 4$ in the example). We assume that each unit at the ith level has a capacity c_i and that if k_i units are up (unfailed) the capacity of the level is

$$C_i = k_i c_i.$$

The capacity C of the total system we then take to be

$$C = \min \{C_1, C_2, \cdots, C_N\}.$$

In other words, we assume capacities can be uniformly assigned to the units at each level such that the system capacity is the minimum cross section of the level capacities. This is essentially an assumption concerning the total organization and structure, including software, of the system. We are assuming that failures at any functional level have a linear constrictive effect and that the surplus of units at one level cannot be substituted for the lack of units at another.

We require the notions of availability and unavailability. Suppose a unit has a mean time to failure of f and a mean time to repair of r, then the *availability* a of that unit, given by

$$a = \frac{f}{r + f},$$

is the probability that the unit is in an operable condition. Similarly the *unavailability* u of the unit, given by

$$u = 1 - a = \frac{r}{r + f},$$

is the probability that the unit is inoperable or undergoing repair. For any functional level, we define the availability function

$$A_i(x) = P\{C_i \geq x\}$$

as the probability that the capacity is greater than or equal to x. Similarly the unavailability function is given by

$$U_i(x) = P\{C_i < x\} = 1 - A_i(x).$$

In an identical manner, we define the availability function, $A(x)$, and unavailability function, $U(x)$, for the entire system. In this situation, the capacities of levels and the capacity of the entire system are random variables. The availability and unavailability functions describe their probability distributions.

The failure-repair effects of the interconnecting hardware can be treated in either of two ways, depending upon how the hardware is implemented. If the hardware is distributed among the units, its failure-repair effects can be combined with those of the unit. If it is not distributed, but centralized, it would have to be included as an additional functional level. See Blaauw[5] for a discussion of these two alternatives.

Before giving the analysis of polymorphic systems, we will discuss the gains achieved by the two essential features of such systems: the load sharing over lower capacity units and the complete interconnection of functional levels.

Under our formulation, a polymorphic system, considered with respect to a particular capacity level, is essentially a series of k_i out of n_i parallel units. The exact reliability analysis of such a configuration is well known. The purpose of this discussion is to provide an approximate analysis that is both more revealing and more useful for initial planning purposes.

Advantages of load sharing over lower capacity units

Consider a task performed by a system of n units, each unit having a capacity, c. Suppose that if k units are operating the system has a capacity $C = kc$. Let each of the units have an unavailability, u. Then for any integer $j (j = 1, 2, \cdots, n)$ the unavailability of the capacity level jc is given by

$$U(jc) = P\{C < jc\} = \sum_{k=n-j+1}^{n} \binom{n}{k} u^k (1 - u)^{n-k}.$$

The probability of fewer than j units operating, i.e., the probability that $n - j + 1$ or more units are under repair, is the sum of the right side.

Now, for any integer m, the following inequalities hold:

$$\binom{n}{m}u^m(1 - u)^{n-m} < \sum_{k=m}^{n} \binom{n}{k}u^k(1 - u)^{n-k} < \binom{n}{m}u^m.$$

The first inequality is obvious since the quantity on the left is one of the terms in the sum on the right. The second inequality is proved in the Appendix. In the case of data processing units, the unavailability is very small and reasonable powers of $1 - u$ are approximately one. Hence

$$\sum_{k=m}^{n} \binom{n}{k}u^k(1 - u)^k \approx \binom{n}{m}u^m.$$

This is an approximation we will use extensively from here on.

Using the above inequalities we have

$$\binom{n}{n - j + 1}u^{n-j+1}(1 - u)^{j-1} < U(jc)$$

$$= \sum_{k=n-j+1}^{n} \binom{n}{k}u^k(1 - u)^{n-k} < \binom{n}{n - j + 1}u^{n-j+1}$$

and

$$U(jc) \approx \binom{n}{n - j + 1}u^{n-j+1}.$$

The latter is a good approximation and, since greater than the actual unavailability at the $j$cth capacity level, a conservative one.

To examine the unavailability function for various load sharing configurations, assume the unit unavailability remains constant as unit capacities change. We will later argue that this is a conservative assumption relative to the points to be made.

One common way of achieving high reliability in an operation is to use two units, either of which is capable of handling the entire job. In this case, the system has only two operating states: either both units are operating and the system has twice the necessary capacity—a state with unavailability approximately $2u$; or only one of the units is operating and the system has just the necessary capacity—a state with unavailability u^2. This situation is illustrated in the "1 out of 2" line of Figure 2, which shows that capacity for full operations is obtained with either one or two identical units in service.

An alternative is to use four units, each capable of handling one half the job. This system would have four operating states: it has twice the capacity with approximate unavailability $4u$, three halves the capacity with approximate unavailability $6u^2$, the necessary capacity with approximate unavailability $4u^3$, and one half the capacity with unavailability u^4. This case is illustrated in the second line of Figure 2. Such a system has the same excess capacity as the two-unit system but a much lower unavailability at the necessary capacity level ($4u^3$ as compared with u^2) and an additional half-capacity level with unavailability u^4.

It is worthwhile to point out again the usefulness in this context of the approximation

$$U(jc) \approx \binom{n}{n = j + 1} u^{n-j+1}$$

It is easy to calculate and is also an upper bound. Hence, it can be used safely (within the confines of the model) to obtain a figure for the unavailability for additional specific configurations.

Advantages of the complete interconnection of functional levels

Suppose that we have a task performed by n units, as suggested in the upper half of Figure 3, and that we are able to break each unit up into N functional levels, as suggested in the lower half of Figure 3. Let the unavailability, before breakup, of the unit be u. After breakup into N levels we will have N unavailabilities $u_1, \cdots, u_N$. In the Appendix we show that $\sum_{i=1}^{N} u_i \approx u$.

We will first assume the breakup is such that the unavailability is divided evenly among the levels; that is, that $u_i = u/N$ for all i. This would occur, for example, if the unit could be divided into N parts each with a mean time to failure of Nf, where f is the mean time to failure of the original unit, and a mean time to repair of r, equal to the mean time to repair of the original unit.

Let C denote the capacity of the original system and $U(x)$ its unavailability function, and let $C = jc$ when there are j units operating. Moreover, let C_i denote the capacity of the ith level of multilevel system, let $U_i(x)$ be its unavailability function, and let $C_i = j_i c$ when j_i of the ith level units are operating. Finally, let $U^*(x)$ be the unavailability function of the entire multilevel system.

From Section 2 we have, for the upper half of Figure 3,

$$U(jc) \approx \binom{n}{n - j + 1} u^{n-j+1}$$

and

$$U_i(jc) \approx \binom{n}{n - j + 1} (u/N)^{n-j+1}.$$

Now in the Appendix we show that

$$U^*(jc) \approx NU_i(jc),$$

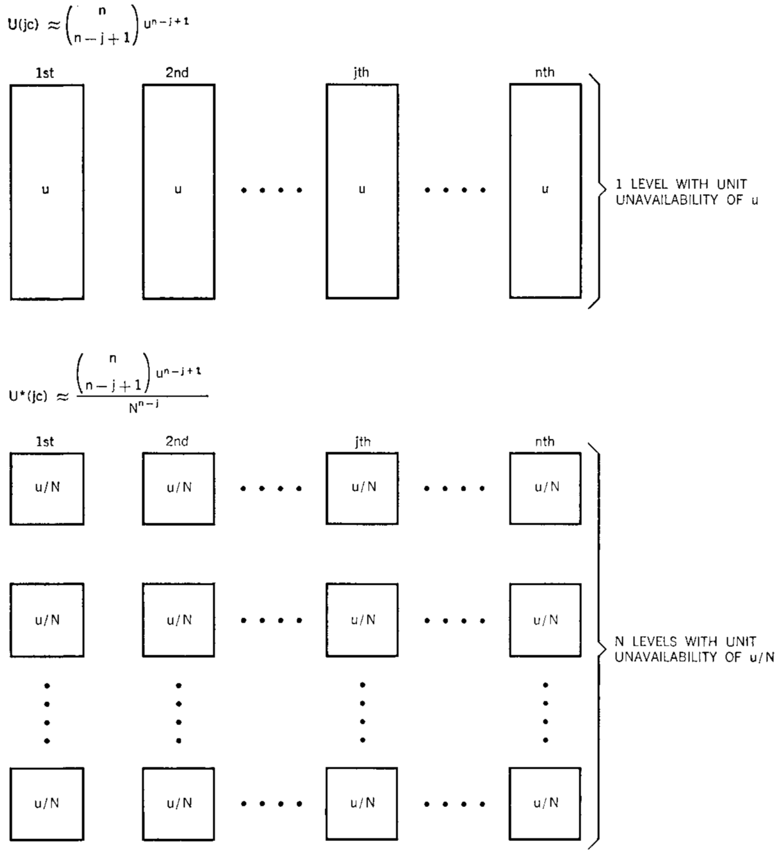
and that this approximation is conservative since the right-hand side is an upper bound. Hence, for the lower half of Figure 3,

$$U^*(jc) \approx \frac{\binom{n}{n - j + 1} u^{n-j+1}}{N^{n-j}}$$

with the right side again an upper bound. Thus

$$\frac{U^*(jc)}{U(jc)} \approx \frac{1}{N^{n-j}}.$$

Figure 3 Alternative single and N-level configuration with approximate unavailability function



Hence, in this special case, the unavailability function is reduced by a power of the number of functional levels with the exponent a linear function of the capacity level: the lower the capacity level, the greater the exponent. Thus the reduction in the unavailability increases as the capacity level decreases. The unavailability of the top capacity level is unaffected by the multiple levels and their interconnection. The unavailability of the level just below full capacity is reduced by a factor $1/N$, the unavailability of the next level by a factor $1/N^2$, etc. This greater reduction in the unavailability as the capacity level decreases is intuitive because it is clear that as more and more failures occur there is greater and greater advantage to the ability to piece subunits together from the different vertical strings.

In the general case, we have unavailabilities $u_1, u_2, \cdots, u_N$ associated with the units at the various functional levels. Hence we have

$$U_i(jc) \approx \binom{n}{n-j+1} u_i^{n-j+1}$$

and

$$U^*(jc) \approx \sum_{i=1}^{N} \binom{n}{n-j+1} u_i^{n-j+1}.$$

Thus

$$\frac{U^*(jc)}{U(jc)} \approx \frac{\sum_{i=1}^{N} \binom{n}{n-j+1} u_i^{n-j+1}}{\binom{n}{n-j+1} u^{n-j+1}} = \sum_{i=1}^{N} (u_i/u)^{n-j+1}. \qquad (1)$$

And with the assumption $\sum_{i=1}^{N} u_i = u$ we have

$$\frac{U^*(jc)}{U(jc)} = \sum_{i=1}^{N} \left(\frac{u_i}{\sum u_i}\right)^{n-j+1}.$$

The approximation $\sum_{i=1}^{N} u_i \approx u$ is based on the assumption that the string of N subunits operating as a series system has the same unreliability as the original integrated unit. This would be the case if in breaking it apart the resulting series system had the same failure and repair characteristics as the original integrated unit. If this is not the case, equation (1) still holds.

Reliability analysis

exact analysis

Consider a polymorphic system with N levels and n_i units of capacity c_i at the ith level. For any x, let $j_i(x)$ be the smallest integer greater than or equal to x/c_i. Then, if C_i is the capacity of the ith level, its availability function is given by

$$A_i(x) = P\{C_i \geq x\} = \sum_{k=j_i(x)}^{n_i} \binom{n_i}{k} (1 - u_i)^k (u_i)^{n-k},$$

and the system availability function is given by

$$A(x) = P\{C \geq x\} = \prod_{i=1}^{N} A_i(x).$$

Thus exact availability curves can be obtained for the various levels and the exact, entire-system availability function obtained by multiplication.

approximate analysis

In the above situation, as was seen earlier,

$$U_i(x) = P\{C_i < x\} \approx \binom{n_i}{n_i - j_i(x) + 1} u_i^{n_i - j_i(x) + 1}.$$

Further, from the Appendix,

$$U(x) = P\{C < x\} \approx \sum_{i=1}^{N} U_i(x).$$

Hence

$$U_i(x) \approx \sum_{i=1}^{N} \binom{n_i}{n_i - j_i(x) + 1} U_i^{n_i - j_i(x) + 1}.$$

Both of the first two approximations are upper bounds; hence, the final approximation is an upper bound. Thus upper-bound approximations to the unavailability curves can be obtained and their sum is a good upper-bound approximation to the unavailability curve for the entire system.

These observations can serve as a guide line for synthesizing a system with a given unavailability function. If one has an N level system, a desired unavailability function $U(x)$ can be achieved by keeping $U_i(x) \leq U(x)/N$ for all i.

concluding remarks

This paper provides an approximate reliability analysis of polymorphic systems and indicates, in a rough way, the reliability gains given by the essential features of such systems. A complete analysis of polymorphic systems would involve many considerations ignored here.

Appendix

Consider first the inequalities stated in Section 2,

$$\binom{n}{m}u^m(1-u)^{n-m} < \sum_{k=m}^{n}\binom{n}{k}u^k(1-u)^{n-k} < \binom{n}{m}u^m.$$

The first is obvious since the quantity on the left is one of the terms in the sum on the right. To see the second, let E_k for $k = 1, \cdots, \binom{n}{m}$ be the event that a particular m of the units are inoperative (others may also be inoperative). Then, for $k = 1, \cdots, \binom{n}{m}$, $P\{E_k\} = u^m$ and

$$\sum_{k=m}^{n}\binom{n}{k}u^k(1-u)^{n-k}$$

$$= P\{E_1 \cup E_2 \cup \cdots \cup E_{\binom{n}{m}}\} < \sum_{i=1}^{\binom{n}{m}} P\{E_i\} = \binom{n}{m}u^m.$$

Next consider a series system of N units with unavailabilities $u_i : i = 1, \cdots, N$. Let D_i be the event that the ith unit is unavailable. Let u be the unavailability of the series system. Then

$$u = P\{D_1 \cup D_2 \cup \cdots \cup D_N\}$$

and

$$P\{\text{exactly one } D_i\} < P\{D_1 \cup D_2 \cup \cdots \cup D_N\} < \sum_{i=1}^{N} P\{D_i\},$$

or

$$\sum_{i=1}^{N} u_i\{\prod_{j\neq i}(1-u_j)\} < u < \sum_{i=1}^{N} u_i.$$

Now, if the u_i's are very small, then $\Pi_{j\neq i}(1-u_j) \approx 1$ for all i, and $u \approx \sum_{i=1}^{N} u_i$. The approximation is conservative since it is an upper bound.

If we divide a unit into k functional levels in such a way that the failure and repair characteristics of the series of subunits are the same as those of the original unit, then, from the previous paragraph, $\sum_{i=1}^{k} u_i \approx u$. This relationship between u and the u_i

is assumed in our analysis of the complete interconnection of a number of functional levels.

In the body of the paper, the unavailability functions of the functional levels are added to obtain an approximation to the unavailability function of the entire system. This again is an application of the above paragraph. The inequalities obtained here could also be used to place bounds on the system unavailability, that is, the approximation is an upper bound and a lower bound can easily be obtained using the bounds stated in this Appendix.

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CITED REFERENCES AND FOOTNOTES

1. A. J. Critchlow, "Generalized multiprocessing and multiprogramming systems," *Proceedings Fall Joint Computer Conference*, 1963.
2. J. P. Anderson, et al., "D825—a multiple computer system for command and control," *Proceedings Fall Joint Computer Conference*, 1962.
3. R. E. Porter, "The RW-400—a new polymorphic data system," *Datamation*, January–February 1960.
4. W. F. Bauer, "Why multi-computers?," *Datamation*, September 1962.
5. G. A. Blaauw, "The structure of System/360, Part V, multisystem organization," *IBM Systems Journal* **3**, No. 2, 181 (1964).
6. On the other hand, we are concerned with neither the economics of polymorphic organization nor the manner in which potential data processing capacity is exploited.
7. E. C. Olsen, *Availability, A Useful Reliability Measure for Real Time Systems*, Masters Thesis, Department of Electrical Engineering, Syracuse University, 1961.