An algorithm for the numeric solution of a common gear-train problem is developed.

A number-theory approach, relatively novel in current engineering practice, is used in deriving the algorithm.

The form of the algorithm obtained is suitable for programming on a digital computer.

Algorithm for a gear-train problem

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A lathe manufacturer planned to incorporate a gear change that would produce a range of metric pitches in addition to the normal range of English pitches. The problem had previously been resolved by using a special 127-tooth change gear to compound the drive to the feedbox. However, such a large gear was awkward to accommodate and involved a loss of production time whenever a change had to be made. It was therefore decided to incorporate four smaller gears in the feedbox, making it possible to use a simple lever change to give direct drive for English pitches or back-geared drive for metric pitches. The problem was to determine the combination of four gears that would give the over-all ratio most closely approximating the one desired.

Two members of the manufacturer's staff worked together, each with a tabulation in ascending order of all the ratios resulting from combinations of any two gears having between 12 and 120 teeth, together with the seven-figure logarithms of these ratios. One person worked up through the entries, subtracting each logarithm in turn from the logarithm of the desired ratio. Each result obtained was then compared by the second person with the logarithms, and the entry having the nearest logarithm was noted, together with the difference. Thus, a set of results was built up, each involving a combination of two pairs of gears and a measure of the difference between their over-all ratio and the desired ratio. Combinations involving gears with less than 16 or more than 60 teeth were discarded. From the remaining results, the closest approximation was selected. The investigation occupied 60 man-hours. Since the problem occurred infrequently in this company, the approach was considered economically feasible.

In other companies, mathematically similar problems arise much more frequently. For example, a pair of mating helical gears of unequal diameters has the same helix angle, but the lead of the helix is different in each case. In the manufacture of such gears, the lead obtained is normally dependent on the lead which the hobbing machine can produce by virtue of the change gears which the operator selects. In normal everyday work, the usual practice is to refer to a table of leads which tabulates those available from certain standard change gears. For really accurate work, however, a computation such as that described above is carried out.

A possible first reaction to the problem is to suggest the compilation of a table giving, in ascending order of over-all ratio, all possible combinations of four gears (subject to the practical consideration that no gear shall have, for example, less than 15 or more than 100 teeth). This once-and-for-all solution is impracticable, since such a table would consist of the order of 10^7 entries.

This paper presents an algorithm whereby the closest possible approximation to a pre-specified ratio can be established with relatively little effort by means of two pairs of gears. This algorithm can be readily programmed so that companies encountering the problem frequently may take advantage of a digital computer.

the problem

It is required to determine a gear train, comprising two pairs of gears, with an over-all reduction ratio that approximates as closely as possible a specified value, subject to the constraint that the number of teeth in each of the four gears must lie within certain specified bounds.

In mathematical terms, we wish to find the best approximation to a given number x $(0<x<1)$ which is of the form $(F_1F_2)/(F_3F_4)$, where F_r $(r = 1, 2, 3, 4)$ are integers (denoted by capital letters throughout this paper) satisfying

$$J \leq F_r \leq K. \tag{1}$$

mathematical description of the algorithm

If x can be expressed in the desired form, or if $x \leq J^2/K^2$, the solution is immediate. It is assumed that x cannot be so expressed, and that $J^2/K^2 < x < 1$.

We obtain values F_r' and F_r'', satisfying (1), such that

$$(F_1'F_2')/(F_3'F_4') < x < (F_1''F_2'')/(F_3''F_4''),$$

and such that there exists no set F_r^* satisfying (1) and

$$(F_1'F_2')/(F_3'F_4') < (F_1^*F_2^*)/(F_3^*F_4^*) < (F_1''F_2'')/(F_3''F_4'').$$

We call $(F_1'F_2')/(F_3'F_4')$ the *best lower approximation*, and $(F_1''F_2'')/(F_3''F_4'')$ the *best upper approximation*.

An irreducible fraction P/Q is called *acceptable* if there exists an integer C such that $CP = F_1F_2$ and $CQ = F_3F_4$.

Any acceptable fractional approximation to x has a denominator less than or equal to K^2. We first obtain a close approximation which satisfies this condition, with the intention of using it as a starting point in a search for the best approximation satisfying all the conditions. To do so, we express x as a simple continued fraction and calculate the successive convergents of this continued fraction, breaking off the calculation with the last convergent having a denominator less than or equal to K^2. Let this convergent be P_N/Q_N.

The Farey series $\mathfrak{F}_M$ of order M is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed M. Clearly, any acceptable fractional approximation to x is a member of $\mathfrak{F}_{K^2}$.

Parenthetically, it is of interest to note that the number of terms in $\mathfrak{F}_{K^2}$ is of the order of $3K^4/\pi^2$.

The major step in our procedure is, now, to generate that part of $\mathfrak{F}_{K^2}$ which neighbors P_N/Q_N.

First, consider the case in which P_N/Q_N is not itself acceptable. We obtain successively higher members R_r/S_r of $\mathfrak{F}_{K^2}$, starting from $R_0/S_0 = P_N/Q_N$ and terminating the process when an acceptable member R_n/S_n is obtained. Since there is no acceptable fraction between P_N/Q_N and x (see Lemma 3, Appendix I), it follows that R_n/S_n corresponds to the best upper approximation, $(F_1''F_2'')/(F_3''F_4'')$. We then repeat the process with successively lower members of $\mathfrak{F}_{K^2}$, starting again from R_0/S_0, to obtain the best lower approximation.

In the particular case in which $R_0/S_0 = P_N/Q_N$ is in itself acceptable, it corresponds to either the best lower approximation (if N is even) or the best upper approximation (if N is odd), and the search for the other required approximation must be directed accordingly.

the algorithm

The algorithm is given in numerical form and in sufficient detail to be used by a person with no previous knowledge either of continued fractions or Farey series.

Step 1. In order to determine P_N/Q_N, form successively[1] integers

$A_1, A_2, \cdots, A_r, \cdots$

by the system of equations

$$\begin{aligned}
1/x &= A_1 + x_1 \qquad (0 \le x_1 < 1) \\
1/x_1 &= A_2 + x_2 \qquad (0 \le x_2 < 1) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
1/x_{r-1} &= A_r + x_r \qquad (0 \le x_r < 1) \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

and calculate simultaneously the integers

$P_1, P_2, \cdots, P_r, \cdots, \qquad Q_1, Q_2, \cdots, Q_r, \cdots$

by the system of equations

$$P_1 = 1, \qquad P_2 = A_2, \qquad \cdots, \qquad P_r = A_r P_{r-1} + P_{r-2};$$

$$Q_1 = A_1, \qquad Q_2 = A_1 A_2 + 1, \qquad \cdots, \qquad Q_r = A_r Q_{r-1} + Q_{r-2}.$$

Terminate the process when either

$$x_r = 0 \qquad (\text{at } r = N, \text{say})$$

or

$$Q_{r+1} > K^2 \qquad (\text{at } r = N, \text{say}).$$

Carry forward to the next step the values

$$P_{N-1},\ Q_{N-1},\ P_N,\ Q_N$$

so obtained and the value of $(-1)^N$.

Step 2. The preliminary step in obtaining the Farey series is to form the next higher fraction after P_N/Q_N having a denominator less than or equal to K^2. Form

$$R_1 = \left\lfloor \frac{K^2 - (-1)^N Q_{N-1}}{Q_N} \right\rfloor P_N + (-1)^N P_{N-1},$$

$$S_1 = \left\lfloor \frac{K^2 - (-1)^N Q_{N-1}}{Q_N} \right\rfloor Q_N + (-1)^N Q_{N-1},$$

where the symbol "$\lfloor\ \rfloor$" is used to denote the *floor* operation,[2] e.g., $\lfloor a \rfloor$ denotes the greatest integer a' such that $a' \leq a$. Then, R_1/S_1 is the desired fraction (see Lemma 1, Appendix I). Carry forward to the next step the values P_N, Q_N (relabeled as R_0, S_0) and the values R_1, S_1 just obtained.

Step 3. The Farey series is developed as follows. We have now two adjacent members R_0/S_0, R_1/S_1 of $\mathfrak{F}_{K^2}$. We obtain successively higher members of $\mathfrak{F}_{K^2}$ by means of the system of equations (see Lemma 2, Appendix I)

$$R_r = \left\lfloor \frac{K^2 + S_{r-2}}{S_{r-1}} \right\rfloor R_{r-1} - R_{r-2},$$

$$S_r = \left\lfloor \frac{K^2 + S_{r-2}}{S_{r-1}} \right\rfloor S_{r-1} - S_{r-2}.$$

For each pair (R_r, S_r) so obtained, we determine the highest prime factor of R_r and the highest prime factor of S_r. If either of these primes exceeds K, we proceed to the calculation of the next successive pair in the sequence. If both R_r and S_r are decomposable into prime factors all less than or equal to K, we determine whether there exists an integer C such that CR_r can be expressed in the form F_1F_2 and CS_r can be expressed in the form F_3F_4. If so, $(F_1F_2)/(F_3F_4)$ is the required best upper approximation $(F_1''F_2'')/(F_3''F_4'')$, and we need not examine further pairs in this sequence. If the condition is not met, we must proceed to the calculation of the next successive pair in the sequence.

The best lower approximation is obtained in a similar manner, using the equations

$$R_r = \left\lfloor \frac{K^2 + S_{r+2}}{S_{r+1}} \right\rfloor R_{r+1} - R_{r+2},$$

$$S_r = \left\lfloor \frac{K^2 + S_{r+2}}{S_{r+1}} \right\rfloor S_{r+1} - S_{r+2},$$

starting with the known values of (R_1, S_1), (R_0, S_0) and proceeding through negative values of r.

concluding remarks

The mathematical technique used, though elementary, does not seem to have been previously applied to the solution of this common engineering problem.

In Appendix II, an example is given to illustrate application of the algorithm. The time to work through this example manually was 2.5 hours—a significant improvement on the 60 man-hours required to achieve similar results by the previous method.

It will be apparent that the algorithm is readily programmable, with an execution time measured in minutes even on the more modest computers.

Appendix I — Derivation of lemmas

Lemma 1. R_1/S_1 is the successor fraction to P_N/Q_N in $\mathfrak{F}_M$, where

$$R_1 = \left\lfloor \frac{M - (-1)^N Q_{N-1}}{Q_N} \right\rfloor P_N + (-1)^N P_{N-1},$$

$$S_1 = \left\lfloor \frac{M - (-1)^N Q_{N-1}}{Q_N} \right\rfloor Q_N + (-1)^N Q_{N-1},$$

and P_{N-1}/Q_{N-1} is the convergent preceding P_N/Q_N in the continued fraction expression of x.

Proof of Lemma 1.

$$\begin{aligned} S_1 &= \left\lfloor \frac{M - (-1)^N Q_{N-1}}{Q_N} \right\rfloor Q_N + (-1)^N Q_{N-1} \\ &\le \left(\frac{M - (-1)^N Q_{N-1}}{Q_N} \right) Q_N + (-1)^N Q_{N-1} \\ &= M. \end{aligned}$$

Hence

$$0 < R_1 < S_1 \le M.$$

Hence R_1/S_1 is a member of $\mathfrak{F}_M$. Also,[3]

$$Q_N R_1 - P_N S_1 = (-1)^N (Q_N P_{N-1} - P_N Q_{N-1}) = 1, \tag{2}$$

and

$$\begin{aligned} Q_N + S_1 &= \left\lfloor \frac{M - (-1)^N Q_{N-1}}{Q_N} \right\rfloor Q_N + Q_N + (-1)^N Q_{N-1} \\ &> \left(\frac{M - (-1)^N Q_{N-1}}{Q_N} - 1 \right) Q_N + Q_N + (-1)^N Q_{N-1} \\ &= M, \end{aligned}$$

so

$$Q_N + S_1 > M. \tag{3}$$

Now, if R_1/S_1 is not the successor to P_N/Q_N in $\mathfrak{F}_M$, there exists a fraction Y/Z for which

$$P_N/Q_N < Y/Z < R_1/S_1, \tag{4}$$

$$Z \leq M. \tag{5}$$

Hence

$$\begin{aligned}
\frac{1}{Q_N S_1} &= \frac{Q_N R_1 - P_N S_1}{Q_N S_1} && \text{by (2)} \\
&= \frac{R_1}{S_1} - \frac{P_N}{Q_N} \\
&= \frac{R_1 Z - S_1 Y}{S_1 Z} + \frac{Q_N Y - P_N Z}{Q_N Z} \\
&\geq \frac{1}{S_1 Z} + \frac{1}{Q_N Z} && \text{by (4)} \\
&= \frac{Q_N + S_1}{S_1 Q_N Z} \\
&> \frac{M}{Q_N S_1 Z} && \text{by (3)} \\
&\geq \frac{1}{Q_N S_1} && \text{by (5)}
\end{aligned}$$

which is impossible. The lemma follows.

Lemma 2. R_r/S_r is the successor fraction to R_{r-1}/S_{r-1} in $\mathfrak{F}_M$, where R_{r-1}/S_{r-1} is the successor fraction to R_{r-2}/S_{r-2} and

$$R_r = \left\lfloor \frac{M + S_{r-2}}{S_{r-1}} \right\rfloor R_{r-1} - R_{r-2},$$

$$S_r = \left\lfloor \frac{M + S_{r-2}}{S_{r-1}} \right\rfloor S_{r-1} - S_{r-2}.$$

Proof of Lemma 2. The proof follows the same lines as that for Lemma 1. We can show:

$$0 < R_r < S_r \leq M,$$

so that R_r/S_r is a member of $\mathfrak{F}_M$, and that[4]

$$S_{r-1}R_r - R_{r-1}S_r = S_{r-2}R_{r-1} - R_{r-2}S_{r-1} = 1. \tag{2$'$}$$

We can also show

$$S_{r-1} + S_r > M. \tag{3$'$}$$

Then, if R_r/S_r is not the successor to R_{r-1}/S_{r-1} in $\mathfrak{F}_M$, there is a fraction Y/Z for which

$$R_{r-1}/S_{r-1} < Y/Z < R_r/S_r \tag{4$'$}$$

and

$$Z \leq M. \tag{5'}$$

Hence

$$\begin{aligned}
\frac{1}{S_r S_{r-1}} &= \frac{S_{r-1}R_r - R_{r-1}S_r}{S_r S_{r-1}} && \text{by} \quad (2') \\
&= \frac{R_r}{S_r} - \frac{R_{r-1}}{S_{r-1}} \\
&= \frac{R_r Z - S_r Y}{S_r Z} + \frac{S_{r-1}Y - R_{r-1}Z}{S_{r-1}Z} \\
&\geq \frac{1}{S_r Z} + \frac{1}{S_{r-1}Z} && \text{by} \quad (4') \\
&= \frac{S_{r-1} + S_r}{S_r S_{r-1} Z} \\
&> \frac{M}{S_r S_{r-1} Z} && \text{by} \quad (3') \\
&\geq \frac{1}{S_r S_{r-1}} && \text{by} \quad (5')
\end{aligned}$$

which is impossible. The lemma follows.

Lemma 3. There is no fraction between x and P_N/Q_N having a denominator less than or equal to K^2.

Proof of Lemma 3. Consider the case in which N is even. Then $P_N/Q_N < x$. By the particular case of Lemma 1 in which $M = Q_{N+1}$, the successor fraction to P_N/Q_N in $\mathfrak{F}_{Q_{N+1}}$ is R^*/S^*, where

$$R^* = \left\lfloor \frac{Q_{N+1} - Q_{N-1}}{Q_N} \right\rfloor P_N + P_{N-1} = P_{N+1},$$

$$S^* = \left\lfloor \frac{Q_{N+1} - Q_{N-1}}{Q_N} \right\rfloor Q_N + Q_{N-1} = Q_{N+1}.$$

Hence, the successor fraction to P_N/Q_N in $\mathfrak{F}_{Q_{N+1}}$ is greater than x. Since $K^2 < Q_{N+1}$, it follows that the successor fraction to P_N/Q_N in $\mathfrak{F}_{K^2}$ is greater than x. A similar argument applies when N is odd. The lemma follows.

Appendix II — Illustration of algorithm

We illustrate application of the algorithm by considering the case $x = 1/\pi$ (with $J = 15$, $K = 45$).

The continued fraction expression for $1/\pi$ is known, so that it is not strictly necessary to carry out the first step. However, for illustrative purposes, suppose that we had been required to find the continued fraction expression, with its convergents, for 1/(3.14159265359). From Step 1 of the algorithm we obtain the results given in Table 1.

Table 1

		A_r	+	x_r	r	P_r	Q_r
$\frac{314159265359}{100000000000}$	=	3	+	$\frac{14159265359}{100000000000}$	1	1	3
$\frac{100000000000}{14159265359}$	=	7	+	$\frac{885142487}{14159265359}$	2	7	22
$\frac{14159265359}{885142487}$	=	15	+	$\frac{882128054}{885142487}$	3	106	333
$\frac{885142487}{882128054}$	=	1	+	$\frac{3014433}{882128054}$	4	113	355
$\frac{882128054}{3014433}$	=	292	+	$\frac{1913618}{3014433}$	5	33102	103993

Table 2

r	R_r	S_r	*Reason for continuing search*
1	558	1753	
0	113	355	
-1	572	1797	1797 has prime factor 599 > 45
-2	459	1442	1442 has prime factor 103 > 45
-3	346	1087	346 has prime factor 173 > 45
-4	579	1819	579 has prime factor 193 > 45
-5	233	732	233 has prime factor 233 > 45
-6	586	1841	586 has prime factor 293 > 45
-7	353	1109	353 has prime factor 353 > 45
-8	473	1486	1486 has prime factor 743 > 45
-9	593	1863	593 has prime factor 593 > 45
-10	120	377	

Since $103993 > 45^2$, we now terminate this process and carry forward to the next step

$$P_{N-1} = 106, \qquad Q_{N-1} = 333,$$

$$P_N = 113, \qquad Q_N = 355,$$

$$(-1)^N = 1.$$

Now, following Step 2 of the algorithm:

$$R_1 = \left\lfloor \frac{45^2 - 333}{355} \right\rfloor 113 + 106 = 4 \times 113 + 106 = 558,$$

$$S_1 = \left\lfloor \frac{45^2 - 333}{355} \right\rfloor 355 + 333 = 4 \times 355 + 333 = 1753.$$

We obtain and carry forward

$$R_0 = 113, \qquad S_0 = 355,$$

$$R_1 = 558, \qquad S_1 = 1753.$$

Now, carrying out Step 3 of the algorithm, the first acceptable pair (R_r, S_r) for positive r is found to be (R_{32}, S_{32}), namely (375, 1178), giving the best upper approximation to $1/\pi$ of $(15 \times 25)/(31 \times 38)$. The process is illustrated in detail in the search for the best lower approximation. We obtain successively the entries in Table 2.

Now

$$120 = 2 \times 2 \times 2 \times 3 \times 5$$
$$377 = 13 \times 29.$$

By taking $C = 2$, we can express R_{-10}/S_{-10} as

$$\frac{240}{754} = \frac{15 \times 16}{26 \times 29};$$

this is the best lower approximation to $1/\pi$.

Then, satisfying the given conditions, we have that the best possible approximations to

$$1/\pi \simeq 0.3183099$$

are

$$\frac{15 \times 16}{26 \times 29} \simeq 0.3183024 \qquad \text{(lower)}$$

and

$$\frac{15 \times 25}{31 \times 38} \simeq 0.3183362 \qquad \text{(upper)}.$$

It is clear that the best lower approximation is, in this case, the better of the two.

CITED REFERENCES AND FOOTNOTE

1. G. H. Hardy and E. M. Wright, *Theory of Numbers*, Clarendon Press, Oxford, 2nd Edition, 1945, pp. 128–135.
2. This notation follows a convention now common among systems engineers to use "$\lfloor\ \rfloor$" in place of the traditional "[ ]". This practice is convenient in permitting simultaneous use of "$\lceil\ \rceil$" to denote the *ceiling* operator ($\lceil x \rceil$ is defined as the smallest integer not exceeded by x).
3. Reference 1, Theorem 150, p. 130.
4. Reference 1, Theorem 28, p. 23.