Finding compact coordinate representations for polygons and polyhedra

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Practical solid modeling systems are plagued by numerical problems that arise from using floating-point arithmetic. For example, polyhedral solids are often represented by a combination of geometric and combinatorial information. The geometric information may consist of explicit plane equations, with floating-point coefficients; the combinatorial information may consist of face, edge, and vertex adjacencies and orientations, with edges defined by face–face adjacencies and vertices by edge–edge adjacencies. Problems arise when numerical roundoff error in geometric operations causes the geometric information to become inconsistent with the combinatorial information. These problems can be avoided by using exact arithmetic instead of floating-point arithmetic. However, some operations, such as rotation, increase the number of bits required to represent the plane equation coefficients. Since the execution time of exact arithmetic operators increases with the number of bits in the operands, the increased number of bits in the plane equation coefficients can cause performance problems. One proposed solution to this performance problem is to *round* the plane equation coefficients without altering the combinatorial information. We show that such rounding is NP-complete.

1. Introduction

To achieve reliable programs that implement geometric computations, one must understand and control the effect of numerical error. One approach to controlling numerical error is to eliminate it by working only with objects and transformations that can be represented with numbers in a representable subfield of the real numbers, such as the field of rational numbers. For linear objects (e.g., points, lines, planes), using rational numbers would allow exact computation of intersections. Unfortunately, the situation is different with rotation, a commonly used geometric transformation: Rotating a line whose equation has rational coefficients can yield a line with irrational coefficients. Any rotation can be approximated arbitrarily closely by a *rational rotation*, a rotation that can be represented by a matrix with rational entries [1, Section 4.3.3]. Unfortunately, when rational representations are used, iteration of geometric transformations, such as rotation, can cause unbounded

growth in the precision (number of bits) needed to represent transformed objects, leading rapidly to unacceptable time required to perform geometric computations. We focus here on the precision growth problem.

Precision growth can arise from a sequence of geometric transformations. As a simple example, consider a sequence of r rotations about the coordinate axes (any rotation can be expressed as a sequence of rotations about the axes), each of which is to be approximated by a rational rotation with an angle of rotation accurate to one part in 2^{-P}, where P is the desired number of bits of accuracy. It is easy to show that after these r rotations have been applied to a point, the precision required per transformed coordinate is $O(rP)$. Precision growth can be limited by interspersing in the sequence *rounding* operations that closely approximate the transformed geometric objects with geometric objects that require less precision to represent. But what does it mean to approximate a geometric object, and how should rounding operations be interspersed in the sequence of transformations? These questions are discussed at some length in [1, Section 4.3].

Sugihara [2] has investigated approximating individual hyperplanes in n-dimensional space. He formulates the problem as follows. Given $n + 1$ positive integers $Q_1, \cdots, Q_{n+1}$, the hyperplane

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + a_{n+1} = 0$$

is approximated by the hyperplane

$$b_1x_1 + b_2x_2 + \cdots + b_nx_n + b_{n+1} = 0,$$

where $a_1, a_2, \cdots, a_{n+1}$ are real numbers and $b_1, b_2, \cdots, b_{n+1}$ are integers such that $|b_i| \leq Q_i$. The Q_i bound the precision of the coefficients of the approximation. The approximation is considered good if b_i/b_j is close to a_i/a_j, for all i and j. Since finding an appropriate set of b_i values is very difficult, Sugihara considers the following problem. Let k be the integer such that the ratio $|a_k|/Q_k$ is maximum. Dividing the original equation by a_k, he obtains the equation

$$w_1x_1 + w_2x_2 + \cdots + w_nx_n + w_{n+1} = 0,$$

where $w_i = a_i/a_k$. He then considers the more tractable approximation problem of finding good rational approximations p_i/q to the w_i, for some integer $q \leq Q_k$. If the rational approximations are chosen such that $|p_i/q| \leq w_i$, then the equation

$$p_1x_1 + p_2x_2 + \cdots + p_nx_n + p_{n+1} = 0$$

approximates the original hyperplane and satisfies the bounds on the precision of the coefficients. In order to generate a *good* approximation, Sugihara proposes several heuristic methods for finding a set of integers $\{q, p_1, \cdots, p_{n+1}\}$ that minimizes

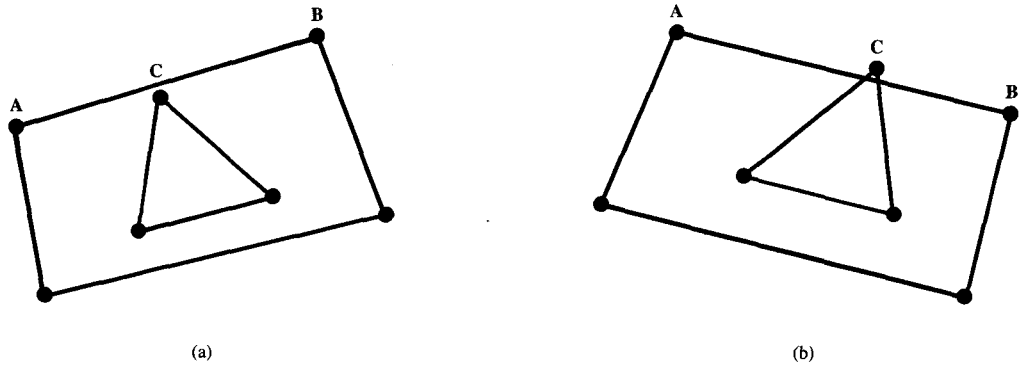
$$\max_i \left| w_i - \frac{p_i}{q} \right| .$$

He evaluates these approximation methods by applying them to a large number of randomly generated lines. Although each individual line is approximated well, nothing constrains the ensemble of approximations to reproduce more global structure (e.g., on which side of one line the intersection of two other lines falls). This is apparent in the pictures shown in [2, Figure 2], in which the rounding process clearly changes the combinatorial structure of the set of lines.

Rounding lines is much more difficult if the rounding must preserve some global structure imposed on the set of lines. For example, **Figure 1** shows a quadrilateral and a triangle before and after a rotation and rounding of individual lines. The rounding has clearly altered the global structure of the set of lines: Before rotation and rounding, point **C** is contained inside the quadrilateral; afterward, **C** is outside the quadrilateral. Such a change could cause failure of an algorithm that assumed that the triangle is surrounded by the quadrilateral, which might be the case in a solid modeler with the figure representing a quadrilateral face of a polyhedron with a triangular hole.

Some of the global structure of geometric objects can be preserved by using the *object reconstruction* approach [1]: If the structure of the set of geometric objects is defined in terms of operations on geometric primitives (e.g., lines or planes), then the primitives can be rounded individually and the structure reimposed. For example, suppose that a simple polygon (a polygon is simple if nonconsecutive edges do not share points) is represented by a constructive solid geometry (CSG) tree of half-planes, that is, as a Boolean combination of half-planes. (Any simple polygon can be represented in this manner [3].) The polygon can then be rounded by rounding each half-plane boundary (line) individually and then re-evaluating the CSG tree to obtain the rounded polygon. Although this sequence ensures that the rounded polygon is also a simple polygon, it does not necessarily preserve other aspects of the polygon's structure, such as the number of sides.

Another approach to avoiding precision growth in a sequence of operations is to compose the sequence of operations, round the composed operation, and then apply the rounded, composed operation. Again, consider rotations as an example. Let $\text{rat}(A)$ denote a rational rotation that approximates the rotation A. If we are to compute an approximation to $A_rA_{r-1}A_{r-2} \cdots A_1\mathbf{p}$, where $\mathbf{p}$ is a point, we should use $\text{rat}(A_rA_{r-1}A_{r-2} \cdots A_1)\mathbf{p}$ instead of $\text{rat}(A_r)\text{rat}(A_{r-1})\text{rat}(A_{r-2}) \cdots \text{rat}(A_1)\mathbf{p}$, thus



(a) (b)

Figure 1

Quadrilateral and triangle (a) before and (b) after finite-precision rotation and rounding.

reducing the precision required from $O(rP)$ to $O(P)$. This is not always possible. For example, if **p** and **q** are vertices of different polygons, we might apply rotation A to **p** and rotation B to **q**. After taking the union of the two polygons, we then apply rotation C to it. Replacing $\text{rat}(C)\text{rat}(A)\mathbf{p}$ and $\text{rat}(C)\text{rat}(B)\mathbf{q}$ with $\text{rat}(CA)\mathbf{p}$ and $\text{rat}(CB)\mathbf{q}$ may change the structure of the resulting polygon.

All three of the approaches we have discussed avoid precision growth by rounding, but provide no guarantees that any notion of structure among a set of geometric objects is preserved. In this paper, we consider the problem of rounding sets of polygons or polyhedra while preserving a certain notion of combinatorial structure of the set (defined precisely below, but, roughly, the nesting relationship among the polygons in the set). Informally, the problem can be formulated as follows. Assume that a polygon is represented by a set of real numbers, called *coordinates*, and let $\mathcal{S}$ be a set of polygons. We would like to round the coordinates of the polygons of $\mathcal{S}$ to obtain a set of polygons $\mathcal{S}'$ that has the same combinatorial structure as $\mathcal{S}$, but with coordinates that can be represented with P bits of precision and are close to the corresponding coordinates of $\mathcal{S}$. By close, we mean that each rounded coordinate must be accurate to $P - K$ bits, for some integer K, $0 \leq K < P$ (that is, the magnitude of the difference between a coordinate and its corresponding rounded coordinate must be less than 2^{K-P}). If it is possible to do this, we call $\mathcal{S}'$ a (P, K)-approximation to $\mathcal{S}$.

Now let us discuss a practical scenario in which the ability to find (P, K)-approximations to sets of polygons would be useful. Suppose we start with a set of P-bit polygons (i.e., polygons with P-bit coordinates) and apply some operation to them (possibly, but not necessarily, a rigid motion), with the result being a set of M-bit polygons, where $M > P$. We then wish to round the set of output polygons back to a set of P-bit polygons while preserving their combinatorial structure. Clearly, we cannot merely truncate coordinates to P bits and expect to preserve the combinatorial structure. Therefore, we are willing to sacrifice a small amount of geometric accuracy in exchange for preserving the structure. Namely, we seek a set of P-bit polygons with the most significant $P - K$ bits correct and the same combinatorial structure—in other words, a (P, K)-approximation to the set of M-bit polygons.

Our main result is that determining the existence of a (P, K)-approximation is NP-complete. This result is stated formally and its applicability and relation to other work discussed in Section 2. The result is proved in Section 3 by reducing the problem of three-coloring a planar graph of degree four to the problem of finding a (P, K)-approximation to a set of simple polygons. Section 4 generalizes from two-dimensional polygons to three-dimensional polyhedra. Since our result is that simultaneously rounding and preserving combinatorial structure is hard, in Section 5 we discuss the use of heuristics and polynomial-time techniques for rounding while preserving *nearly* the same combinatorial structure.

2. Rounding is hard

In this section we show that the problem of determining whether a collection of simple polygons has a

(P, K)-approximation is NP-complete. This result generalizes easily to polyhedra. We first define the problem precisely and state the result and then discuss the applicability of the result in practice and its relation to other work in the area of coordinate representations.

• *Combinatorial structure*

We say that a polygon is *bounded* by lines $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \cdots, \mathbf{L}_k$, $k \geq 3$, if it has vertices $\mathbf{L}_1 \cap \mathbf{L}_2, \mathbf{L}_2 \cap \mathbf{L}_3, \cdots, \mathbf{L}_{k-1} \cap \mathbf{L}_k, \mathbf{L}_k \cap \mathbf{L}_1$ in counterclockwise order. A polygon is *simple* if it has no self-intersections. A set of simple polygons is *simple* if no two members intersect. We define the *combinatorial structure* of a simple set of simple polygons to be (a) the number of members, (b) the number of sides of each member, and (c) the nesting relationship among the members. The nesting relationship can be represented by a forest of trees in which a polygon P is a descendant of a polygon Q if and only if P is nested in Q.

Two sets of polygons, S_1 and S_2, have the same combinatorial structure if and only if both are simple sets of simple polygons, there is a one-to-one correspondence between the polygons of S_1 and the polygons of S_2, corresponding polygons are bounded by the same number of lines, and the forests are isomorphic under the correspondence. For example, the two sets of simple polygons shown in Figures 1(a) and 1(b) do not have the same combinatorial structure because (a) is a simple set of simple polygons and (b) is not (the quadrilateral and triangle intersect).

Henceforth, we use "polygon" to mean simple polygon and "set of polygons" to mean simple set of simple polygons. Also, we denote by n the total number of bounding lines of all the members of a set of polygons.

• *(P, K)-approximation*

Assume that P and K are positive integers such that $0 \leq K < P$, and define $\epsilon = 2^{-P}$ and $\eta = 2^{K-P}$. The set $\mathbf{BF}_P$ of *P-bit binary fractions* (or more simply *P-bit numbers*) is defined as

$$\mathbf{BF}_P = \{2^{-P} q \mid q \text{ is an integer and } -2^P \leq q \leq 2^P\}.$$

This is the set of numbers in the range $[-1, 1]$ that terminate within P bits of the "binary point." A *line* is defined by the equation $ax + by = c$, where a, b, and c are real. A *P-bit line* has coefficients $a, b, c \in \mathbf{BF}_P$.

A *(P, K)-approximation* to a real number a is a P-bit number a' satisfying $|a' - a| < \eta$. It follows that a and a' have the same $P - K$ most significant bits. A line $a'x + b'y = c'$ is a *(P, K)-approximation* to a line $ax + by = c$ if a', b', and c' are (P, K)-approximations to a, b, and c, respectively. A simple polygon P' is a *(P, K)-approximation* to a simple polygon P if it has the same number of lines and if each line of P' is a (P, K)-approximation to the corresponding line in P. A set of simple polygons S' is a *(P, K)-approximation* to a set S if S' has the same combinatorial structure as S and if each simple polygon of S' is a (P, K)-approximation to the corresponding polygon of S.

• *NP-completeness*

Theorem 1 The language of simple sets of simple polygons with at least one (P, K)-approximation is NP-complete.

This theorem is proved in Section 3 and generalized to polyhedra in Section 4.

• *Applications of Theorem 1*

It is important to understand what Theorem 1 implies with regard to practical applications. Suppose we wish to rotate (or apply some other Euclidean transformation to) a polygon or polyhedron. We start with a P-bit polygon, rotate it exactly using rational arithmetic, and then round the higher-precision result to a nearby (P, K)-approximation. The theorem does not necessarily imply that this rounding step will be difficult because, in this situation, we are restricting the input to those sets of simple polygons *that result from the Euclidean transformation of P-bit polygons*. This may be an easier problem than the more general situation covered in the theorem, in which the set of input polygons is not restricted. Nevertheless, the theorem does imply that any procedure we might be able to devise for rounding restricted sets of polygons (e.g., those that result from Euclidean transformation of P-bit polygons) must exploit properties of the restriction. There can be no general polynomial time procedure for rounding polygons (unless, of course, the $P \neq NP$ conjecture is false).

Unfortunately, a transformation-dependent rounding procedure may be hard to find, for the following reason. The distance between closest *representable* points varies greatly from one part of the plane to another (as we show below), where we define a point to be representable if it is the intersection of two P-bit lines. Thus, even a pure translation may cause problems if it moves a cluster of vertices with representable locations to a region of the plane of low "density." As is shown in Section 3, the proof of Theorem 1 depends on the existence of objects that expand greatly when approximated. Such objects may arise from a sequence of transformations and set operations on polygons that creates objects that have P-bit approximations but which expand if a subsequent translation moves them to lower-density regions of the plane.

To see how much the distance between closest representable points can vary, consider first the origin. Only lines of the form $ax + by = 0$ pass through the

origin. Every other P-bit line $ax + by = c$ with $c \neq 0$ lies at least

$$\frac{|c|}{\sqrt{a^2 + b^2}} \geq \frac{2^{-P}}{\sqrt{2}}$$

distant. Therefore, the origin is the only representable point in a circle of radius $2^{-P}\sqrt{2}$.

Now consider nine integers, A_i, B_i, C_i, $i = 1, 2, 3$, with magnitude approximately equal to but not exceeding 2^P such that

$$\Delta_{123} = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 1.$$

Such values exist, since we can assign the points (A_i, B_i, C_i), $i = 1, 2, 3$, to be any basis for the integer lattice in three dimensions [4, Section 13.9], and there are an infinite number of such bases. It can be shown that the three P-bit lines $2^{-P}A_i x + 2^{-P}B_i y = 2^{-P}C_i$, $i = 1, 2, 3$, bound a triangle of diameter roughly 2^{-3P}. In particular, the distance from the intersection of the first two lines ($i = 1, 2$) to the third line ($i = 3$) is

$$2^{-P}\Delta_{123}\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}^{-1}.$$

When A_1, B_1, A_2, B_2 are assumed to have the appropriate signs, this distance is close to 2^{-3P}. Thus, the minimum distance between representable points can vary from 2^{-3P} to $2^{-P}/\sqrt{2}$. There is clearly considerable unevenness in the distribution of representable points.

• *Relation to other work*

Before proceeding to the proof, let us compare this result to other work. Mnev [5] and Goodman, Pollack, and Sturmfels [6] examine the problem of finding coordinate representations for order types. An *order type* is an assignment of an orientation for every triple of points **A**, **B**, and **C**: The orientation is *positive* if triangle **ABC** is counterclockwise, *negative* if **ABC** is clockwise, and *zero* if **A**, **B**, and **C** are collinear. Mnev shows that determining whether there exists a set of points with a given order type is equivalent to the existential theory of the real numbers. Goodman et al. show that even if such a set of points exists, a coordinate representation may require storage exponential in the number of points.

In contrast to the work of Mnev and of Goodman et al., our work considers lines instead of points, uses the notion of combinatorial structure instead of order type, and seeks a (P, K)-approximation to some given representation instead of seeking any coordinate representation of a combinatorial structure. These differences are motivated by practical considerations.

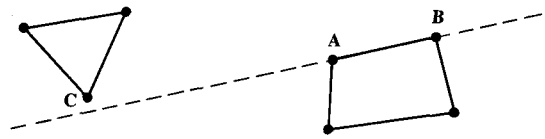


Figure 2

Order type vs. combinatorial structure.

We consider lines instead of points because we wish to generalize to polyhedra, which are commonly represented by the plane equations of their faces. By well-known duality relationships [7], there is a direct connection between results on lines and results on points, so this difference is of little consequence.

On the other hand, the concept of combinatorial structure of a set of simple polygons is less stringent than the order type of points. For example, in **Figure 2**, point **C** could be moved to the other side of line **AB** without changing the combinatorial structure. The order type would be changed in this case because the orientation of triangle **ABC** would change from positive to negative. We believe that our definition of combinatorial equivalence is more natural for most practical problems.

One way to pose the problem we are considering is the following: Given a combinatorial structure for a set of simple polygons, find a coordinate representation for the polygons subject to the constraint that the coordinate representation is a (P, K)-approximation to some specified representation. This constraint is imposed because in practice rounding must both preserve combinatorial structure and incorporate some notion of "nearness" to the unrounded object. Indeed, rounding would be easy without this constraint, for it can easily be shown that a combinatorial structure has a coordinate representation if and only if each polygon appears exactly once in the forest. Furthermore, an $O(\log n)$-bit coordinate representation can be found in linear time, where n is the total number of sides of all the polygons.

Finally, we note that, as in the results of Mnev and of Goodman et al., determining the existence of a (P, K)-approximation is difficult in two or more dimensions and easy in one dimension. The one-dimensional problem, generating a (P, K)-approximation to a forest of nested closed intervals, can be solved in linear time.

3. Proof

To prove Theorem 1, we must show that the problem of finding a (P, K)-approximation to a set of simple polygons is contained in NP and that it is NP-hard. Establishing that the problem is in NP is easy. Given a set of simple polygons with n distinct lines, there are up to 2^{3nK} possible (P, K)-approximations to the set, each of size $O(nP)$ bits. To prove that a particular set of simple polygons has a (P, K)-approximation, we nondeterministically generate one of the approximations and check it in polynomial time. To check an approximation, we first verify in time $O(n)$ that each coordinate is indeed a (P, K)-approximation of the corresponding coordinate in the original set of polygons. Next, in time $O(n \log n)$ we verify that the set of polygons is simple and determine the nesting relationships among the polygons in the approximation (using a standard sweep-line algorithm [8]). Since the correspondence between the lines in the original set of simple polygons and the lines in a (P, K)-approximation to it is known, it is then only necessary to test that the two forests are isomorphic. This can be done in $O(n)$ time. Therefore, the problem is in NP.

We show that the problem is NP-hard, thus completing the proof of Theorem 1, by reducing the NP-complete problem of three-coloring planar graphs having no vertex degree greater than four [9, Section A1.1] to the problem of finding a (P, K)-approximation to a set of simple polygons. For any such graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the reduction consists of two steps:

1. Embed $\mathcal{G}$ in an orthogonal grid, with graph vertices placed at grid vertices and graph edges following grid edges. Using the algorithm of Tamassia and Tollis [10], we can embed $\mathcal{G}$ in a cV by cV grid in $O(V)$ time, where c is a constant and $V = |\mathcal{V}|$.
2. Construct, in time polynomial in V, a set of simple polygons S such that there exists a (P, K)-approximation S' of S if and only $\mathcal{G}$ can be three-colored.

The set of polygons constructed in the second step is the union of subsets of polygons, each subset being drawn from a small library of basic *component* types. To help explain the construction, we group the component types into "levels" in a manner analogous to the way components are organized on a computer chip. We use four levels, with a number of component types defined at each level:

- **DEVICE**: sponge, slider, adder.
- **SSI** (small-scale integration): transmission line, splitter-inverter, AND gate.
- **MSI** (medium-scale integration): transmission-line crossing.
- **LSI** (large-scale integration): graph.

The purpose of embedding $\mathcal{G}$ in a grid is to simplify the "chip wiring."

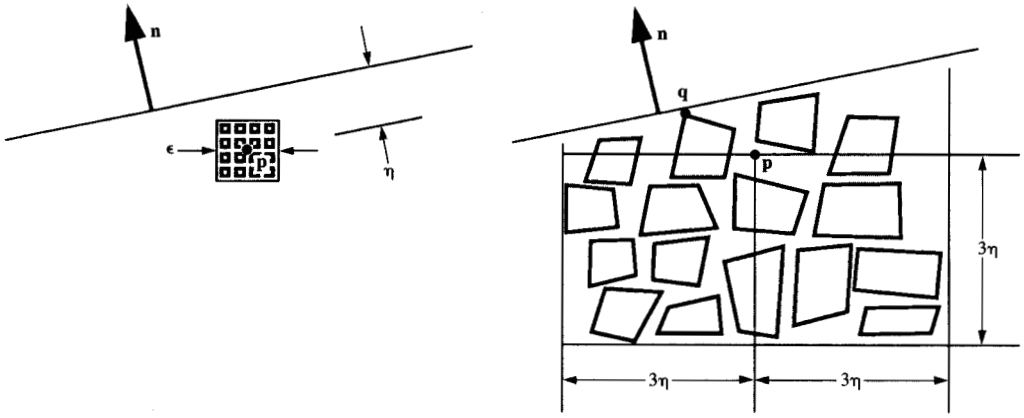
A component, at any level, is characterized by its *terminals* and its input-output behavior with respect to its terminals. Terminals are geometric points that enable us to reason about the interaction of the components but do not belong to the set of simple polygons. We say that a (P, K)-approximation to a component *covers* a terminal if that terminal is in the interior of one of the polygons of that component; otherwise, that terminal is *uncovered.* The input-output behavior of a component is the list of legal assignments of the values "covered" or "uncovered" to each of its terminals. A particular assignment is legal if and only if there exists some (P, K)-approximation to the component that covers the terminals according to the assigned values.

Components may be placed so that a terminal of one coincides with a terminal of another. Since components are not allowed to intersect (all sets of polygons are simple), in any (P, K)-approximation to two components, one at most can cover a common terminal. Components come in two versions: horizontal and vertical. We describe below the construction of horizontal components; the vertical versions can be obtained by a 90° rotation. In our constructions, we place components so that each terminal can be covered by at most one horizontal component and at most one vertical component. For each (P, K)-approximation, each terminal has the value 0, h, or v, if it is uncovered, covered by a horizontal component, or covered by a vertical component, respectively. The coloring of the graph $\mathcal{G}$ we are simulating is represented by the values of certain terminals. We describe the construction from the bottom up, presenting at each level the characteristics and construction of the components at that level.

As a form of shorthand and to make the input-output behavior of a component more intuitive, we introduce the notion of *pushing* on a terminal. We say that we *push* on a terminal of a particular component when we assign the value *uncovered* to that terminal. That is, we restrict our consideration to all legal assignments in which the value of that terminal is "uncovered." We say that that component *pushes* on another terminal if in all members of that restricted set of assignments, the value of that terminal must be "covered."

- *Device level*

At the lowest level, the graph implementation consists of three types of components (sets of simple polygons): sponges, sliders, and adders.



(a) (b)

Figure 3

(a) Sponge and (b) its (P, K)-approximation.

Sponges

One can think of a sponge as a tiny object that expands when approximated (see **Figure 3**). Let $\mathbf{p}$ be a point and $\mathbf{n}$ a nonzero vector such that $|\mathbf{n}_y| \geq |\mathbf{n}_x|$. A *horizontal* $(\mathbf{p}, \mathbf{n})$-*sponge* is a set of simple polygons with the following properties:

- The sponge's polygons lie in a square of side ϵ aligned with the coordinate axes and centered at $\mathbf{p}$. (As stated in the subsection on (P, K)-approximation, $\epsilon \equiv 2^{-P}$ and $\eta \equiv 2^{K-P}$.)
- In every (P, K)-approximation to the sponge, there exists a polygon vertex $\mathbf{q}$ at least $[(\eta - \epsilon)|\mathbf{n}_y|/|\mathbf{n}|]$ distant from $\mathbf{p}$ along the direction of the vector $\mathbf{n}$. Expressed analytically, $(\mathbf{q} - \mathbf{p}) \cdot \mathbf{n} \geq (\eta - \epsilon)|\mathbf{n}_y|$.
- Every (P, K)-approximation to the sponge lies in a square of side 6η aligned with the coordinate axes and centered at $\mathbf{p}$.
- There exists at least one (P, K)-approximation to the sponge that extends no farther than $[(\eta + \epsilon)|\mathbf{n}_y|/|\mathbf{n}|]$ from $\mathbf{p}$ in the direction $\mathbf{n}$. That is, for all vertices $\mathbf{q}$ in this approximation, $(\mathbf{q} - \mathbf{p}) \cdot \mathbf{n} < (\eta + \epsilon)|\mathbf{n}_y|$.

A *vertical* $(\mathbf{p}, \mathbf{n})$-*sponge* is defined analogously for $|\mathbf{n}_x| \geq |\mathbf{n}_y|$. When we refer to a $(\mathbf{p}, \mathbf{n})$-*sponge*, we mean either a horizontal or vertical $(\mathbf{p}, \mathbf{n})$-sponge, as appropriate.

A $(\mathbf{p}, \mathbf{n})$-sponge consists of a set of squares with sides parallel to the coordinate axes. We now give an algorithm for constructing a sponge for the case $\mathbf{n}_y \geq \mathbf{n}_x \geq 0$; constructions for the other cases can be obtained straightforwardly by reflections and rotations by multiples of 90°. Let $\xi = 2^{-(P+3K+4)}$ and define the sets of horizontal and vertical lines:

$$\mathbf{hl}_i: y = \mathbf{p}_y + i\xi, \qquad i = 0, 1, \cdots, 2^{3K+3};$$

$$\mathbf{vl}_j: x = \mathbf{p}_x + j\xi, \qquad j = 0, 1, \cdots, 2^{3K+3}.$$

Because ξ is so small, the point $\mathbf{p}$ lies within ϵ of each of these lines, each line $\mathbf{hl}_i$ has the same set of (P, K)-approximations as the line $y = \mathbf{p}_y$, and each line $\mathbf{vl}_j$ has the same set of (P, K)-approximations as the line $x = \mathbf{p}_x$. Since the K least significant bits of all three coefficients of any line can be changed, $x = \mathbf{p}_x$ and $y = \mathbf{p}_y$ each have 2^{3K} (P, K)-approximations.

The following algorithm generates a $(\mathbf{p}, \mathbf{n})$-sponge:

```
sponge-set = ∅
for i = 0 to 2^(3K+1)
    for j = 0 to 2^(3K+1)
        /* (η − ε)-invariant */
        add the square bounded by hl_2i, vl_2j, hl_2i+1,
          vl_2j+1 to sponge-set.
        /* (η + ε)-invariant */
        if sponge-set is a (p, n)-sponge, return it.
```

The set *sponge-set* can contain no more than 2^{6K+2} squares. Each square has no more than 2^{12K} (P, K)-approximations (because each side has no more than 2^{3K} approximations). Therefore, there are no more

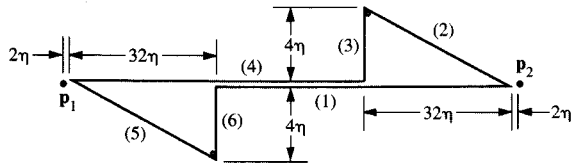


Figure 4

Slider.

than $(2^{12K})^{(2^{6K+2})}$ (P, K)-approximations to the squares in *sponge-set*. Since K is a constant, we can test in (a somewhat daunting) constant time whether the current *sponge-set* is a $(\mathbf{p}, \mathbf{n})$-sponge.

We prove correctness using two invariants:

- $(\eta - \epsilon)$-invariant: There exists at least one (P, K)-approximation to *sponge-set* that does not extend farther than $[(\eta - \epsilon)|\mathbf{n}_y|/|\mathbf{n}|]$ in the direction of $\mathbf{n}$ (for all vertices $\mathbf{q}$ in the approximation, $(\mathbf{q} - \mathbf{p}) \cdot \mathbf{n} < (\eta - \epsilon)|\mathbf{n}_y|$).
- $(\eta + \epsilon)$-invariant: There exists at least one (P, K)-approximation to *sponge-set* that extends at least $[(\eta - \epsilon)|\mathbf{n}_y|/|\mathbf{n}|]$, but no farther than $[(\eta + \epsilon)|\mathbf{n}_y|/|\mathbf{n}|]$, in the direction of $\mathbf{n}$.

The proof of correctness is as follows.

- The first invariant clearly holds for the empty set.
- If the first invariant holds for *sponge-set*, then the second invariant holds when the square bounded by $\mathbf{hl}_{2i}$, $\mathbf{vl}_{2j}$, $\mathbf{hl}_{2i+1}$, $\mathbf{vl}_{2j+1}$ is added to *sponge-set*. The (P, K)-approximation implied by the first invariant does not intersect the square bounded by the lines $x = \mathbf{p}_x$, $y = \mathbf{p}_y + \eta - \epsilon$, $x = \mathbf{p}_x + \epsilon$, and $y = \mathbf{p}_y + \eta$, because this square lies at least $[(\eta - \epsilon)|\mathbf{n}_y|/|\mathbf{n}|]$ in the direction of $\mathbf{n}$ and no farther than $[(\eta + \epsilon)|\mathbf{n}_y|/|\mathbf{n}|]$. But this square is a (P, K)-approximation to the square $\mathbf{hl}_{2i}$, $\mathbf{vl}_{2j}$, $\mathbf{hl}_{2i+1}$, $\mathbf{vl}_{2j+1}$, so the second invariant is satisfied.
- If the *sponge-set* satisfies the second invariant but fails to be a $(\mathbf{p}, \mathbf{n})$-sponge, it must be true that *not all* (P, K)-approximations have a vertex lying $[(\eta - \epsilon)|\mathbf{n}_y|/|\mathbf{n}|]$ in the direction of $\mathbf{n}$. In other words, the first invariant holds.

The proof of termination is as follows. Each time around the inner loop, the second invariant implies that there exists at least one (P, K)-approximation to *sponge-set*. If the algorithm terminates without concluding that *sponge-set* is a $(\mathbf{p}, \mathbf{n})$-sponge, then there are 2^{6K+2} squares in *sponge-set*. But this set is too large to have a valid (P, K)-approximation. To see why, consider that there are only 2^{3K} possible approximations each for the set of horizontal and for the set of vertical lines. Therefore, only 2^{6K} points in the plane are eligible to be vertices of polygons in the (P, K)-approximation. A set of 2^{6K+2} disjoint polygons needs more vertices than that.

It remains for us to verify that each point of every (P, K)-approximation to the sponge lies within the square of side 6η centered at $\mathbf{p}$. Let $ax + by = c$ be a line, where $a^2 + b^2 = 1$. Let $a'x + b'y = c'$ be a (P, K)-approximation to that line, and let (X, Y) be a point on the latter line inside the unit square centered at the origin. How far can this point lie from the original line? We know that

$$|aX + bY - c| \le |a'X + b'Y - c'| + |(a - a')X| + |(b - b')Y| + |(c - c')| \le 0 + \eta + \eta + \eta = 3\eta.$$

Since each vertex in the (P, K)-approximation to a $(\mathbf{p}, \mathbf{n})$-*sponge* lies on a (P, K)-approximation to a vertical line and to a horizontal line through $\mathbf{p}$, each vertex lies within the $6\eta \times 6\eta$ square centered at $\mathbf{p}$.

Sliders

This section defines a *slider*, which is a component that can "transmit information" from a terminal $\mathbf{p}_1$ to another terminal $\mathbf{p}_2$. Sliders are used below to construct *transmission lines*, which transmit colors among vertices. As depicted in **Figure 4** (which is foreshortened),* a $\mathbf{p}_1\mathbf{p}_2$-*slider* is a hexagon with two sponges inside. The hexagon has the form of a long shaft with a right-triangular "speartip" at each end. The length of the shaft is arbitrary, and it is chosen so that the hexagon has the following properties:

- The slider covers neither terminal $\mathbf{p}_1$ nor $\mathbf{p}_2$ and lies at least η distant from each terminal.
- Every (P, K)-approximation to the slider covers at least one of the terminals.
- There exists at least one (P, K)-approximation that covers $\mathbf{p}_1$ but not $\mathbf{p}_2$, and at least one that covers $\mathbf{p}_2$ but not $\mathbf{p}_1$.

In other words, if we push on one terminal, the slider pushes on the other.

We may think of the slider as a computational device that amplifies the expansion of the slider's sponges under

* In this and all other figures, dimensions are considered to be accurate within a small multiple of ϵ. Thus, $3\eta \pm 4\epsilon$ is labeled as 3η. The correct line equations are given in the text.

(P, K)-approximation. Amplification is achieved through the use of speartips with a high aspect ratio, one part in eight in this case (Figure 4 is foreshortened). Displacing by η either the hypotenuse or the longer leg parallel to itself has the effect of displacing their intersection in the x-direction by 8η. Initially, the shorter legs have length about 4η, and the longer legs about 32η. In any (P, K)-approximation, the sponges force each hypotenuse to be displaced parallel to itself by about η, causing the spearheads to extend about an additional 8η to cover the terminals, as shown in **Figure 5**. However, the shaft can be displaced upward by η, restoring the right spearhead to its original length and uncovering $\mathbf{p}_2$, as shown in **Figure 6**, or the shaft can be displaced downward to uncover $\mathbf{p}_1$. Because of the length chosen for the shaft, it is not possible to uncover both terminals at the same time.

Let us first define the slider for $\mathbf{p}_1 = (-\lambda, 0)$ and $\mathbf{p}_2 = (\lambda, 0)$; generalization is straightforward. For simplicity, we assume that λ is an integral multiple of η. Since there must be room for both sponges to expand simultaneously in the horizontal direction by 3η, the tip length is about 32η, and the distance from the tip point to terminal must be at least η. Thus, λ must be at least 36η. To allow for the space necessary to position the sponges properly, we require that $\lambda \geq 38\eta$. The six lines, as labeled in Figure 4, are defined by the following equations:

line

(1) $0x + -1y = \epsilon$,

(2) $\frac{1}{8}x + 1y = \frac{\lambda - 2\eta}{8} + \epsilon$,

(3) $-1x + 0y = -\lambda + 34\eta + \epsilon$,

(4) $0x + 1y = \epsilon$,

(5) $-\frac{1}{8}x + -1y = \frac{\lambda - 2\eta}{8} + \epsilon$,

(6) $1x + 0y = -\lambda + 34\eta + \epsilon$.

The sponge in the uppermost corner of the hexagon is a $((\lambda - 34\eta, 4\eta), \mathbf{n})$-sponge, and the one in the lowermost corner is a $((-\lambda + 34\eta, -4\eta), -\mathbf{n})$-sponge, with $\mathbf{n} = (1/8, 1)$.

Assuming that the lines and their (P, K)-approximations are parallel, we can easily verify that the slider has the desired properties. The sponges expand by η (actually $\sqrt{64/65}\eta$, the amount required to shift line (2) *vertically* by a distance η) in the desired direction and expand by no more than 3η horizontally and vertically. Because the spearheads have height 4η initially, the vertical expansion does not interfere with the motion of the shaft. Without loss of generality, we always choose

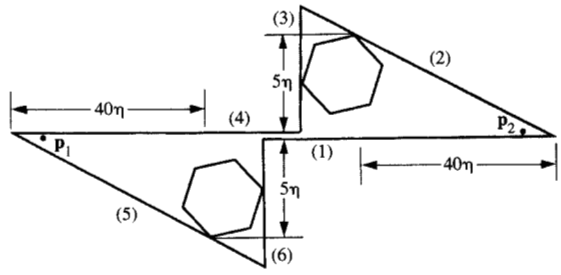


Figure 5

One possible (P, K)-approximation to a slider.

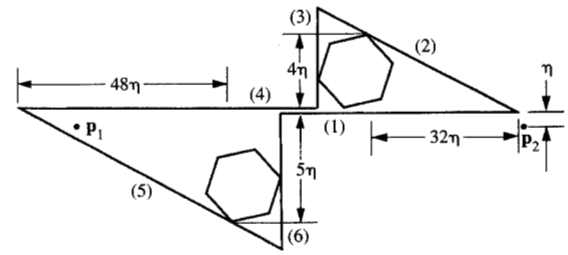


Figure 6

Another (P, K)-approximation to a slider.

the shaft to have length greater than 6η; thus, the horizontal displacement cannot eliminate the shaft.

We claim that any change in orientation of the lines has negligible effect for the length of sliders we consider, a few hundred η at most. The magnitude of the slope of line (2) is (1/8)/1. Approximation can increase the slope magnitude to $((1/8) + \eta)/(1 - \eta)$ at most, which is $(1/8)(1 + 9\eta + 9\eta^2 + \cdots)$. When terms higher than first order are neglected, this change in slope corresponds to a change in orientation of $(9/8)\eta = 1.125\eta$ radians. Similar reasoning shows that the change in orientation of line (1) is bounded by η radians. Thus, for $\lambda = 1000\eta$, a change in the orientation of lines (1) and (2) moves the intersection of lines (1) and (2) by no more than $2.123\eta\lambda = 2125\eta^2$. For $\eta = 2^{-20}$, this error amounts to about 0.002η, which is negligible, as claimed.

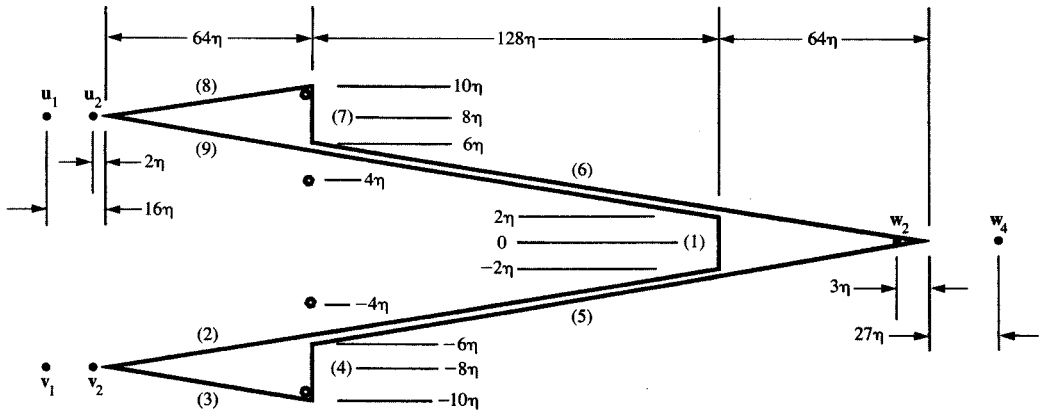


Figure 7

Adder.

In general, a slider can be positioned anywhere in the plane and oriented either horizontally or vertically. Translating the slider to be centered at a point (X, Y) simply requires changing each of its lines $ax + by = c$ to $ax + by = c + aX + bY$.

Adder

An *adder*, as depicted in **Figure 7** (foreshortened), has four *input* terminals $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_1$, and $\mathbf{v}_2$, and two *output* terminals $\mathbf{w}_2$ and $\mathbf{w}_4$. For a (P, K)-approximation to the adder, we say that its first input is zero if both $\mathbf{u}_1$ and $\mathbf{u}_2$ are covered, one if only $\mathbf{u}_2$ is covered, and two if neither $\mathbf{u}_1$ nor $\mathbf{u}_2$ is covered. The second input is defined analogously for $\mathbf{v}_1$ and $\mathbf{v}_2$. The adder has the property that if the sum of the inputs is at least 2, $\mathbf{w}_2$ is covered, and if the sum is at least 4, $\mathbf{w}_4$ is also covered.

An adder is implemented by a nine-sided polygon with four sponges. Let us first consider an adder centered at the origin. It is bounded by the following lines:

line

(1) $-1x + 0y = -64\eta$,

(2) $-\frac{1}{32}x + 1y = -4\eta + \epsilon$,

(3) $-\frac{1}{32}x + -1y = 12\eta + \epsilon$,

(4) $1x + 0y = -64\eta + \epsilon$,

(5) $\frac{1}{32}x + -1y = 4\eta + \epsilon$,

(6) $\frac{1}{32}x + 1y = 4\eta + \epsilon$,

(7) $1x + 0y = -64\eta + \epsilon$,

(8) $-\frac{1}{32}x + 1y = 12\eta + \epsilon$,

(9) $-\frac{1}{32}x + -1y = -4\eta + \epsilon$.

Wedged in the uppermost and lowermost corners are two sponges—a $((-64\eta, 10\eta), (-1/32, 1))$-sponge and a $((-64\eta, -10\eta), (-1/32, -1))$-sponge, respectively—which displace lines (8) and (3) outward and parallel to themselves by about η in each (P, K)-approximation. To keep lines (2), (5), (6), and (9) from moving more than η inward, we add two more sponges: a $((-64\eta, 4\eta), (1/32, 1))$-sponge and a $((-64\eta, -4\eta), (1/32, -1))$-sponge. Now, suppose that we push on $\mathbf{u}_2$. Since lines (6) and (9) are prevented from moving inward, they must move outward so that the intersection of (the approximations of) lines (8) and (9) lies to the right of $\mathbf{u}_2$. This then causes the intersection of (the approximations of) lines (5) and (6) to move to the right of $\mathbf{w}_2$, pushing on $\mathbf{w}_2$. By analogous reasoning, if we also push on $\mathbf{v}_2$, line (5) is also forced to move outward, moving the intersection of lines (5) and (6) still farther to the right, pushing on $\mathbf{w}_4$.

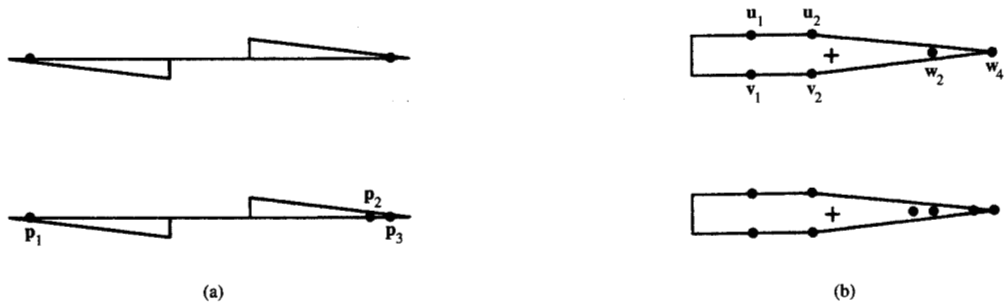


Figure 8

SSI symbols: (a) slider and (b) adder.

Similar arguments can be used to show that the desired behavior can be obtained for the other combinations of inputs.

To make the adder perform as described, the terminals must be placed appropriately. This requires consideration both of the geometry of the adder itself and of the way it will be integrated into higher-level "circuits." We see in the next section that $\mathbf{u}_2$ and $\mathbf{v}_2$ must be placed sufficiently far to the left of the intersection of lines (8) and (9) and the intersection of lines (2) and (3), respectively, so that they can be covered by slider tips; it suffices to shift $\mathbf{u}_2$ and $\mathbf{v}_2$ left by 2η. Terminals $\mathbf{u}_1$ and $\mathbf{v}_1$ are placed 16η to the left of the intersections of lines (8) and (9) and lines (2) and (3), respectively. To compensate for shifting $\mathbf{u}_2$ and $\mathbf{v}_2$, $\mathbf{w}_2$ is shifted 3η to the left of the intersection of lines (5) and (6). The coordinates of the control points are

$$\mathbf{u}_1 = (-144\eta, 8\eta),$$

$$\mathbf{u}_2 = (-130\eta, 8\eta),$$

$$\mathbf{v}_1 = (-144\eta, -8\eta),$$

$$\mathbf{v}_2 = (-130\eta, -8\eta),$$

$$\mathbf{w}_2 = (125\eta, 0),$$

$$\mathbf{w}_4 = (155\eta, 0).$$

Centering the adder at some point other than the origin is accomplished as with sliders.

- *SSI level*

The device-level components described in the preceding subsection are integrated into higher-level components by placing horizontal and vertical devices so that they share a common terminal. Recall that a terminal can be covered by at most one device at a time (because polygons are not permitted to intersect), and a terminal can have three logic values, denoted 0, h, or v, depending on whether no device, a horizontal device, or a vertical device is covering it.

The top row of **Figure 8** shows symbols used in SSI diagrams for sliders and adders. The output of each of these devices can be duplicated by placing extra terminals appropriately. This is illustrated on the bottom row of Figure 8: An extra terminal $\mathbf{p}_3$ is placed 3η to the right of $\mathbf{p}_2$ and an extra terminal is placed 3η to the left of each of the original control points $\mathbf{w}_2$ and $\mathbf{w}_4$. The specifications given in the subsections on adders and sliders are designed to permit these extra terminals.

Transmission lines

Figure 9(a) depicts a *transmission line*, which consists of a string of horizontal and vertical sliders sharing common terminals. If the value of $\mathbf{p}_1$ is 0, then $\mathbf{p}_2$ must have value h, $\mathbf{p}_3$ must have value v, $\mathbf{p}_4$ must have value h, and so on. Thus, asserting a zero value for $\mathbf{p}_1$ transmits information over the line. Transmission lines have no "tensile strength," however; if $\mathbf{p}_1$ is h, we can conclude nothing about the other values; for example, $\mathbf{p}_2$ can take on any of the three logic values. The MSI symbol for a transmission line is a sequence of alternating horizontal and vertical line segments, like the ones shown in **Figure 9(b)**.

AND gates

Figure 10 illustrates the SSI implementation of an *AND gate* and its MSI symbol. Pushing on $\mathbf{p}_1$ and $\mathbf{p}_2$ causes the gate to push on $\mathbf{p}_3$. In fact, the gate is symmetrical with respect to its three terminals: Pushing on any two causes the gate to push on the third. In other words, any

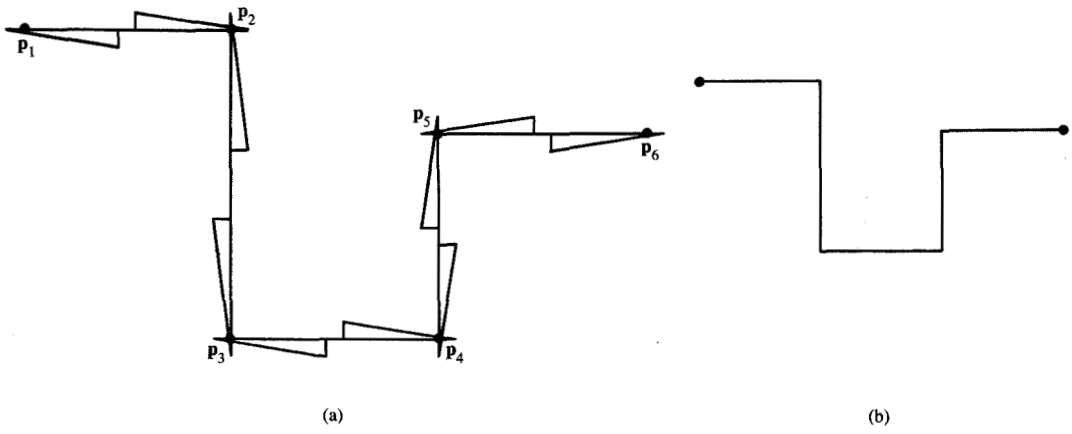


Figure 9

(a) SSI implementation of a transmission line and (b) its MSI symbol.

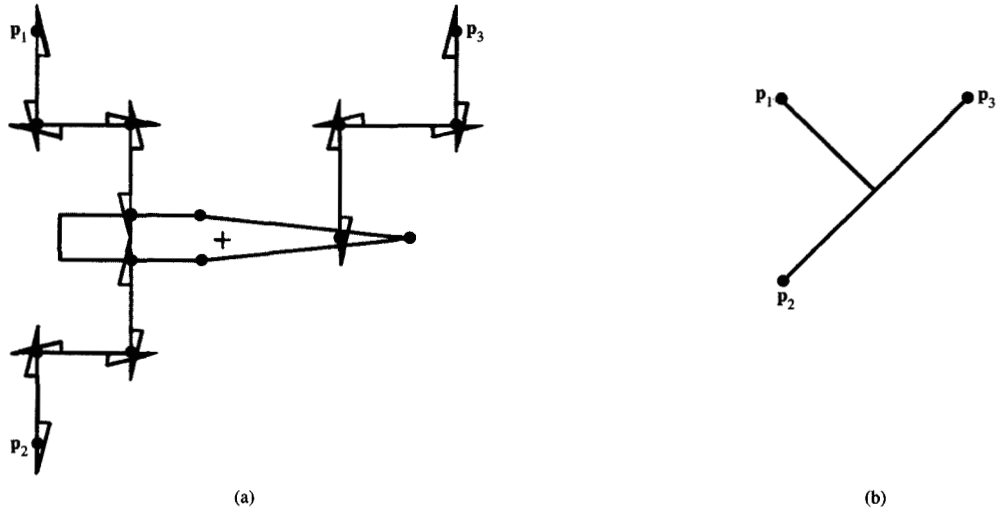


Figure 10

(a) SSI implementation of an AND gate and (b) its MSI symbol.

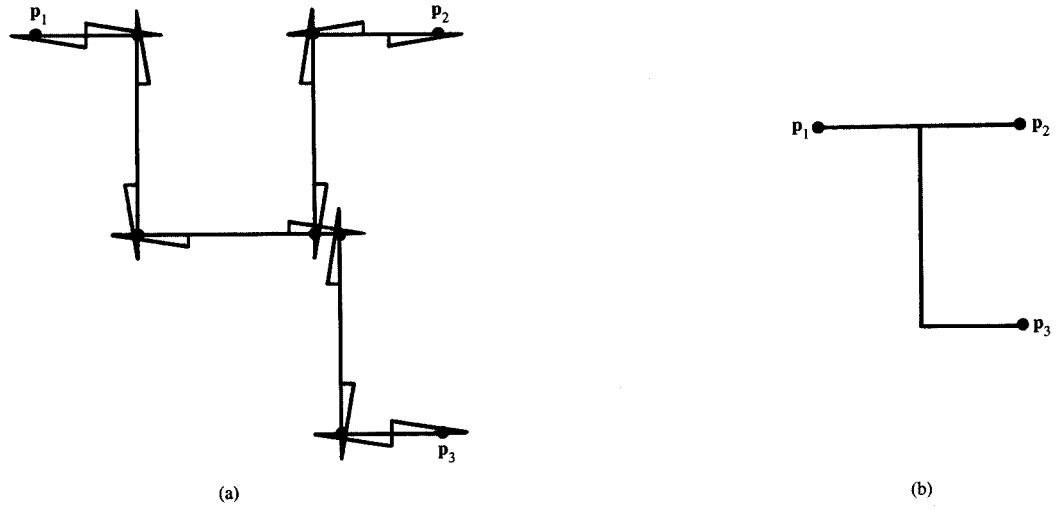
(P, K)-approximation of an AND gate must cover at least one and at most two of its terminals.

Splitter-inverters

Figure 11 illustrates the horizontal SSI implementation of a *splitter* and its MSI symbol. Pushing on $\mathbf{p}_1$ causes this component to push on $\mathbf{p}_2$ and $\mathbf{p}_3$, thus duplicating its input. If we think of $\mathbf{p}_2$ as the input, pushing on $\mathbf{p}_2$ has the effect of freeing $\mathbf{p}_3$ to be pushed on. Since we cannot actually *pull* on a terminal, this is as close as we can come to an *inverter*.

- *MSI: Transmission-line crossing*

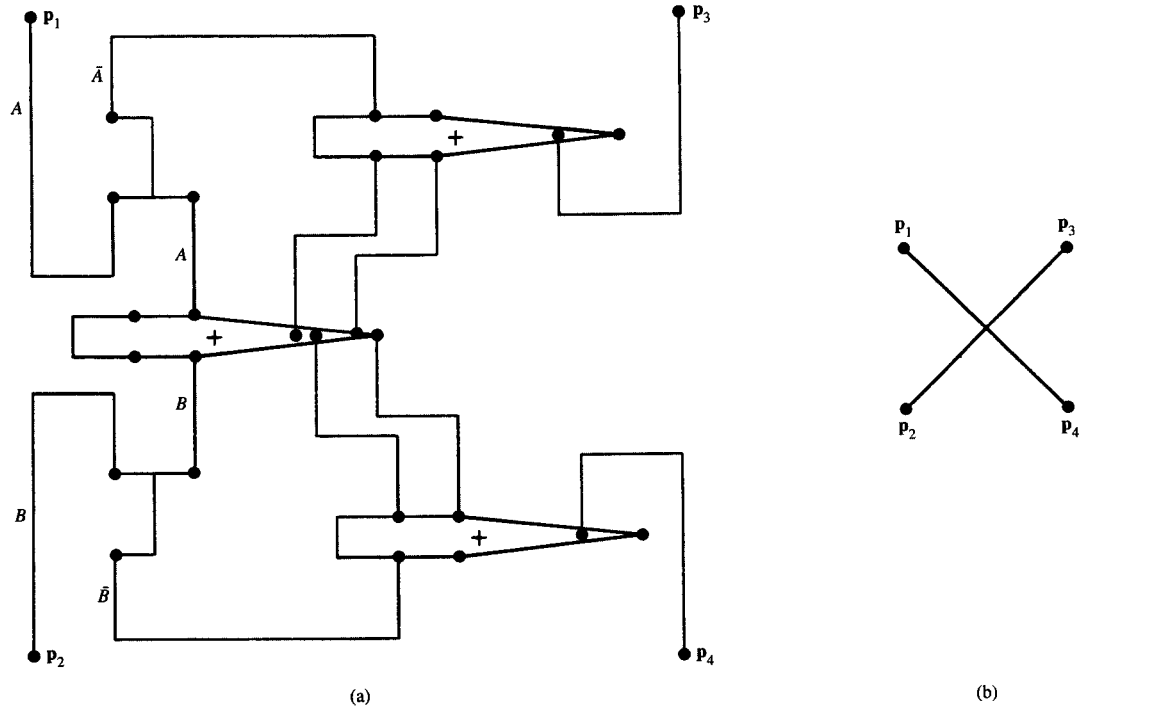
Even though the graph to be simulated is planar, transmitting color information among vertices requires transmission lines to cross. **Figure 12** illustrates the MSI



(a) (b)

Figure 11

(a) SSI implementation of a splitter-inverter and (b) its MSI symbol.



(a) (b)

Figure 12

(a) MSI implementation of a transmission-line crossing and (b) its LSI symbol.

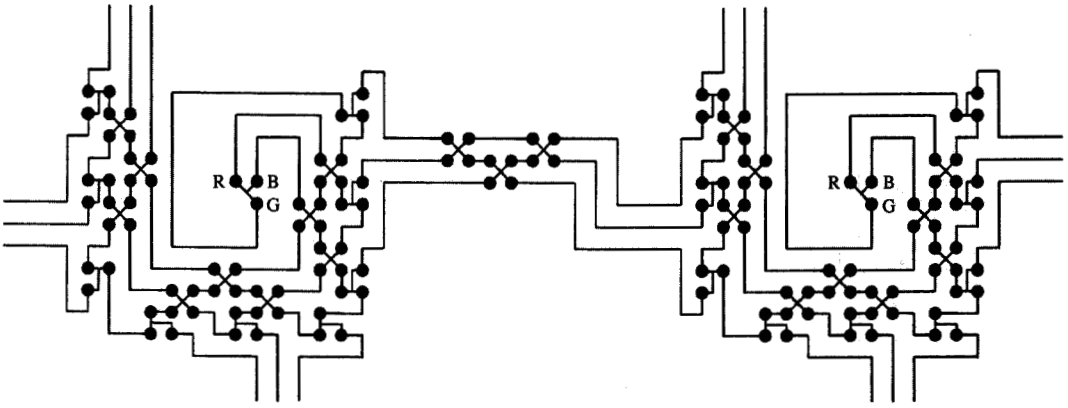


Figure 13

LSI implementation of two neighboring vertices of a graph.

implementation of a *transmission-line crossing* and its LSI symbol. A crossing consists of three adders, two splitter-inverters, and a number of transmission lines. We now show that if we push on $\mathbf{p}_1$, the crossing pushes on $\mathbf{p}_4$; independently, if we push on $\mathbf{p}_2$, the crossing pushes on $\mathbf{p}_3$. Let A be the first input of the left adder, and let B be its second input. If we push on $\mathbf{p}_1$, $A = 1$; otherwise $A = 0$, and similarly for $\mathbf{p}_2$ and B. The left adder computes $A + B$: It pushes on $\mathbf{w}_2$ if either is 1, and it pushes on $\mathbf{w}_4$ if both are 1. The upper adder computes $\overline{A} + A + B = 1 - A + A + B = 1 + B$, which depends on B only. Similarly, the lower adder computes $1 + A$.

• *LSI: Implementing a graph*

Each vertex in $\mathcal{V}$ is represented by an AND gate surrounded by circuitry required to transmit its state to other vertices. It follows from the definition of the logical AND that an AND gate must cover at least one of its three terminals. Let us label its terminals R, B, and G. If the gate covers terminal R, we say the vertex is colored red; if it covers B but not R, its color is blue; else it covers only G, and it is green. Information about the color of a vertex is transmitted to neighboring vertices by three transmission lines. For example, if a vertex is blue, it pushes on the transmission line leading out of the B terminal. Since it is not possible to push on both ends of a transmission line simultaneously, none of the neighboring vertices can be blue.

Figure 13 illustrates the LSI implementation of two neighboring vertices of a graph. Nine splitters and nine crossings suffice to transmit information about a vertex's color in four directions (remember, this graph has degree four); an additional three crossings may be required for each simulated edge.

We now argue that P and K can be chosen so that appropriate vertices and transmission lines can be constructed to simulate any graph $\mathcal{G}$. Recall that the algorithm of Tamassia and Tollis [10] can be used to embed $\mathcal{G}$ in a $cV \times cV$ grid, for some constant c. For some constants k_1 and k_2, the "circuitry" for a vertex takes no more space than $k_1\eta \times k_1\eta$, and the "circuitry" for a transmission line is no wider than $k_2\eta$. In total, the width of the embedding is $\max(k_1, k_2)\,\eta cV$, which must be less than unity. Therefore, $\eta = \min\{2^{-20}, [\max(k_1, k_2)\,cV]^{-1}\}$, since, from the subsection on sliders, $\eta < 2^{-20}$. The value of ϵ must be small enough that the addition of ϵ to line coefficients has a negligible effect. Setting $\epsilon = 2^{-20}\eta$ easily suffices. This leads to $P = \lceil -\log \epsilon \rceil$ and $K = \lceil -\log \epsilon/\eta \rceil = 20$.

To complete the proof of the theorem, we point out that before any rounding, none of the terminals is covered by any slider. If a valid (P, K)-approximation exists, the set of covering choices for the slider terminals corresponds to a choice of colors. Thus, an approximation exists if and only if a coloring

exists. Therefore, finding a (P, K)-approximation is NP-complete.

4. Generalization to polyhedra

With suitable generalizations of the definitions, Theorem 1 also applies to sets of simple polyhedra. To understand this, observe that if it did not, membership in the language of sets of simple polygons could be determined by "thickening" polygons into polyhedra in the z-direction.

In particular, we could replace each line $ax + by = c$ with the *plane* $ax + by = c$. This replacement would transform each simple polygon into an infinite cylinder in the z-direction. To make each cylinder finite, we would terminate it at the plane $z = 1$ and the plane $z = -1$. If a set of simple polygons has a (P, K)-approximation, then the resulting set of cylinders has a (P, K)-approximation: Simply convert each polygon in the planar (P, K)-approximation into a cylinder. Conversely, if the set of cylinders has a (P, K)-approximation, then we can generate a (P, K)-approximation to the original set of polygons by taking the cross section $z = 0$. Thus, the set of polygons has a (P, K)-approximation if and only if the corresponding set of cylinders has a (P, K)-approximation. Therefore, the polyhedral rounding problem is NP-hard.

To show that the polyhedral rounding problem is in NP, we have to be able to check a potential (P, K)-approximation in polynomial time. Even using naive methods, it requires no more than $O(n^3)$ time to verify that a set of polyhedra is simple and to determine the nesting relationship. Comparing the forest of nesting relationships with that of the original set of polyhedra can be done in $O(n)$ time.

5. Approximation methods

Since finding a (P, K)-approximation to a set of simple polygons is an NP-complete problem, hence prohibitively expensive to execute, it is important to consider other approaches to finding compact coordinate representations of polygons and polyhedra. We discuss four alternative approaches in this section.

Perhaps the most obvious approach is to avoid the problem altogether by using one of the techniques we discussed in Section 1. To round an object defined in terms of a constructive solid geometry (CSG) tree of set operations on geometric primitives, the object-reconstruction technique rounds individual geometric primitives independently and then reconstructs the object according to its CSG definition. This technique need not preserve the combinatorial structure of the resulting object. When rounding is to be used to control precision growth resulting from applying a *sequence* of Euclidean transformations to a set of polygons or polyhedra, it may be possible to compose the transformations and then round the composed transformation before applying it to the set of polygons or polyhedra. Unfortunately, if the transformations are not applied in a single sequence but instead are distributed throughout a CSG tree, rounding of transformations may also alter the structure of the final object.

A second alternative is to place restrictions on the set of input polygons to make rounding easier. Clearly, the sets of simple polygons used in the proof of Theorem 1, particularly the sponges, are not typical. Suppose then that we are willing to assume that the sets of polygons we wish to round result from, say, applying a rigid motion to sets of polygons having some minimum separability properties (e.g., no two vertices are too close and no vertex is too close to an edge). Theorem 1 does not imply that the problem of finding a (P, K)-approximation for this restricted input is NP-complete, but it does imply that any algorithm that solves the restricted problem must exploit the input restrictions. Another alternative is an algorithm that can find a (P, K)-approximation for any set of polygons which has a $(P, K/2)$-approximation. It might be possible to find a polynomial-time algorithm for these or other restricted problems. This is a topic for future research.

Another avenue of exploration, which is related to the previous one, is to turn to heuristics. Here, a heuristic is an algorithm that accepts arbitrary sets of input polygons and in polynomial time either computes the desired (P, K)-approximation or reports that it cannot do it. In the latter case, we know nothing about whether or not the desired approximation exists, only that this particular algorithm cannot find it. The difference between this approach and the previous one is that we might not be able to characterize concisely the sets of input polygons for which the algorithm will succeed. However, if the algorithm succeeds for most inputs encountered in practice, it might nevertheless be useful in the following scenario. Suppose that all operations on polygons are implemented in arbitrary (but finite) precision arithmetic. When the coordinates of the polygons become too long for efficient computation, we run the rounding operation to try to find a (P, K)-approximation for some suitable values of P and K. If the algorithm fails to find an approximation, we increase P and/or K and try again. Either we succeed after some small number of attempts, or we continue working with full precision, but with some performance penalty. If the algorithm succeeds for most inputs, the overall performance penalty might be acceptable. This, too, is a topic for future research.

The fourth alternative is very different, in a fundamental way. In Section 2 we argue that the notion of combinatorial structure is more natural than the

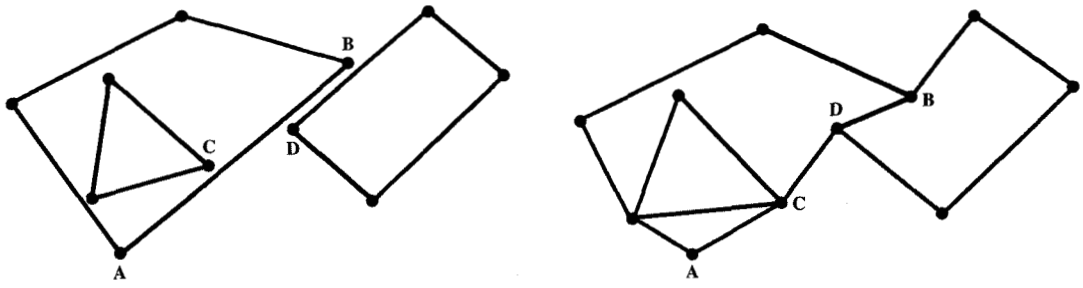


Figure 14

Polygonal rounding based on shortest paths.

stronger notion of order type. Perhaps there is an even weaker type of structure that is still useful for practical applications. It might be possible to devise a polynomial-time algorithm that preserves this weaker structure. (This differs from Sugihara's hyperplane rounding algorithm, summarized in Section 1, in that his algorithm does not attempt to preserve any definable structure, aside from the number of hyperplanes.) Milenkovic [11] gives a polynomial-time algorithm that replaces each edge of a polygon with the shortest polygonal path that has nearly the same combinatorial relationship with every other edge. By "nearly" we mean that some new vertices may be introduced, causing what was originally an edge to become a path. This process is illustrated in **Figure 14**, in which edge **AB** becomes path **ACDB**. This approach allows the combinatorial structure to change, but it does not allow polygons to interpenetrate. Whether or not this technique preserves enough structure for practical applications is another topic for future research.

6. Conclusion

We have shown that to round polygons and polyhedra while preserving combinatorial structure is a difficult problem. We have not shown, however, that rotation with limited precision growth is necessarily difficult, because this would require showing that the polygon used in the reduction can result from the rotation of a P-bit set of simple polygons. However, we have shown that one cannot solve the problem of rotations by solving the general problem of rounding. Any technique for rotation must either increase the number of bits required to represent each coefficient, or it must change the combinatorial structure, or it must exploit the fact that the polygons or polyhedra to be rounded result from a rotation. A technique based on the third approach may exist, but it will not generalize to other operations that require rounding.

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