

Some bipartite states do not arise from channels

M. B. Ruskai

It is well known that the action of a quantum channel on a state can be represented, using an auxiliary space, as the partial trace of an associated bipartite state. Recently, it was observed that for the bipartite state associated with the optimal average input of the channel, the entanglement of formation is simply the entropy of the reduced density matrix minus the Holevo capacity. It is natural to ask if every bipartite state can be associated with some channel in this way. We show that the answer is negative.

Background

Recently, Matsumoto, Shimono, and Winter (MSW) [1] pointed out an important connection between the channel capacity and entanglement of formation that allows one to draw some conclusion about the additivity of the latter from that of the former. There has also been speculation that the additivities of channel capacity and of entanglement of formation are equivalent. A connection between the additivity of entanglement of formation and the multiplicativity of the p-norm measure of purity has also been given by Audenaert and Braunstein [2].

In view of this it is natural to ask whether every bipartite state ρ_{AB} can be associated with a channel in the sense of MSW. In particular, is its reduced density matrix, ρ_A , the optimal average output of a channel whose capacity is related to the entanglement of formation of ρ_{AB} , as described in [1]? We give a more precise formulation of this statement and show that the answer is negative.

Recall that a state is represented by a density matrix ρ , i.e., a positive semidefinite operator with $\text{Tr} \rho = 1$. The Von Neumann entropy of ρ is $S(\rho) = -\text{Tr} \rho \log \rho$. By an ensemble $\mathcal{E} = \{\pi_i, \rho_i\}$, we mean a set of density matrices ρ_i and associated probabilities π_i . By a channel Φ , we mean a completely positive, trace-preserving (CPT) map. The Holevo capacity of the map Φ is

$$C_{\text{Holv}}(\Phi) = \sup_{\mathcal{E}} \left\{ S[\Phi(\rho)] - \sum_i \pi_i S[\Phi(\rho_i)] \right\}, \quad (1)$$

where the supremum is taken over all ensembles $\mathcal{E} = \{\pi_i, \rho_i\}$ and $\rho = \sum_i \pi_i \rho_i$. The optimal average input is the state $\rho_{\text{opt}} = \sum_i \pi_i \rho_i$ associated with the ensemble which attains the supremum in (1). The optimal average output is then $\Phi(\rho_{\text{opt}})$.

The key to the MSW construction is the following result of Stinespring [3], which was subsequently used by Lindblad [4] in his work on relative entropy.

Theorem 1

Given a CPT map Φ on $\mathcal{B}(\mathcal{H})$, one can find an auxiliary space $\mathcal{H}_B$, a density matrix ν_B , and a unitary map U_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that

$$\Phi(\rho) = \text{Tr}_B[U_{AB}^\dagger \rho \otimes \nu_B U_{AB}], \quad (2)$$

where Tr_B denotes the partial trace and we have identified $\mathcal{H}$ with $\mathcal{H}_A$.

Although we are interested in the case in which ρ is a density matrix, representation (2) is valid for all operators in $\mathcal{B}(\mathcal{H})$. When $\Phi(\rho) = \sum_k A_k^\dagger \rho A_k$, it is easy to construct a representation of the form (2), as discussed in Section III.D of [5]. In fact, $U_{AB}^\dagger \rho \otimes \nu_B U_{AB}$ is the block matrix that has the form $\sum_{jk} A_j^\dagger \rho A_k \otimes |j\rangle\langle k|$.

Given a bipartite state γ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$, its entanglement of formation (EoF) satisfies

Note: This paper is dedicated to Charles H. Bennett on the occasion of his sixtieth birthday.

$$\text{EoF}(\gamma_{AB}) = \inf \left\{ \sum_k \pi_k S(\text{Tr}_B \gamma_k) : \{\pi_k \gamma_k\} \text{ ensemble with } \gamma_{AB} = \sum_k \pi_k \gamma_k \right\}. \quad (3)$$

Although it is customary to define the EoF using ensembles for which all γ_k are pure states, there is no loss of generality in allowing the presence of arbitrary states.

Theorem 2 (MSW)

Let Φ be a CPT map with a representation $\mathcal{H}_B, \nu_B, U_{AB}$ as in Theorem 1. The Holevo capacity of Φ satisfies

$$C_{\text{Holv}}(\Phi) = \sup \{S(\text{Tr}_B \gamma_{AB}) - \text{EoF}(\gamma_{AB}) : \gamma_{AB} = U_{AB}^\dagger \rho \otimes \nu_B U_{AB}\}, \quad (4)$$

where ρ is a density matrix on $\mathcal{H} = \mathcal{H}_A$. Moreover, the state $\tilde{\gamma}_{AB}$ which attains this supremum satisfies $\text{Tr}_B \tilde{\gamma}_{AB} = \Phi(\rho_{\text{opt}})$, where ρ_{opt} is the optimal input of Φ .

It follows immediately that for the state $\tilde{\gamma}_{AB}$,

$$\text{EoF}(\tilde{\gamma}_{AB}) = S[\Phi(\rho_{\text{opt}})] - C_{\text{Holv}}(\Phi). \quad (5)$$

It is thus natural to ask the following.

Question 3

Given a bipartite state γ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$, is there a CPT map Φ on $\mathcal{B}(\mathcal{H}_A)$ such that $\text{Tr}_B \gamma_{AB}$ is the optimal average output state of Φ and

$$\text{EoF}(\gamma_{AB}) = S(\text{Tr}_B \gamma_{AB}) - C_{\text{Holv}}(\Phi)? \quad (6)$$

We show that the answer to this question is negative.

Our counterexample is based on a result in [6] and [7] which comes from the fact that when $\rho = \sum_k \pi_k \rho_k$,

$$S(\rho) - \sum_k \pi_k S(\rho_k) = \sum_k \pi_k H(\rho_k, \rho), \quad (7)$$

where the *relative entropy* is defined as

$$H(\omega, \rho) = \text{Tr}[\omega \log \omega - \omega \log \rho]. \quad (8)$$

Theorem 4

Let Φ be a CPT map with optimal average input ρ_{opt} . Then,

$$C_{\text{Holv}}(\Phi) = \sup_{\omega} H[\Phi(\omega), \Phi(\rho_{\text{opt}})]. \quad (9)$$

It follows as an immediate corollary that if $\mathcal{E} = \{\pi_k, \rho_k\}$ is any optimal ensemble for the channel Φ , then $H[\Phi(\rho_k), \Phi(\rho_{\text{opt}})] = C_{\text{Holv}}(\Phi)$ is independent of k ; i.e., the optimal average output is “equidistant” in the sense of relative entropy from all outputs $\Phi(\rho_k)$ in the ensemble.

Counterexample

An affirmative answer to Question 3 above would imply that if the ensemble $\{\pi_k, |\psi_k\rangle\langle\psi_k|\}$ is optimal for $\text{EoF}(\gamma_{AB})$, then $\{\pi_i, \text{Tr}_B |\psi_k\rangle\langle\psi_k|\}$ would be an optimal *output* ensemble for the corresponding map Φ . It would then follow from the equidistance corollary to Theorem 4 that

$$H(\omega_k, \gamma_A) = -S(\omega_k) - \text{Tr}_A \omega_k \log \gamma_A = C, \quad (10)$$

where $\omega_k = \text{Tr}_B |\psi_k\rangle\langle\psi_k|$ and C is a constant that is independent of k . (In fact, C is the Holevo capacity of Φ if such a channel exists.) We next present an example of a bipartite qubit state which does not satisfy (10).

It may be that counterexamples to (10) can already be found in the literature on entanglement. However, we take advantage of a result of Wootters [8] to construct a rather simple qubit counterexample that does not require extensive numerical computation. In [8], Wootters obtained an explicit formula for the EoF of a bipartite qubit state using a quantity designated as the concurrence. Moreover, he showed that the EoF could be achieved using an ensemble of at most four pure states $|\psi_k\rangle\langle\psi_k|$, for which $S(\text{Tr}_B |\psi_k\rangle\langle\psi_k|) = \text{EoF}(\gamma_{AB})$. Thus, for such an ensemble, (10) holds if and only if

$$\text{Tr}_A[(\text{Tr}_B |\psi_k\rangle\langle\psi_k|) \log \gamma_A] \quad (11)$$

is independent of k . We can assume without loss of generality that γ_A is diagonal, with eigenvalues $\frac{1}{2}(1 \pm x)$. Then $\text{Tr} \omega \log \gamma_A$ depends only on the diagonal elements of ω which can be written as $\frac{1}{2}(1 \pm d)$. In fact,

$$\begin{aligned} \text{Tr} \omega \log \gamma_A &= \frac{1+d}{2} \log \frac{1+x}{2} + \frac{1-d}{2} \log \frac{1-x}{2} \\ &= \log 2 - \frac{1}{2} \left[\log(1-x^2) + d \log \frac{1+x}{1-x} \right], \end{aligned} \quad (12)$$

which depends linearly on d for a fixed value of x not equal to 0. Thus, when $\gamma_A \neq \frac{1}{2}I$, (11) is independent of k if and only if all $\omega_k = \text{Tr}_B |\psi_k\rangle\langle\psi_k|$ have the same diagonal elements. However, we also know that $S(\omega_k) = \text{EoF}(\gamma_{AB})$ is independent of k , which implies that all ω_k have the same eigenvalues. Thus, all of the reduced density matrices $\text{Tr}_B |\psi_k\rangle\langle\psi_k|$ must have the form

$$\frac{1}{2} \begin{pmatrix} 1+d & e^{i\theta_k} t \\ e^{-i\theta_k} t & 1-d \end{pmatrix}$$

for some fixed value of t . It is not hard to find an example for which this does not hold.

Let $|\beta_0\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$ and $|\beta_3\rangle = (\sigma_z \otimes I)|\beta_0\rangle = (1/\sqrt{2})(|00\rangle - |11\rangle)$ be the indicated maximally entangled Bell states, and

$$\gamma_{AB} = \frac{5}{8} |\beta_0\rangle\langle\beta_0| + \frac{1}{16} |\beta_3\rangle\langle\beta_3| + \frac{1}{4} |01\rangle\langle 01| + \frac{1}{16} |10\rangle\langle 10|. \quad (13)$$

The reduced density matrix is $\gamma_A = \text{Tr}_B \gamma = \frac{1}{2}(I + \frac{3}{16}\sigma_z) \neq \frac{1}{2}I$. One can show that the optimal EoF decomposition of (13) has the form $\gamma_{AB} = \sum_{k=1}^4 \pi_k |\psi_k\rangle\langle\psi_k|$, with $|\psi_1\rangle = a_0|\beta_0\rangle + a_3|\beta_3\rangle$ so that $\omega_1 = \text{Tr}_B(|\psi_1\rangle\langle\psi_1|)$ is diagonal in the basis $|0\rangle, |1\rangle$ (i.e., $t = 0$ in the matrix above). However, the remaining ψ_k are superpositions which contain Bell states $|\beta_k\rangle$ and the product states $|01\rangle, |10\rangle$ in a form which necessarily yields a reduced density matrix ω_k that is *not* diagonal. By the discussion above, this implies a negative answer to Question 3.

To actually find the entanglement of formation and optimal decomposition of γ_{AB} , let

$$\tilde{\gamma} = \frac{5}{8}|\beta_0\rangle\langle\beta_0| + \frac{1}{16}|\beta_3\rangle\langle\beta_3| + \frac{1}{4}|10\rangle\langle 10| + \frac{1}{16}|01\rangle\langle 01| \quad (14)$$

be the density matrix with all spins flipped. The concurrence μ can be expressed in terms of the eigenvalues of $(\sqrt{\gamma} \tilde{\gamma} \sqrt{\gamma})^{1/2}$. Following Wootters [8], one finds

$$\mu = \frac{5}{8} - \frac{1}{16} - \frac{1}{8} - \frac{1}{8} = \frac{5}{16}.$$

Let

$$h(x) = -\frac{1+x}{2} \log \frac{1+x}{2} - \frac{1-x}{2} \log \frac{1-x}{2}. \quad (15)$$

Then, proceeding as described in [8], one finds

$$\begin{aligned} \text{EoF}(\gamma_{AB}) &= h(\sqrt{1-\mu^2}) = h\left(\frac{\sqrt{231}}{16}\right) \approx 0.1689 < 0.9745 \\ &\approx h\left(\frac{3}{16}\right) = S(\gamma_A). \end{aligned}$$

The optimal ensemble has weights

$$\pi_1 = 0.1527, \pi_2 = \pi_3 = \pi_4 = 0.2824$$

associated with the projections for the following pure states:

$$\begin{aligned} |\psi_1\rangle &= 0.8101|\beta_0\rangle + 0.5863|\beta_3\rangle; \\ |\psi_2\rangle &= -0.7870|\beta_0\rangle + 0.1087|\beta_3\rangle + 0.5432|01\rangle \\ &\quad + 0.2716|10\rangle; \\ |\psi_3\rangle &= 0.7870|\beta_0\rangle - 0.1087|\beta_3\rangle + 0.5432e^{i\pi/3}|01\rangle \\ &\quad + 0.2716e^{-i\pi/3}|10\rangle; \\ |\psi_4\rangle &= -0.7870|\beta_0\rangle + 0.1087|\beta_3\rangle - 0.5432e^{-i\pi/3}|01\rangle \\ &\quad - 0.2716e^{i\pi/3}|10\rangle. \end{aligned}$$

One can easily verify that the diagonal elements of $\omega_1 = \text{Tr}_B |\psi_1\rangle\langle\psi_1|$ are not equal to those of $\omega_k = \text{Tr}_B |\psi_k\rangle\langle\psi_k|$ for $k = 2, 3, 4$. One can also compute the reduced density matrices ω_k and infer that ω_1 is diagonal in the basis $|0\rangle, |1\rangle$, but the others are not. Alternatively, one

can observe that if a reduced density matrix is diagonal in the basis $|0\rangle, |1\rangle$, then any purification must have the form

$$|\Psi\rangle = a|0\rangle \otimes |\phi_1\rangle + b|1\rangle \otimes |\phi_2\rangle, \quad (16)$$

with $|\phi_1\rangle, |\phi_2\rangle$ orthogonal. By rewriting

$$\begin{aligned} |\psi_2\rangle &= -0.4796|00\rangle - 0.6333|11\rangle + 0.5432|01\rangle \\ &\quad + 0.2716|10\rangle \\ &= -|0\rangle \otimes (0.4796|0\rangle - 0.5432|1\rangle) + |1\rangle \otimes (0.2716|0\rangle \\ &\quad - 0.6333|1\rangle), \end{aligned}$$

one sees that ψ_2 does not have the form (16) with orthogonal $|\phi_k\rangle$. The actual reduced density matrices have the form

$$\begin{aligned} \omega_1 &= \frac{1}{2} \left[I + \frac{\sqrt{231}}{16} \sigma_z \right] = \begin{pmatrix} 0.9750 & 0 \\ 0 & 0.0250 \end{pmatrix}, \\ \omega_k &= \begin{pmatrix} 0.5251 & e^{i\theta_k} 0.4743 \\ e^{-i\theta_k} 0.4743 & 0.4749 \end{pmatrix}, \end{aligned}$$

$$\text{with } \theta_2 = \pi, \theta_3 = \frac{\pi}{3}, \theta_4 = -\frac{\pi}{3}.$$

Remarks on representations

In the canonical method of constructing a representation of the form (2), the reference state ν_B is pure, and d_B , the dimension of $\mathcal{H}_B$, is equal to the number of Kraus operators. Then the lifted state $\gamma_{AB} = U_{AB} \rho \otimes \nu_B U_{AB}^\dagger$, for which $\Phi(\rho) = \text{Tr}_B \gamma_{AB}$ has rank at most $d = d_A$, the dimension of the original Hilbert space. This is clearly a very restricted class of bipartite states. Moreover, generically, $d_B > d$, since many maps require the maximum number d^2 of Kraus operators. Thus, the canonical representation yields only bipartite states that are far more singular than typical states.

Some maps $\Phi(\rho)$ can be represented in the form (2) using a mixed reference state ν_B . Indeed, given a mixed bipartite state ν_B and unitary operator U_{AB} , (2) can be used to define a channel Φ . Unfortunately, relatively little is known about such mixed-state representations. It has been suggested that one might be able to represent a channel Φ in the form (2) using a space of dimension $d_B = d$ if mixed states are used. However, this is known not to be true in general [9, 10].

Now suppose that a qubit channel Φ can be represented (possibly using a mixed reference state ν_B) in the form (2) using an auxiliary qubit space $\mathcal{H}_B$ so that $d_B = 2$. Then the argument given before (11) can be used to show that for the optimal input distribution $\{\pi_k \rho_k\}$,

$$\text{Tr} \Phi(\rho_k) \log \Phi(\rho_{\text{opt}}), \quad (17)$$

must be independent of k . One can easily check that the three-state examples given in [11] do *not* have this property. Most non-unital qubit maps for which the translation of the image of the Bloch sphere lacks symmetry also violate (17). This implies that such channels require an auxiliary space with $d_B > 2$. One expects $d_B = 4$, consistent with the fact that such maps also require four Kraus operators. This gives another, somewhat indirect, proof that some CPT maps require an auxiliary space with $d_B > d$.

It would be of some interest to characterize the maps which admit mixed-state representations with $d_B = d$, as well as the bipartite states corresponding to the optimal average output.

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Mary Beth Ruskai *Department of Mathematics, Tufts University, Medford, Massachusetts 02155 (marybeth.ruskai@tufts.edu).* After receiving her Ph.D. degree in physical chemistry from the University of Wisconsin in 1969, Dr. Ruskai held postdoctoral positions as Battelle Fellow in Mathematical Physics at the University of Geneva from 1969 to 1971, and with Elliott Lieb at the Massachusetts Institute of Technology from 1972 to 1973. From 1977 to 2002, she was on the faculty at the University of Massachusetts at Lowell. She now holds an appointment as Research Professor in the Department of Mathematics at Tufts University, and is a Walton Visitor at the Communications Network Research Institute of the Dublin Institute of Technology. Her other visiting positions include appointments at the Rockefeller University, the University of Vienna, Bell Laboratories, the Courant Institute, and the Technical University of Berlin. Dr. Ruskai's current research interests are on mathematical questions in quantum information processors, particularly those pertaining to channel capacity and quantum error correction.