A walk along the branches of the extended Farey tree

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The rational numbers can be presented as the set of vertices of a degree-three tree. If p/q and p'/q' are two rational numbers written in lowest terms, the difference $pq' - p'q$ depends only on the shape of the path joining p/q to p'/q' on this tree.

1. Introduction

Back in elementary school, many of us thought that life would be much simpler if adding fractions required simply adding the numerators to get the numerator of the sum, and adding the denominators to get the denominator of the sum. In fact, the pairing

$$\left(\frac{p}{q}, \frac{p'}{q'}\right) \mapsto \frac{p + p'}{q + q'} \tag{1}$$

is useful in number theory, in topology, and in dynamical systems theory under the name *Farey sum* of p/q and p'/q'. More precisely, this is so when p/q and p'/q' are both in the unit interval, are in lowest terms, and are *Farey neighbors*, i.e., are not separated, on the real line, by any fraction with denominator smaller than $\max(q, q')$. [For example, we would associate 10/13 with the pair (3/4, 7/9) because no other rational number with denominator smaller than 9 can be found in the interval (3/4, 7/9); however, we would not associate 2/7 with the pair (1/5, 1/2) because both 1/4 and 1/3 are found in the interval (1/5, 1/2).] The Farey sum operation allows one to present the set of all rational numbers in (0, 1) as the set of vertices of an infinite rooted tree known as the *Farey tree*. The use of formula (1) can be generalized to allow the construction of a free tree of degree three, which we call the *extended Farey tree* or *$\mathbb{Q}$-tree*, having the set $\mathbb{Q}$ of rational numbers as its set of vertices, as we show in Section 2.

The $\mathbb{Q}$-tree embeds in the plane in such a way that its set of ends, naturally labeled by real numbers (in general, not rational), forms a nondecreasing sequence when read from left to right. Using the embedded tree, we give a topological interpretation of the numerator $pq' - p'q$ of the difference of two fractions p/q and p'/q', before reduction to lowest terms, and an equivalent interpretation of the same quantity in terms of symbols associated with the abstract tree. This extends the following reinterpretation of $n - m$, the difference of two integers: *Consider the real line marked by the lattice $\mathbb{Z}$ of integer points. When going from n to m along the line, initialize a counter to zero; add 1 for each lattice point reached if walking toward $-\infty$ (to the left); subtract 1 if walking toward $+\infty$ (to the right). The final number on the counter is $n - m$.*

Keeping in mind that n and m can be rewritten as $n/1$ and $m/1$ (as rational numbers written in lowest terms), we wish to rewrite $n - m$ as $n \cdot 1 - m \cdot 1$. The aim of this paper is to use the $\mathbb{Q}$-tree to interpret topologically, as we did for $n - m$, the quantity $pq' - p'q$ associated with the pair $(p/q, p'/q')$ of rational numbers reduced to lowest terms. In other words, we give a pictorial perspective to the solutions of the Diophantine equations $pq' - p'q = k$, where the case $k = 1$ is solved by consecutive terms of Farey series.

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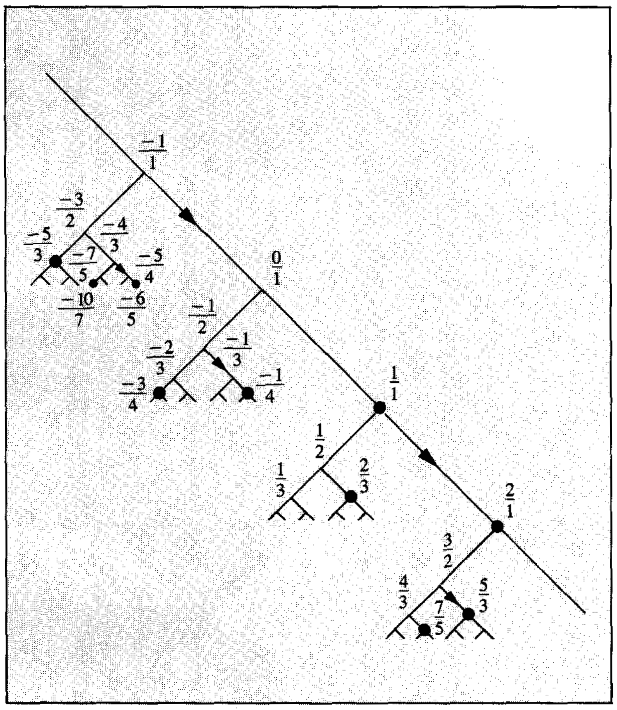

Figure 1

Illustration of the main results: The difference $pq' - p'q$ depends only on the shape of the oriented path from p/q to p'/q' on the $\mathbb{Q}$-tree.

In Section 2, we construct the $\mathbb{Q}$-tree and formulate what we call the $\mathbb{Q}$-tree theorem. Proofs and complementary results are provided in Section 3; the proofs given there also furnish algorithms to compute $pq' - p'q$ along the $\mathbb{Q}$-tree paths emanating from the vertex p/q. Sections 4 and 5 give less algorithmic but more structural and natural proofs of the result in Section 2. More precisely, Section 4 is an elementary proof, which depends on a nice homogeneity property of the $\mathbb{Q}$-tree. Section 5 relates the $\mathbb{Q}$-tree theorem to the action of $PSL(2, \mathbb{Z})$ on trees, which is related to a well-developed theory (e.g., [1, 2]). In Section 5, the $\mathbb{Q}$-tree is labeled by the tiles of a tessellation of the hyperbolic plane, introduced by H. J. S. Smith [3] and studied at length by G. Humbert [4–6]. For another discussion of the relations between the Farey tree and $PSL(2, \mathbb{Z})$, see [7]. See also [8] for related material.

2. Definitions and the $\mathbb{Q}$-tree theorem

We first describe an extension of the elementary concepts in the theory of Farey sequences from $\mathbb{Q} \cap [0, 1]$ to $\mathbb{Q}$. For the classical theory, see, e.g., [9], pp. 23–26, or [10], pp. 7–11. Then, the $\mathbb{Q}$-tree theorem is stated in two different forms, once the required language is in place.

Since the formal language makes comprehension of our elementary result artificially difficult, we have tried to capture this result in **Figure 1**, which displays the embedded $\mathbb{Q}$-tree. Vertices are labeled with rational numbers. Five paths are marked. The three red paths [from $(-5)/3$ to $1/1$, from $(-3)/4$ to $(-1)/4$, and from $(-10)/7$ to $(-6)/5$] have the same shape and the same value of $pq' - p'q$: $(-5) \cdot 1 - 1 \cdot 3 = (-3) \cdot 4 - (-1) \cdot 4 = (-10) \cdot 5 - (-6) \cdot 7 = -8$. Similarly, the two blue paths have the same value of $pq' - p'q$: $2 \cdot 1 - 2 \cdot 3 = 7 \cdot 3 - 5 \cdot 5 = -4$.

In the sequel, all rational numbers are written in lowest terms (with nonnegative denominators), except when otherwise stated, and with the proviso that both n and n/1 are considered as being written in lowest terms.

We write $[x]$ for the integer part of x, and $\{x\}$ for its fractional part, so $\{x\} = x - [x]$. For rational number $x = p/q$, we define

$$N(x) \equiv p, \qquad D(x) \equiv q,$$

and if $x' = p'/q'$ is rational, we define

$$\langle x, x' \rangle \equiv pq' - p'q.$$

For any integer n, and for $x \in [n, n+1]$, we write

$$\{x\}_n = \begin{cases} \{x\} & \text{if } n \le x < n+1, \\ 1 & \text{if } x = n+1. \end{cases}$$

Then, for x and x' both in $[n, n+1]$, it is easy to verify that

$$\langle x, x' \rangle = \langle \{x\}_n, \{x'\}_n \rangle. \tag{2}$$

As a consequence, all classical results concerning Farey sequences extend readily to fractions in $\mathbb{Q}$.

Remark It is convenient to extend the set of rational numbers to contain the ideal number 1/0. Since this convention allows us to shorten many discussions, we set $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{1/0\}$. The usefulness of adjoining 1/0 to $\mathbb{Q}$ in the context of Farey theory was recognized, e.g., by P. Bachmann [11], E. Lucas [12], and A. Denjoy [13][1].

Extended Farey sequences The *extended Farey sequence of order i* is the ordered set $\mathcal{H}_i$ of fractions x in $\hat{\mathbb{Q}}$, written in lowest terms, whose denominators do not exceed i. Hence,

$$\cdots = \mathcal{H}_{-3} = \mathcal{H}_{-2} = \mathcal{H}_{-1} = \emptyset,$$

$$\mathcal{H}_0 = \frac{1}{0},$$

[1] As a matter of fact, we could have chosen to adjoin $(-1)/0$, instead of 1/0, to $\mathbb{Q}$ for much of the discussion: A geometrical meaning of the ambiguity appears in Section 5. However, a reason to choose 1/0 as we did, and not $(-1)/0$, that is meaningful to our main purpose of exploring the $\mathbb{Q}$-tree, appears in Section 4. See also [13].

$$\mathcal{H}_1 = \mathbb{Z} \cup \frac{1}{0},$$

$$\mathcal{H}_2 = \cdots, \frac{-3}{2}, \frac{-1}{1}, \frac{-1}{2}, \frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{3}{2}, \cdots, \frac{1}{0},$$

and

$$\mathcal{H}_3 = \cdots, \frac{-5}{3}, \frac{-3}{2}, \frac{-4}{3}, \frac{-1}{1}, \frac{-2}{3}, \frac{-1}{2}, \frac{-1}{3}, \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \cdots, \frac{1}{0}.$$

This extension of the definition of Farey sequence to all of $\hat{\mathbb{Q}}$ is preferred here to the more usual one (see Section 6), because our main object is the extended Farey tree, constructed below. With our definition, $\mathcal{H}_1$ and $\mathcal{H}_2$ (but not $\mathcal{H}_3$) appear at the first stages of the construction of this tree.

Farey neighbors We say that two fractions p/q and p'/q' are *Farey neighbors* if they are consecutive in some extended Farey sequence, or equivalently, as is easy to verify, if

$$|pq' - p'q| = \left|\left\langle \frac{p}{q}, \frac{p'}{q'} \right\rangle\right| = 1.$$

• *Farey sum*

It is clear that if p/q and p'/q' are Farey neighbors, there exists some integer n such that both numbers belong to $[n, n + 1]$. Given Farey neighbors p/q and p'/q' in $[n, n + 1]$, we define their *Farey sum* as

$$\frac{p}{q} \oplus \frac{p'}{q'} \equiv \frac{p + p'}{q + q'}.$$

Equivalently, we have

$$\frac{p}{q} \oplus \frac{p'}{q'} = n + \left(\left\{ \frac{p}{q} \right\}_n \oplus_0 \left\{ \frac{p'}{q'} \right\}_n \right), \tag{3}$$

where, for Farey neighbors p_0/q_0 and p_1/q_1 in $[0, 1]$, we define

$$\frac{p_0}{q_0} \oplus_0 \frac{p_1}{q_1} \equiv \frac{p_0 + p_1}{q_0 + q_1}.$$

Formula (3) is the key to an immediate generalization of Farey theory from $[0, 1]$ to $\hat{\mathbb{Q}}$.

The well-known uniqueness of the Farey sum decomposition for numbers in $\mathbb{Q} \cap [0, 1]$ (the combination of Theorems 28–30 in [9]; see also [14]) combined with (2) and (3) gives Theorem 1.

• *Theorem 1*

Any noninteger rational number admits a unique Farey sum decomposition. Specifically, given p/q in $(n, n + 1)$, there exists a unique pair of Farey neighbors $(p_0/q_0, p_1/q_1)$ in $\hat{\mathbb{Q}}^2$ such that

$$\frac{p}{q} = \frac{p_0}{q_0} \oplus \frac{p_1}{q_1}.$$

Furthermore, the pair $(p_0/q_0, p_1/q_1)$ belongs to $[n, n + 1]^2$.

The numbers p_0/q_0 and p_1/q_1 are called the *Farey parents* of p/q. In the sequel, each time we use the symbol p/q, we associate it with the ordered pair $(p_0/q_0, p_1/q_1)$ of its Farey parents, so that

$$\frac{p}{q} = \frac{p_0}{q_0} \oplus \frac{p_1}{q_1},$$

with $p_0/q_0 < p_1/q_1$. To represent $n/1$, we use the convention

$$\frac{n}{1} \equiv \frac{n - 1}{1} \oplus \frac{1}{0}.$$

Young and old parents For any rational number, one of its two Farey parents, called the *young Farey parent*, has a bigger denominator than the other, called the *old Farey parent*. For instance, the young parent of $(n + 1)/1$ is $n/1$, and its old parent is $1/0$. This terminology will make more sense after we use it in the description of the $\mathbb{Q}$-tree. To avoid inflating our list of definitions even more, we freely use genealogical relations that are coherent with the ones precisely defined so far.

• *$\mathbb{Q}$-tree*

The *extended Farey tree* or *$\mathbb{Q}$-tree*, $\mathcal{T}$, is defined as the free tree of degree three with vertices labeled by rational numbers in such a way that p/q and p'/q' are consecutive (or bound an edge) on the tree if and only if one of them is the young Farey parent of the other.

Once the ordered set of integers has been used to consecutively label all vertices along an infinite path of the tree $\mathcal{T}$, the remaining vertices of $\mathcal{T}$ can be labeled inductively. In the induction process, for any p/q, the young parent of p/q is used to label a vertex one or more steps after the old parent has been used to label a vertex.[2]

The restriction of $\mathcal{T}$ to the set $\mathbb{Q} \cap (0, 1)$ is often called the *Farey tree* (see Section 6).

Walking on the $\mathbb{Q}$-tree We say that a path on the $\mathbb{Q}$-tree *goes down* an edge if it goes from the parent to the child; otherwise it *goes up*. The path *goes left* along an edge if the label of the end point of this edge is smaller than the

[2] The label 1/0 appears morally first, but at infinity.

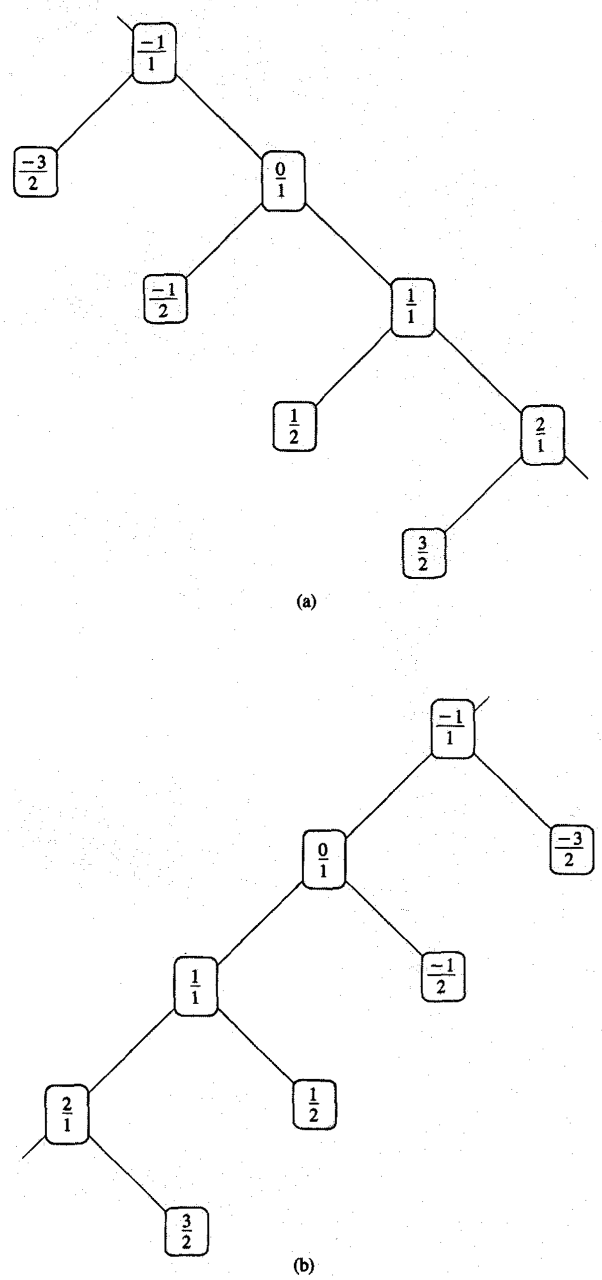

Figure 2

Initialization of the construction of the $\mathbb{Q}$-tree: (a) $\mathbb{Q}$ is augmented by 1/0; (b) $\mathbb{Q}$ is augmented by $(-1)/0$.

label at its starting point; otherwise, we say it *goes right*. Thus, an edge can be labeled unequivocally by one of the following *elementary symbols*:

$$D_l, D_r, U_l, U_r.$$

We say that D or U is the *principal part* of an elementary symbol, and l or r its *index*.

For any pair $(p/q, p'/q') \in \mathbb{Q}^2$, the path from p/q to p'/q' is completely described by its symbol,

$$\Sigma_{\left(\frac{p}{q}, \frac{p'}{q'}\right)} \equiv U_{a_1} \cdots U_{a_m} D_{b_1} \cdots D_{b_n},$$

where both a_i and b_j are in $\{l, r\}$, and $m \geq 0$ and $n \geq 0$.

• *$\mathbb{Q}$-tree theorem (symbolic version)*
For $(x, x') \in \mathbb{Q}^2$, the quantity $\langle x, x' \rangle$ depends only on the symbol $\Sigma_{(x,x')}$.[3]

• *Fifteen examples*
According to the $\mathbb{Q}$-tree theorem, we can unambiguously assign the number $\langle x, x' \rangle$ to the symbol $\Sigma_{(x,x')}$. We call the number $\langle x, x' \rangle$ the *value* of the symbol $\Sigma_{(x,x')}$. The list of values $v(\Sigma)$ corresponding to the 15 shortest symbols is as follows:

$$v(\varnothing) = 0, \tag{4a}$$

$$v(U_l) = v(D_l) = 1, \tag{4b}$$

$$v(U_r) = v(D_r) = -1, \tag{4c}$$

$$v(U_r U_l) = v(D_l D_r) = 1, \tag{4d}$$

$$v(U_l U_r) = v(D_r D_l) = -1, \tag{4e}$$

$$v(U_l U_l) = v(D_l D_l) = 2, \tag{4f}$$

$$v(U_r U_r) = v(D_r D_r) = -2, \tag{4g}$$

$$v(U_l D_l) = 3, \tag{4h}$$

$$v(U_r D_r) = -3, \tag{4i}$$

where (4b) and (4c) are simply consequences of the definition of generalized Farey neighbors.

• *Embedding the $\mathbb{Q}$-tree in $\mathbb{R}^2$*
To limit the amount of formalism, we use figures to indicate how to inductively construct a topological embedding of the $\mathbb{Q}$-tree in the real plane $\mathbb{R}^2$: **Figure 2** shows the starting configuration (integer vertices) together with the next generation[4], while **Figure 3** explains by examples how to grow the tree. The upper path in the figure is used to label the vertex (already labeled) $(a + e)/(b + f)$. The young Farey parent of that vertex is the one labeled e/f. The path is used to find the old Farey parent. Because the path from e/f to the vertex to be labeled goes to the *right*, the path used to find the old Farey parent starts to the *right* of vertex e/f and moves upward to the *right* of edges until it is forced to turn down. The old Farey parent, a/b, is the label of the vertex at

[3] That is, $\langle x, x' \rangle$ does not depend on the values x and x'—only on the symbol $\Sigma_{(x,x')}$.
[4] Figure 2(b) shows how the diagram must be changed if $\mathbb{Q}$ is augmented by $(-1)/0$ instead of 1/0; see footnote 1.

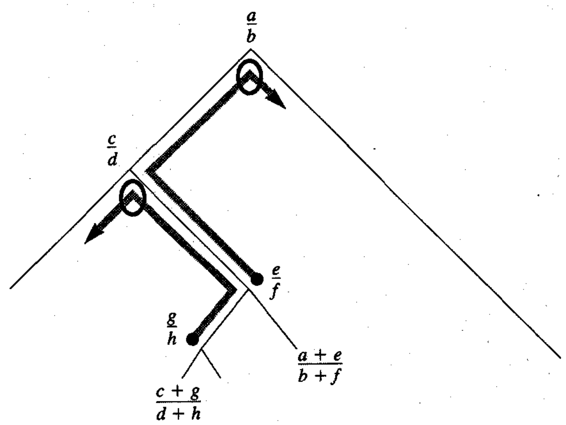

Figure 3

Examples of how to grow the $\mathbb{Q}$-tree.

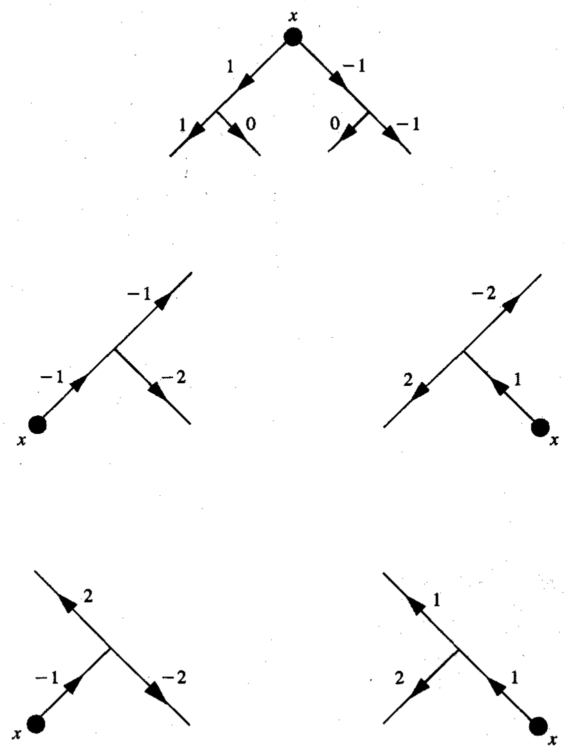

Figure 4

All five possible initializations for Algorithm 2 of Section 3.

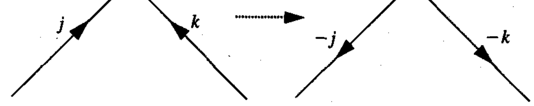

Figure 5

Implementation of Algorithm 2 of Section 3.

which the downturn occurs. The Farey sum of a/b and e/f is used as the new label. Similarly, the lower path in Figure 3 is used to label the vertex (already labeled) $(c + g)/(d + h)$. Because the path from g/h to the vertex to be labeled goes to the *left*, the path used to find the old parent starts to the *left* of vertex g/h and moves upward to the *left* of edges until it is forced to turn down. The old Farey parent is c/d in this case.

There are, of course, infinitely many geometrical realizations of this topological embedding. Some examples, based on hyperbolic geometry, are discussed in Section 5, but in these cases the edges do not go left and right or up and down as desired. As we have already mentioned, our $\mathbb{Q}$-tree theorem is illustrated in Figure 1, which suggests a different geometrical realization, using Euclidean geometry: All edges at generation n have length $1/2^n$. The contents of Figure 1 can be formulated as follows, with the word "shape" referring to the way in which a sequence of directed edges, say oriented line segments, goes up and down, and left and right, with no distances involved.

- *$\mathbb{Q}$-tree theorem (graphical version)*

For $(x, x') \in \mathbb{Q}^2$, $\langle x, x' \rangle$ depends only on the shape of the path going from x to x' on the embedded $\mathbb{Q}$-tree.

3. Evaluation of the paths

We prove the $\mathbb{Q}$-tree theorem by giving two equivalent algorithms for computing the value associated with a symbol.

In the first algorithm, knowing the value of a symbol Σ of length n, and the value of its truncation of length $n - 1$, we compute the value of all symbols of length $n + 1$ beginning with Σ. In the second (equivalent) algorithm, we construct a directed, weighted tree with marked vertex x, so that in order to determine the value of the symbol $\Sigma_{(x,x')}$, we sum the weights along the path from x to x'. To avoid formal writing, Algorithm 2 is illustrated in **Figures 4** and **5**. Checking that the steps of these algorithms are correct amounts to elementary algebra, and is essentially left to the reader, except for one case (chosen at random) of the first algorithm.

- *Algorithm 1*

Given two symbols,

$$\Sigma_n = S_1 S_2 \cdots S_{n-1} S_n$$

and

$$\Sigma_{n+1} = S_1 S_2 \cdots S_n S_{n+1},$$

with

$$S_i \in \{D_l, D_r, U_l, U_r\},$$

we say that Σ_{n+1} is obtained from Σ_n by either

- *following* ($\Sigma_{n+1} = \mathbf{F}\Sigma_n$) if S_n and S_{n+1} have the same principal part and the same index;
- *changing* ($\Sigma_{n+1} = \mathbf{C}\Sigma_n$) if S_n and S_{n+1} have the same principal part and different indices; or
- *turning* ($\Sigma_{n+1} = \mathbf{T}\Sigma_n$) if S_n and S_{n+1} have different principal parts.

We denote the truncation of Σ_n by $\Sigma_{n-1} = S_1 S_2 \cdots S_{n-1}$. We can then calculate the column vector

$$\begin{bmatrix} v(\Sigma_{n+1}) \\ v(\Sigma_n) \end{bmatrix}$$

from the column vector

$$\begin{bmatrix} v(\Sigma_n) \\ v(\Sigma_{n-1}) \end{bmatrix}$$

by multiplying the latter by the one of the following matrices that corresponds to the preceding list of directions:

$$\mathbf{F} \mapsto \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{C} \mapsto \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{T} \mapsto \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}.$$

For example, if S_n and S_{n+1} have the same principal part and the same index, then

$$\begin{bmatrix} v(\Sigma_{n+1}) \\ v(\Sigma_n) \end{bmatrix} = \mathbf{F} \begin{bmatrix} v(\Sigma_n) \\ v(\Sigma_{n-1}) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v(\Sigma_n) \\ v(\Sigma_{n-1}) \end{bmatrix}.$$

Thus, it is a straightforward process to calculate Σ_{n+1}, beginning with the values of Σ_1 and Σ_2 from Equations (4).

We now justify the formula $\mathbf{F}$, the proofs for the other two formulas being similar. From Equation (5), we have

$$\begin{bmatrix} v(\Sigma_{n+1}) \\ v(\Sigma_n) \end{bmatrix} = \begin{bmatrix} 2v(\Sigma_n) - v(\Sigma_{n-1}) \\ v(\Sigma_n) \end{bmatrix}.$$

Thus, it is only necessary to verify that $v(\Sigma_{n+1}) = 2v(\Sigma_n) - v(\Sigma_{n-1})$. We assume that both Σ_n and Σ_{n-1} terminate with U_r; all other cases would be treated similarly. Then, choose any u_0/v_0 so that the symbols Σ_n and Σ_{n+1} can be followed when starting from u_0/v_0. Let

- u_1/v_1 be the vertex reached by following the symbol Σ_{n-1}, starting from u_0/v_0;
- u_2/v_2 be the vertex reached by following the symbol Σ_n, starting from u_0/v_0;
- u_3/v_3 be the vertex reached by following the symbol Σ_{n+1}, starting from u_0/v_0.

Thus, we have

$$\left\langle \frac{u_0}{v_0}, \frac{u_2}{v_2} \right\rangle = v(\Sigma_n)$$

and

$$\left\langle \frac{u_0}{v_0}, \frac{u_1}{v_1} \right\rangle = v(\Sigma_{n-1}).$$

On the other hand, from the 15 examples, we have

$$\left\langle \frac{u_1}{v_1}, \frac{u_3}{v_3} \right\rangle = -2$$

and

$$\left\langle \frac{u_2}{v_2}, \frac{u_3}{v_3} \right\rangle = -1,$$

which allows us to solve for u_3 and v_3 in terms of u_1, v_1, u_2, and v_2. It is then easy to verify that

$$\left\langle \frac{u_0}{v_0}, \frac{u_3}{v_3} \right\rangle = 2v(\Sigma_n) - v(\Sigma_{n-1}).$$

- *Algorithm 2*

All of the possibilities used in the initialization for the construction of a directed weighted tree with marked vertex x are given in Figure 4, and the way to compute the remaining weights inductively is indicated in Figure 5; weights are integers associated with directed edges. To compute $\langle x, x' \rangle$, we sum the weights along the path from x to x'. Like the first one, this second algorithm has a matrix representation, the relevant transformations, as seen in Figure 5, being

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} = \begin{bmatrix} j \\ j+k \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} = \begin{bmatrix} j+k \\ j \end{bmatrix},$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} = \begin{bmatrix} -k \\ -j \end{bmatrix}.$$

Figure 6 displays an example.

4. Copies of the Farey tree in the $\mathbb{Q}$-tree

For any rational number, Theorem 1 asserts the existence of a unique decomposition

$$\frac{p}{q} = \frac{p_0}{q_0} \oplus \frac{p_1}{q_1}$$

of p/q, as a Farey sum of its two parents $p_0/q_0 < p_1/q_1$. Let us denote p_0/q_0 by $(0, 1)$ and p_1/q_1 by $(1, 0)$. Then p_0/q_0, p_1/q_1, p/q, and all of the descendants of p/q can be written uniquely as $(mp_0 + np_1)/(mq_0 + nq_1)$ and represented by the pair of integers (n, m). **Figure 7** shows how the pairs can be organized on a tree. If the symbol (n, m) is replaced by n/m, one obtains the piece of $\mathcal{T}$ generated from the pair $0/1 < 1/0$, which thus reappears everywhere in $\mathcal{T}$, justifying the choice we made of $1/0$ to extend $\mathbb{Q}$ (see footnote 1). On the basis of this observation, the proof of the $\mathbb{Q}$-tree theorem is now straightforward: Just notice that if p/q is the youngest common ancestor of x and x', setting

$$x = \frac{ap_0 + bp_1}{aq_0 + bq_1},$$

$$x' = \frac{cp_0 + dp_1}{cq_0 + dq_1},$$

we have

$$\begin{aligned} \langle x, x' \rangle &= (ap_0 + bp_1)(cq_0 + dq_1) - (aq_0 + bq_1)(cp_0 + dp_1) \\ &= (bc - ad)(p_1 q_0 - p_0 q_1) \\ &= bc - ad \end{aligned}$$

independently of x, x', but depending on the path that joins them on $\mathcal{T}$.

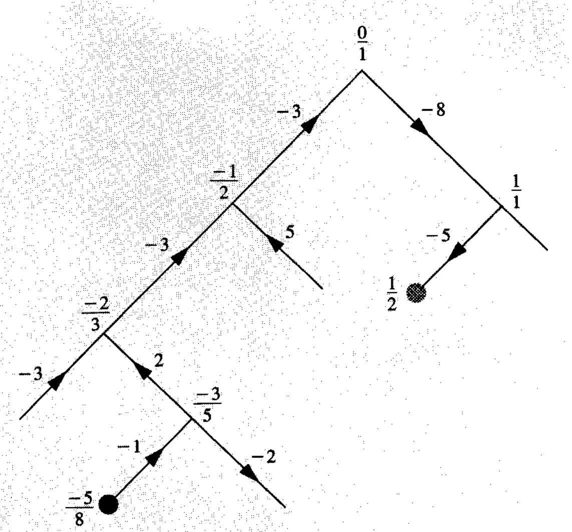

Figure 6

Example illustrating Algorithm 2 of Section 3. The sum of the weights in the path from vertex $(-5)/8$ to vertex $1/2$ is $-18 = (-5) \cdot 2 - 1 \cdot 8$.

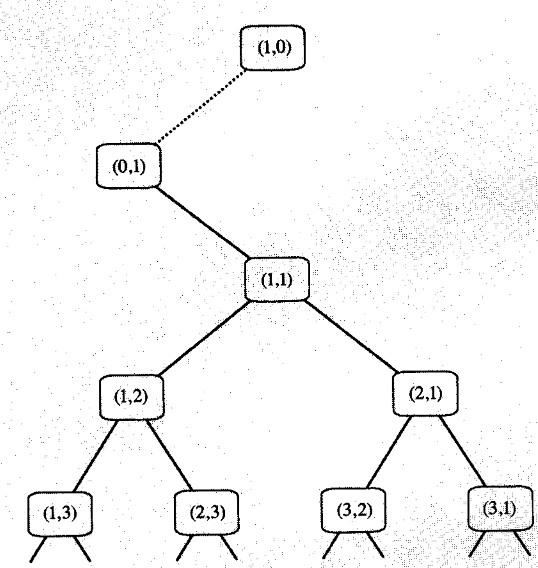

Figure 7

Illustration of the construction in Section 4.

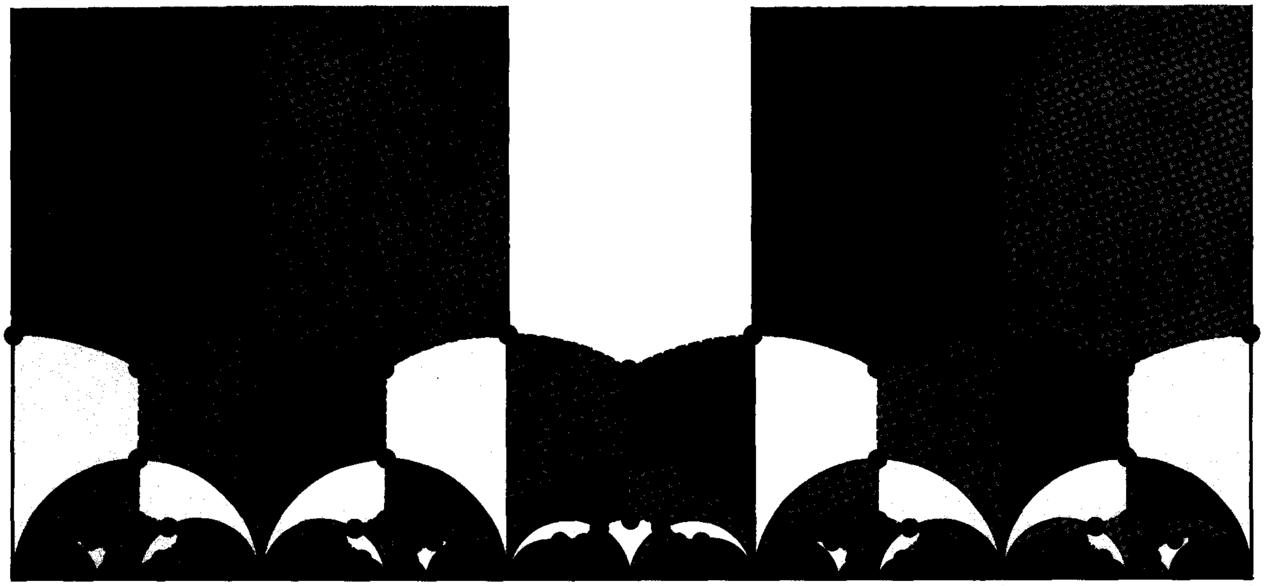

Figure 8

Tesselations $\mathbb{M}$ and $\mathbb{T}$.

5. *PSL*(2, $\mathbb{Z}$) and the $\mathbb{Q}$-tree

- *The tiling of H. J. S. Smith*

As usual, let $SL(2, \mathbb{R})$ stand for the set of 2×2 matrices of real numbers with determinant equal to 1. It is well known that the group $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\pm I$, which acts on $\mathbb{C}$ by the Möbius transforms as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}: z \mapsto \frac{az + b}{cz + d},$$

is in fact the set of the orientation-preserving isometries of Poincaré's model of the hyperbolic plane, i.e., the upper half of the complex plane $\mathbf{H} = \{x + iy \in \mathbb{C}: y > 0\}$, equipped with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

For this metric, geodesics are pieces of circles orthogonal to the real axis, or pieces of vertical lines. For a discrete subgroup Γ of $PSL(2, \mathbb{R})$, a *fundamental region R* is defined by the following properties:

- Its interior does not contain any pair $(x, g(x))$, $g \in \Gamma$.
- Its closure contains at least a point of each orbit.

The set of images of any fundamental region R by the elements of Γ provides a tessellation of $\mathbf{H}$, called the *tiling by* Γ, with *fundamental region R*.

Figure 8 represents two tilings of $\mathbf{H}$ together:

- The tiling $\mathbb{M}$, corresponding to the coloring, is a tiling by the *modular group* $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\pm I$, with fundamental region

$$R \equiv \{z: 0 \le \mathbb{R}e(z) < 1, |z| \ge 1, |z - 1| \ge 1\}.$$

- The tiling $\mathbb{T}$, corresponding to the solid lines, we call the $\mathbb{Q}$-tiling. Each of its tiles is made of three tiles of $\mathbb{M}$. [**Figure 9** is an embedding of $\mathcal{T}$, invariant by $PSL(2, \mathbb{Z})$. The dotted lines are the same as in Figure 8, but more easily seen.]

The $\mathbb{Q}$-tiling, obtained by joining all pairs of Farey neighbors by a geodesic in $\mathbf{H}$, is not a tiling by a subgroup of $PSL(2, \mathbb{R})$; in fact, it corresponds to an order-two extension (by $z \mapsto -\bar{z}$) of the congruence subgroup $\Gamma(2)$ of $PSL(2, \mathbb{Z})$ [see, e.g., [15], p. 82, for a fundamental domain of $\Gamma(2)$].

Remark The relevance of $\mathbb{T}$ in the study of continued fractions was recognized by H. J. S. Smith [3] in 1877 and studied in detail by A. Hurwitz [16] and G. Humbert [4–6].

Marked tiles Each tile of $\mathbb{T}$ is an *ideal triangle*, i.e., a triangle with all vertices at infinity (the real axis is at infinity for $\mathbf{H}$). The middle vertex of the ideal triangle can be chosen to label the tile, but there is an ambiguity for the upper tiles with vertical sides, regarding how to label the upper ideal vertex. This is the geometrical ambiguity in the choice among 1/0 and (−1)/0 we mentioned in footnote 1. We choose 1/0 as ∞. With each tile, let us associate the collection of its vertices, ordered so that one goes around the tile counterclockwise. A threefold ambiguity then remains, corresponding to the original vertex of the triplet. Hence, for any p/q, we have three *marked tiles*,

$$T^r_{\frac{p}{q}} = \left(\frac{p_1}{q_1}, \frac{p_0}{q_0}, \frac{p}{q}\right),$$

$$T^l_{\frac{p}{q}} = \left(\frac{p_0}{q_0}, \frac{p}{q}, \frac{p_1}{q_1}\right),$$

$$T^m_{\frac{p}{q}} = \left(\frac{p}{q}, \frac{p_1}{q_1}, \frac{p_0}{q_0}\right),$$

with r, l, m standing respectively for *right*, *left*, and *middle*.

In the following computations from [6], we set

$$\frac{a \cdot 0 + b}{c \cdot 0 + d} = \frac{b}{d},$$

and NOT

$$\frac{a \cdot 0 + b}{c \cdot 0 + d} = \lim_{\epsilon \to 0} \frac{a\epsilon + b}{c\epsilon + d},$$

which matters (only) when $d = 0$. A simple computation yields

$$M^r_{\frac{p}{q}} = \begin{bmatrix} p_1 & p_0 \\ q_1 & q_0 \end{bmatrix}: T^r_{\frac{1}{1}} \mapsto T^r_{\frac{p}{q}},$$

$$M^l_{\frac{p}{q}} = \begin{bmatrix} -p_0 & p_0 + p_1 \\ -q_0 & q_0 + q_1 \end{bmatrix}: T^r_{\frac{1}{1}} \mapsto T^l_{\frac{p}{q}},$$

$$M^m_{\frac{p}{q}} = \begin{bmatrix} p_0 + p_1 & -p_1 \\ q_0 + q_1 & -q_1 \end{bmatrix}: T^r_{\frac{1}{1}} \mapsto T^m_{\frac{p}{q}},$$

from which it is then easy to prove the following result of G. Humbert (cf. [6], pp. 105–110).

Theorem 2

For any s, s' in r, l, m, and any x, x' in $\hat{\mathbb{Q}}$, there is a single matrix $M \in PSL(2, \mathbb{Z})$ such that $M \cdot T^s_x = T^{s'}_{x'}$. More precisely, $M = M^{s,s'}_{x,x'} \equiv M^{s'}_{x'} \cdot (M^s_x)^{-1}$.

$PSL(2, \mathbb{Z})$ and degree-three trees The pieces of the boundary of $\mathbb{M}$ that do not belong to the boundary of $\mathbb{T}$ (i.e., the dotted lines in Figure 8 and Figure 9) form a (geometrical) degree-three tree $\mathcal{T}_g$ on which $PSL(2, \mathbb{Z})$ acts with fundamental domain PQ (see [1], p. 35 and [2], pp. 21–24). Notice that each vertex of $\mathcal{T}_g$ belongs to the interior of a single tile of $\mathbb{T}$, and that each tile of $\mathbb{T}$

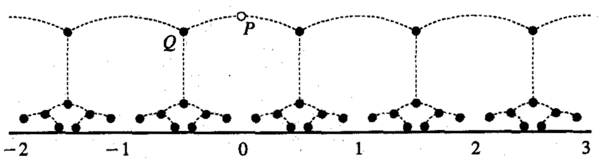

Figure 9

Embedding of $\mathcal{T}$, invariant by $PSL(2, \mathbb{Z})$. The arc PQ is a fundamental domain for the action of $PSL(2, \mathbb{Z})$ on this tree.

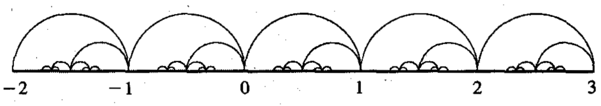

Figure 10

Another embedding of $\mathcal{T}$ in **H**.

contains a vertex of $\mathcal{T}_g$ (see the black dots in Figures 8 and 9). Hence, from the previous discussion about the tiles of $\mathbb{T}$, the vertices of $\mathcal{T}_g$ are in one-to-one correspondence with the rational numbers, and we can easily verify that the labeling so defined is such that $\mathcal{T}_g$ is a geometrical realization of $\mathcal{T}$.

Remark The boundary of $\mathbb{T}$ also contains an embedding of $\mathcal{T}$, shown in **Figure 10**.

$PSL(2, \mathbb{Z})$ action and the $\mathbb{Q}$-tree Theorem 2 yields an action of $PSL(2, \mathbb{Z})$ on $\mathcal{T} \times \mathbb{Z}/3\mathbb{Z}$, defined by matrix multiplication, once the three matrices $M^r_{p/q}$, $M^l_{p/q}$, $M^s_{p/q}$ are used to label the vertex p/q of $\mathcal{T}$. In general, $M^{s,s'}_{x,x'}(M^{s''}_{x''}) = M^{s'''}_{x'''}$, with no simple general rule for s''' and x''', because the order relation of numbers with their young Farey parents can be found in either way. However, it is easy to verify the following result.

Lemma 1

Consider any s in r, l, m, and any x, x' in $\hat{\mathbb{Q}}$. Assume that x is the young Farey parent of y and z, with $y < x < z$, and that x' is the young Farey parent of y' and z', with $y' < x' < z'$. Then

$$M^{s,s}_{x,x'}(M^{s''}_x) = M^{s''}_{x'},$$

$$M^{s,s}_{x,x'}(M^{s''}_y) = M^{s''}_{y'},$$

and

$$M^{s,s}_{x,x'}(M^{s''}_{z}) = M^{s''}_{z'}.$$

Now let $\mathcal{T}^s_x$ stand for the rooted sub-tree of $\mathcal{T}$ labeled by x and all its descendants, each vertex p/q being labeled by $M^s_{p/q}$. Iterating Lemma 1, we obtain the following equality.

- *Theorem 3*

$$M^{s,s}_{x,x'}(\mathcal{T}^{s''}_{x}) = \mathcal{T}^{s''}_{x'}.$$

- *$PSL(2, \mathbb{Z})$ and the $\mathbb{Q}$-tree theorem*

The $\mathbb{Q}$-tree theorem can be deduced from Theorem 3, specialized to the case $s = r$. More precisely, let

$$M^{r,r}_{x,x'} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then, for u, v among the descendants of x, and u', v' the corresponding points among the descendants of x', we have

$$\frac{au + b}{cu + d} = u'$$

and

$$\frac{av + b}{cv + d} = v'.$$

Now, for any rational numbers p/q and p'/q', we have

$$\left\langle \frac{a\dfrac{p}{q} + b}{c\dfrac{p}{q} + d}, \frac{a\dfrac{p'}{q'} + b}{c\dfrac{p'}{q'} + d} \right\rangle$$

$$= (ap + bq)(cp' + dq') - (cp + dq)(ap' + dq')$$

$$= (ad - bc)(pq' - p'q)$$

$$= (pq' - p'q),$$

so that

$$\langle u', v' \rangle = \langle u, v \rangle.$$

6. Historical notes

- *Farey sequences*

The *Farey sequence of order i* is usually defined as the ordered set $\mathcal{F}_i$ of fractions in x in $\mathbb{Q} \cap [0, 1]$ whose denominators $D(x)$ do not exceed i. This is usually extended to $\hat{\mathbb{Q}}$ as the ordered set $\mathcal{F}'_i$ of fractions x in $\hat{\mathbb{Q}}$, with $\max(|(N(x)|, D(x)) \leq i$ (see, e.g., [11, 17, 18]). In 1816, J. Farey (a geologist), studying the privately circulated early version of the table of "complete decimal quotients" by Henry Goodwyn, Esq., of Blackheath, noticed in [19] that if all "vulgar fractions" with denominator smaller than n are written in order, with $a/b < a'/b' < a''/b''$, then $a' = a + a''$ and $b' = b + b''$ (Theorem 29 in [9]). A wrong claim of proof was made by an anonymous person with the signature "S.A." in the following volume of the same journal [20]. Farey also presented his remark anonymously in [21]. His remark was shortly afterward proved by A. Cauchy [22], who mentioned only the French version [21] but was possibly familiar with [19], since he associated the name of Farey with the subject, followed in that regard by most mathematicians since. Cauchy in fact proved that with the previous notations, $a'b - ab' = 1$, from which he deduced the property noticed by Farey.[5] Cauchy did not limit himself to [0, 1] but considered, without giving them a name, what we call extended Farey sequences; it occurred to us that the name "Cauchy sequences" would not necessarily be well accepted in this context. In 1879, J. W. L. Glaisher [23] (see, in particular, the historical comment, pp. 329–336) examined the history of the subject and concluded that at least part of the credit should go to H. Goodwyn. He seems to have ignored the following fact: Most of the Farey–Cauchy theory was presented in 1802 by *le Citoyen* Haros[6] in [24] (see in particular the bottom of p. 367 and the top of p. 368 of [24]). We refer the reader to L. E. Dickson's book ([25], pp. 155–158, 162) for the early history of Farey sequences up to 1919. Important applications were made by A. Hurwitz to Diophantine approximation [26] and the reduction of binary quadratic forms [16], and a relation to the Riemann hypothesis was found by J. Franel [27] (see also [28, 29]). The Hardy–Littlewood circle method (see, e.g., [30]) makes essential use of a Farey series dissection of [0, 1]; it led, in particular, to the celebrated result of I. M. Vinogradov [31] that every sufficiently large odd number is expressible as the sum of three prime numbers. For more concrete applications, see, e.g., [32].

- *Farey tree*

The first occurrences we could find of this tree are in [33–35], where the Farey tree is presented as a way to analyze applications of Diophantine approximation to dynamics, and in [36], where the tree is implicit but parenthood is mentioned explicitly to describe a dynamical result. For recent constructions, uses of the Farey tree in dynamics, and lists of applications of Farey theory to dynamical systems theory, see for instance [7, 8, 14, 37].

The Farey tree is strongly reminiscent of what were called *Brocot sequences* in treatises on number theory in

[5] Theorem 28 in [9]. See p. 24 of [9] for the proof of the equivalence of the two properties, and pp. 24–26 for proofs of both results.

[6] *Citoyen* is the French word for citizen: The paper was published in Messidor, year X of the Republican calendar; Dickson refers to "C. Haros," as well as most authors after him, but we could not find any evidence that C stands for the initial of Haros's first name.

France around the turn of the century, after the clock maker who described them in 1862 [38] (see also [12] and [39]). The *Brocot sequence of order i* is the ordered set $\mathcal{B}_i$ of fractions x in $\mathbb{Q} \cap [0, 1]$ (or in $\hat{\mathbb{Q}}^+$) such that $\mathcal{B}_0 = \{0/1, 1/1\}$ (or $\mathcal{B}_0 = \{0/1, 1/1, 1/0\}$), and $\mathcal{B}_{i+1}$ is obtained from $\mathcal{B}_i$ by including the Farey sum of all pairs of successive elements in $\mathcal{B}_i$. Thus, in $\hat{\mathbb{Q}}^+$,

$$\mathcal{B}_1 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}, \frac{1}{0}\right\}$$

and

$$\mathcal{B}_2 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{0}\right\}.$$

In [13], Denjoy generalizes this definition, and in particular extends it to $\hat{\mathbb{Q}}$[7]. Brocot sequences and their generalizations are a primary object of study in the theory of the [?] function of Minkowski [40] (see also [7, 13, 41]), which is defined as follows: *If the path on the Farey tree from* 1/2 *to* $x \in [0, 1]$ *is labeled* $a_1, a_2, a_3, \cdots$, *then* $[?](x) = 0.a_1a_2a_3 \cdots$ *in base* 2. We also notice that the ordered set of denominators of the fractions appearing at depth n of the Farey tree form the Stein diatomic sequences [42] (see also [43–45]).

- *Farey tiling*

The tiling $\mathbb{T}$ is often called the *Farey tiling* in the recent literature, e.g., [17, 18], although G. Humbert (cited in [17] and [18]) did his best to associate $\mathbb{T}$ with the name Smith. It is well known that some of the relations established between the $\mathbb{Q}$-tiling and the theory of continued fractions can be reformulated in terms of the $\mathbb{Q}$-tree (see, e.g., [14, 33, 34], as well as [46–48] for related results).

There are, of course, many aspects of Farey theory and its applications not discussed here (see, e.g., [49–56]).

Conclusion

As J. Farey said at the end of his first paper on the subject: *"I am not acquainted, whether this curious property of vulgar fractions has been before pointed out?"*

Acknowledgments

Recent discussions of one of the authors (C. T.) with Lisa Goldberg, Matt Grayson, and George Zettler about the Farey tree played a crucial role in discovering (perhaps rediscovering) the elementary result reported in this paper. Section 4 is a modification of a simplified proof suggested by Don Coppersmith, who is gratefully acknowledged. Thanks also to Linda Keen for a helpful discussion.

[7] We could have started with such a definition right away, but extended Farey sequences, which do not involve Farey sums, are more elementary than the sequences defined by the Brocot construction.

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Received June 4, 1993; accepted for publication March 15, 1994

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