Geometric tolerancing: II. Conditional tolerances

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In a companion paper [1], we examined the representation of geometric tolerances in solid models from the perspective of certain functional requirements. We showed that assembly and material bulk requirements can be specified as virtual boundary requirements (VBRs). Here, we study the related issue of deriving equivalent alternative specifications. Specifically, we first explore the reasons for converting VBRs to another form of tolerances designated as conditional tolerances (CTs). We then develop a theoretical basis for converting VBRs to CTs and derive CTs for some common and practical VBRs. We thereby demonstrate the difficulties in finding a general-purpose algorithm for such conversions and also show that some of the CT formulas used in current practice are incorrect.

Introduction

In current practice [2], the geometrical specifications of mechanical parts take the form of *dimensions* and *tolerances*. The goal of such specifications is to describe a class of functionally acceptable mechanical parts that are geometrically similar. The approach used is to describe the geometry of a nominal mechanical part whose surface features are mathematically perfect in form, and the extent to which the geometry of any candidate part may deviate from that of the nominal part.

The description of the nominal geometry often takes the form of a set of annotated two-dimensional projections of the nominal mechanical part, or more recently, complete and unambiguous solid-geometric representations in a computer. In either case, dimensions are assertions of geometric nature on a set of features (both surface features and derived features such as axes and median planes) of the nominal mechanical part. Dimensions specify the form (implicitly) (e.g., *shank* of **pin** in Example 1 of the companion paper [1] is a cylindrical surface) or the size (e.g., diameter of *shank* is 10 mm) of a single feature, or the positional (locational and/or orientational) relationship among a set of features (e.g., *shank* and *lip* of **pin** are perpendicular).

A tolerance specification, or, briefly, a *tolerance*, on a set of features is a measure of the extent to which the set of features of an actual candidate part may deviate geometrically from the set of features of the nominal part. Conceptually, a tolerance specification can be interpreted to define a certain region of space (a tolerance zone) within which it should be possible to contain the set of features of an actual candidate part [2, 3]. Thus, it serves to provide a theoretical inspection procedure. In practice, the set of features of the candidate part is approximated closely (through a process of measuring and fitting of features) by a set of features of the corresponding mathematically perfect form, and the geometric parameters derived from them are compared with the tolerance specification. Here, the tolerance specification is interpreted to provide bounds on the allowable variation of geometric parameters derived from the measurement of actual features.

As in the case of dimensions, tolerances can be specified on the form of a single feature (e.g., the form tolerance on *shank* of **pin** is 0.1 mm) or its size (e.g., the size tolerance on the diameter of *shank* is 0.25 mm), or the positional (locational and/or orientational) relationship among a set of features (e.g., the orientation tolerance for *shank* and *lip* is 0.1 mm). In addition, tolerances may be broadly qualified as either *unconditional* or *conditional* on the basis of the dependence of the allowable geometrical variations they represent on the geometric parameters of actual features.

An *unconditional* tolerance on a set of features specifies geometric variations that do not depend on any geometric parameters derived from fitting of the actual set of features. For example, an unconditional tolerance on the position of a cylindrical hole specifies that the allowable variation in the position of the (fitted) hole is independent of the size of the (fitted) hole. In practice, most tolerance specifications, including *Regardless of Feature Size* (RFS) tolerances [2], are unconditional.

In contrast, a *conditional* tolerance (CT) on a set of features specifies geometric variations that are dependent on some geometric parameters derived from fitting of the actual set of features. Thus, a CT on the position of a cylindrical hole states that the allowable variation in the position of the (fitted) hole depends on the size of the (fitted) hole. The *Maximum Material Condition* (MMC) and *Least Material Condition* (LMC) tolerances [2] used in practice are in fact CT specifications. Typically, virtual boundary requirements (VBRs), when translated into allowable variations in geometric parameters of features, give rise to CTs.

VBRs capture certain classes of functional requirements and facilitate the design of functional gauges where appropriate [1]. It is necessary, however, to derive alternative specifications (as nearly equivalent to the VBRs as possible) based on the concept of CTs. Some of the reasons for this are as follows.

Part geometry verification VBRs are not always verifiable using functional gauges. Even when applicable, building functional gauges can be very expensive, particularly for small volume production. Use of modern inspection tools such as coordinate measuring machines and vision-based parts measurements systems [4] would require that VBRs be translated into CTs.

Part fabrication process planning Selection of the appropriate equipment and the subsequent specification of process parameters for fabricating a part require knowledge of the geometric parameters of the features of the part and of allowable variations from their nominal values. For example, if a drilling machine with an xy table is used to drill *hole* in **washer** of Example 1 in [1], it is important to know the allowable variations in the x and the y locations, and the orientation of the drill axis with respect to the fixtures. The VBRs do not provide such information directly.

Part fabrication process control In-process measurements that can point back to drifting process parameters are necessary for process control. For example, in the case of drilling a hole in a plate, the process controller monitors the location, orientation, and size parameters associated with the drilled hole and must decide which parameter is drifting toward its control limit. Merely verifying compliance with the VBRs does not provide this information.

Statistical tolerancing Tolerance analysis and synthesis based on statistical approaches require a set of parameters, each of which assumes a range of values subject to a probability distribution [5]. Since the parameters associated with VBRs are deterministic and do not vary from instance to instance of the manufactured part, they are not directly suitable for statistical tolerancing. On the other hand, converting VBRs into tolerance specifications on feature parameters can help identify the relevant set of parameters to focus on, as well as establish their ranges of variations over which probability distributions can be used.

We have seen a number of reasons for converting VBRs into tolerance specifications on the geometric parameters of features. We next make a number of observations regarding current industrial tolerancing practices related to CTs. Figures 9 and 10 in [1] indicate how the example parts of the companion paper are usually specified. In addition to the derivation of virtual surfaces, the specifications are used to derive CTs on the orientation or the position of axes and median planes in simple cases (see Figure 9 in [1] and the discussions following Results 2 and 5 in this paper). Such interpretations are not provided in more complex cases (see [6]). Even when provided, the interpretations may not be equivalent to the VBRs, as shown later. Furthermore, when the tolerance is on position, the manner in which deviations in orientation and location can combine to produce the positional deviation is usually not known.

The problem of deriving alternative tolerance specifications from the given primary tolerance specifications has received very little attention in the literature (see [7, 8]). Our earlier work [9] raised this issue as a topic to be studied in detail. Here, we examine the problem of converting geometric tolerance representations and develop a theoretical basis for the conversion of VBRs to CTs. We derive the CTs for a number of VBRs that occur frequently. We thereby demonstrate the difficulties in finding a general-purpose algorithm for such conversions, and also show that some of the CT expressions in current use are incorrect. Finally, we discuss some open research issues in converting VBRs to CTs.

Formalization

In this section, we provide a formal basis for the conversion of VBRs to CTs. We first derive the necessary and sufficient conditions for the satisfaction of VBRs, in terms of virtual boundary parameters (a_i in Definition 7 of [1]) and fitting parameters for actual surface features (b_i in Definition 7 of [1]). We then develop formally the manner in which a well-defined set of geometric parameters can be associated with actual surface features (using the concept of surrogate surface features that are of perfect form and are conservative approximations to actual surface features), the concept of tolerance zones (that correspond to VBRs) in the space of such geometric parameters, and the conditions under which such zones are to be considered CT zones. We use the definitions and notations developed in [1].

- *Conversion of virtual boundary requirements*

In this subsection, we state and prove a number of important properties of projection and offsetting. Using these properties, we show that fitting surfaces and the actual surface features they fit touch each other at least at one point. This leads to the proof of an important relationship between the virtual boundary parameters and fitting parameters of actual surface features.

We start with a property of projection.

Property 1

Let A be a subset of E^3 and $\mathbf{p}$ a point with $d(\mathbf{p}, A) > 0$. If $\mathbf{q} \in P(\mathbf{p}, A)$, then $\mathbf{q} \in P(\mathbf{r}, A)$ where $\mathbf{r} \in \mathcal{L}(\mathbf{p}, \mathbf{q})$.

Proof For any $\mathbf{q}' \in A$ we have, from the triangle inequality, $d(\mathbf{r}, \mathbf{q}') + d(\mathbf{p}, \mathbf{r}) \geq d(\mathbf{p}, \mathbf{q}')$. From the definition of projection, we have $d(\mathbf{p}, \mathbf{q}') \geq d(\mathbf{p}, \mathbf{q})$. Hence, $d(\mathbf{r}, \mathbf{q}') \geq [d(\mathbf{p}, \mathbf{q}) - d(\mathbf{p}, \mathbf{r})] = d(\mathbf{r}, \mathbf{q})$, which implies that $\mathbf{q}$ is one of the members of cl A closest to $\mathbf{r}$, leading to the desired result. □

Some properties of offsetting follow.

Property 2

Let A be a regular subset of E^3 and a a scalar such that $a \geq 0$. Then $d(\mathbf{p}, \partial A) = a$ for any $\mathbf{p} \in \partial(A \downarrow^* a)$.

Proof From the definitions of regularized shrinking and regular sets, we have $\partial(A \downarrow^* a) = \partial(\overline{\overline{A}^* \uparrow^* a}^*) = \partial(\overline{A}^* \uparrow^* a)$. Hence, $\mathbf{p} \in \partial(\overline{A}^* \uparrow^* a)$, and from [10], $d(\mathbf{p}, \partial\overline{A}^*) = a = d(\mathbf{p}, \partial A)$. □

Property 3

Let A be a regular subset of E^3, $\mathbf{p}$ a point with $d(\mathbf{p}, A) = b > 0$, and a a scalar such that $0 \leq a \leq b$. Then $d(\mathbf{p}, A \uparrow^* a) = b - a$.

Proof From [10] it follows that

$$d(\mathbf{p}, A \uparrow^* a) = d\left[\mathbf{p}, \bigcup_{\mathbf{q} \in A} \mathcal{B}(\mathbf{q}, a)\right]$$

$$= \min\{d[\mathbf{p}, \mathcal{B}(\mathbf{q}, a)] : \mathbf{q} \in A\}$$

$$= \min\{[d(\mathbf{p}, \mathbf{q}) - a] : \mathbf{q} \in A\}$$

$$= b - a. \quad \square$$

Property 4

Let A be a regular subset of E^3, $\mathbf{p} \in \mathrm{i}A$ a point such that $d(\mathbf{p}, \partial A) = b > 0$, and a a scalar such that $0 \leq a < b$. Then $d[\mathbf{p}, \partial(A \downarrow^* a)] = b - a$ and $\mathbf{p} \in \mathrm{i}(A \downarrow^* a)$. Furthermore, if $\mathbf{q} \in P(\mathbf{p}, \partial A)$, then $\mathbf{r} \in P[\mathbf{p}, \partial(A \downarrow^* a)]$, where $\mathbf{r} = \partial(A \downarrow^* a) \cap \mathcal{L}(\mathbf{p}, \mathbf{q})$.

Proof $\mathbf{p} \in \mathrm{i}A$ implies $\mathbf{p} \notin \overline{A}^*$. Then, from [10] it follows that $d(\mathbf{p}, \overline{A}^*) = d(\mathbf{p}, \partial\overline{A}^*) = d(\mathbf{p}, \partial A) = b > 0$. From Property 3, $d(\mathbf{p}, \overline{A}^* \uparrow^* a) = (b - a) > 0$. Hence, $\mathbf{p} \in \mathrm{i}(\overline{\overline{A}^* \uparrow^* a}^*)$. That is, $\mathbf{p} \in \mathrm{i}(A \downarrow^* a)$. Furthermore, $d(\mathbf{p}, \overline{A}^* \uparrow^* a) = d[\mathbf{p}, \partial(\overline{A}^* \uparrow^* a)] = d[\mathbf{p}, \partial(\overline{\overline{A}^* \uparrow^* a}^*)] = d[\mathbf{p}, \partial(A \downarrow^* a)] = (b - a)$.

Since $\mathbf{r} \in \mathcal{L}(\mathbf{p}, \mathbf{q})$ and $\mathbf{q} \in P(\mathbf{p}, \partial A)$, we have $\mathbf{q} \in P(\mathbf{r}, \partial A)$ from Property 1. From Property 2 we have $d(\mathbf{r}, \partial A) = a$, since $\mathbf{r} \in \partial(A \downarrow^* a)$. Therefore, $d(\mathbf{r}, \mathbf{q}) = a$, which implies that $d(\mathbf{p}, \mathbf{r}) = b - a = d[\mathbf{p}, \partial(A \downarrow^* a)]$. From the definition of projection, it follows that $\mathbf{r} \in P[\mathbf{p}, \partial(A \downarrow^* a)]$. □

Given two arbitrary scalars r_1 and r_2, it is trivially true that if $r_1 = r_2$, then $O(S; r_1) = O(S; r_2)$. We need the following properties and lemmas to understand what happens when $r_1 \neq r_2$.

Property 5

$O(S; r_1)$ and $O(S; r_2)$ are regular.

Proof Follows from the fact that S is regular. See [11]. □

Property 6

$\forall r > 0, S \subset S \uparrow^* r$.

Proof Let $\mathbf{p} \in S$. Then $d(\mathbf{p}, S) = 0 < r$. Hence, by definition of regularized growing, $\mathbf{p} \in S \uparrow^* r$. Now pick a small positive ε such that $0 < \varepsilon < r$. Then $\forall \mathbf{p} \ni d(\mathbf{p}, S) = \varepsilon$, $\mathbf{p} \notin S$ but $\mathbf{p} \in S \uparrow^* r$. Hence we obtain the proper subset property. □

Property 7

$\forall r > 0, S \supset S \downarrow^* r$.

Proof From Property 6 we know that $\overline{S}^* \subset \overline{S}^* \uparrow^* r$. Applying regularized complementation on both sides, we have $S \supset \overline{\overline{S}^* \uparrow^* r}^*$. Hence $S \supset S \downarrow^* r$. □

Lemma 1

If $r_1 < r_2$, then $O(S; r_1) \subset O(S; r_2)$.

Proof There are three cases. In the first case, $r_1, r_2 \le 0$; i.e., both offsets are obtained by regularized shrinking. Let $|r_1| = |r_2| + \varepsilon$ for some positive $\varepsilon > 0$. Since the regularized shrinking operation is associative [11], $S \downarrow^* |r_1| = (S \downarrow^* |r_2|) \downarrow^* \varepsilon \subset S \downarrow^* |r_2|$, using Property 7. From the definition of regularized offset, it follows that $O(S; r_1) \subset O(S; r_2)$.

In the second case, $r_1 \le 0$, $r_2 > 0$; i.e., one is obtained by regularized shrinking, and the other is obtained by regularized growing. From Properties 6 and 7, we have $S \downarrow^* |r_1| \subset S \subset S \uparrow^* r_2$. Then the result follows directly from the definition of regularized offset.

In the third case, $r_1, r_2 > 0$; i.e., both offsets are obtained by regularized growing. Let $r_2 = r_1 + \varepsilon$ for some positive $\varepsilon > 0$. Since the regularized growing operation is associative [11], $S \uparrow^* r_2 = (S \uparrow^* r_1) \uparrow^* \varepsilon \supset S \uparrow^* r_1$, using Property 6. From the definition of regularized offset, it follows that $O(S; r_1) \subset O(S; r_2)$. □

Lemma 2

If $\exists\ \mathbf{p} \in \partial O(S; r_2) \ni \mathbf{p} \in O(S; r_1)$, then $r_1 \ge r_2$.

Proof By contradiction. Let $r_1 < r_2$. Then from Lemma 1, $O(S; r_1) \subset O(S; r_2)$. Since both the regularized offsets are closed, this means that $\partial O(S; r_2) \subset \mathrm{i}[\overline{O(S; r_1)}^*]$. But this violates the condition satisfied by $\mathbf{p}$. □

From the requirements to be satisfied by measures of closeness used in fitting, we can prove the following important lemmas, which show that the boundary of every fitting half-space touches the corresponding actual feature at least at one point.

Lemma 3

If H is a fitting half-space for an actual surface feature F_A, then $\exists\, \mathbf{p} \in F_A \ni \mathbf{p} \in \partial H$.

Proof We first address the external fitting case. From the definition of fitting, $F_A \subset H$ and $\mathcal{C}(F_A, \partial H)$ is the smallest it can be. The proof is by contradiction. Assume that for any $\mathbf{q} \in F_A$, $\mathbf{q} \in \mathrm{i}H$. Then $d(\mathbf{q}, \partial H) > 0$. Let

$$\min_{\mathbf{q} \in F_A} d(\mathbf{q}, \partial H) = r > 0$$

and let

$$H' = \left(H \downarrow^* \frac{r}{2}\right).$$

Then, from Property 4, $\mathbf{q} \in \mathrm{i}H'$. Thus, H' satisfies the containment condition for fitting.

Note that since H is a fitting half-space for F_A, it satisfies the material-side condition. That is, $\exists\, \mathbf{q}' \in \mathcal{P}(\mathbf{q}, \partial H) \wedge \exists\, \mathbf{r} \in [\mathrm{cl}\ \mathcal{L}(\mathbf{q}, \mathbf{q}')] \cap F_A \ni \mathbf{q}' \in \mathcal{V}_E(\mathbf{r}, F_A)$. Using Property 4, we conclude that $\mathbf{q}'' = [\partial H' \cap \mathcal{L}(\mathbf{q}, \mathbf{q}')] \in \mathcal{P}(\mathbf{q}, \partial H')$. From Property 2, $d(\mathbf{q}'', \mathbf{q}') = (r/2)$, since $\mathbf{q}'' \in \partial H'$, while $d(\mathbf{r}, \mathbf{q}') \ge r$, by assumption. Therefore, $\mathbf{q}'' \in \mathcal{L}(\mathbf{r}, \mathbf{q}')$ and $\mathbf{r} \in \mathrm{cl}\ \mathcal{L}(\mathbf{q}, \mathbf{q}'')$. Since $\mathbf{r} \in \mathrm{cl}\ \mathcal{L}(\mathbf{q}, \mathbf{q}') \cap F_A$, it follows that $\mathbf{r} \in \mathrm{cl}\ \mathcal{L}(\mathbf{q}, \mathbf{q}'') \cap F_A$. Furthermore, $\mathbf{q}'' \in \mathcal{V}_E(\mathbf{r}, F_A)$, since $\mathcal{L}(\mathbf{r}, \mathbf{q}') \subset \mathcal{V}_E(\mathbf{r}, F_A)$. Thus, H' satisfies the material-side condition also.

Also from Property 4, we have

$$d(\mathbf{q}, \partial H') = \left[d(\mathbf{q}, \partial H) - \frac{r}{2}\right] < d(\mathbf{q}, \partial H).$$

Thus, $\partial H'$ is closer to every point of F_A than ∂H. From the requirements to be satisfied by the criterion for individual measures of closeness, it follows that $\mathcal{C}(F_A, \partial H') < \mathcal{C}(F_A, \partial H)$, which contradicts the fact that H is a fitting half-space for F_A. Hence, $\exists\, \mathbf{p} \in F_A \ni \mathbf{p} \in \partial H$.

The proof is similar for the internal fitting case. □

Lemma 4

For a set of actual surface features $T_A \equiv \{F_{A1}, \cdots, F_{Ak}\}$, if $\mathbf{H} \equiv \{H_1, \cdots, H_k\}$ is a rigid collection of fitting half-spaces, then $\forall i\, \exists\, \mathbf{p}_i \in F_{Ai} \ni \mathbf{p}_i \in \partial H_i$.

Proof Follows from the definition of overall measure of closeness as the sum of the individual measures of closeness, the requirements to be satisfied by the criterion for the individual measures of closeness, and Lemma 3. □

We now state necessary and sufficient conditions for the satisfaction of VBRs, in terms of virtual boundary parameters and fitting parameters for actual surface features.

Theorem 1

Let $\mathcal{A}$ be a virtual boundary requirement asserted on T_N and characterized by $\mathbf{a}$. Let $\mathbf{b}$ and $\mathcal{M}$ be such that $\mathcal{M}O(T_N; \mathbf{b})$ is a rigid collection of fitting half-spaces (satisfying appropriate spatial constraints with respect to the datum system, if any, referred to in $\mathcal{A}$ and established on the actual solid) for T_A. Then, $T_A \copyright \mathcal{A}$ if and only if, for all i, $b_i \le a_i$ for assembly, and $b_i \ge a_i$ for material bulk.

Proof We start with the assembly requirement. To establish necessity, assume that $T_A \copyright \mathcal{A}$. Then from the containment condition we have $\forall i$, $F_{Ai} \subset \mathcal{M}O(H_{F_{Ni}}; a_i)$. From Lemma 4 it follows that $\forall i$, $\exists\, \mathbf{q}_i \in F_{Ai} \ni \mathbf{q}_i \in \partial\, \mathcal{M}O(H_{F_{Ni}}; b_i)$. From these facts, we deduce that $\forall i$, $\exists\, \mathbf{q}_i \in \partial O(H_{F_{Ni}}; b_i) \ni \mathbf{q}_i \in O(H_{F_{Ni}}; a_i)$. Using Lemma 2, we conclude that $\forall i$, $b_i \le a_i$.

To show sufficiency, assume that $\forall i$, $b_i \le a_i$. From Lemma 1 we observe that $O(H_{F_{Ni}}; b_i) \subseteq O(H_{F_{Ni}}; a_i)$. From the containment condition for fitting, we have $F_{Ai} \subset \mathcal{M}O(H_{F_{Ni}}; b_i)$, from which we deduce that $F_{Ai} \subset \mathcal{M}O(H_{F_{Ni}}; a_i)$. The position condition is automatically satisfied by the existence of $\mathcal{M}O(T_N; \mathbf{b})$. Thus $T_A \copyright \mathcal{A}$.

The proof is similar for material bulk requirement. □

- *Conditional tolerances*

In this subsection, we describe an approach for associating a well-defined set of geometric parameters with a set of actual features subject to a given set of tolerance assertions. We define an alternative representation for the tolerance assertions, in terms of these geometric parameters. We discuss the concept of CTs by characterizing further the nature of the set of allowable parameter values, and close with some remarks on deriving CTs from VBRs.

The set of parameters associated with actual features should reflect the purpose of deriving the alternative tolerance representation. For this reason, we consider certain grouping, fitting, and limiting operations on sets of actual features to create perfect-form surface patches that serve as *conservative* and close approximations (*surrogates*) to actual features in verifying their satisfaction of given tolerance assertions. Here, the grouping operations preserve some of the spatial relationships among the nominal features in the fitting process. The type of tolerance assertion being verified determines the side of the actual surface feature in which the fitting entity should lie.

We start with a definition of a grouping of a set of surface features.

Definition 1

Let $T \equiv \{F_1, \cdots, F_k\}$ be a set of surface features. A *grouping* $\mathbf{G} \equiv \{G_1, \cdots, G_l\}$ associated with T is a subset of the power set of T such that $\varnothing \notin \mathbf{G}$; for $i \neq j$, $G_i \cap G_j = \varnothing$; and

$$\bigcup_i G_i = T. \quad \square$$

Observe that each G_i is a set of surface features and therefore can play roles similar to that of T, wherever appropriate. Two extreme examples of grouping are $l = k$ with $G_i = \{F_i\}$ and $l = 1$ with $G_1 = T$. In the former case, none of the positional relationships among the members of the corresponding set of nominal surface features are preserved in the subsequent fitting process. In the latter case, all of the angular relationships are maintained.

Next, we define the notion of fitted and limited surface features for a member G_{N} of a grouping $\mathbf{G}_{\mathrm{N}}$ associated with a member T_{N} of the nominal tolerance set $\mathbf{T}_{\mathrm{N}}$.

Definition 2

Let $G_{\mathrm{N}} \equiv \{F_{\mathrm{N}1}, \cdots, F_{\mathrm{N}m}\}$ be a set of nominal surface features and $G_{\mathrm{A}} \equiv \{F_{\mathrm{A}1}, \cdots, F_{\mathrm{A}m}\}$ be the corresponding set of actual surface features. Let $\mathcal{M}$ be a rigid-body transformation and $\mathbf{a} \equiv \{a_1, \cdots, a_m\}$ be a set of scalars such that $\mathcal{M}O(G_{\mathrm{N}}; \mathbf{a})$ is a rigid collection of *externally* (*internally*) fitting half-spaces for G_{A} for *assembly* (*material bulk*). A *rigid collection of surrogate surface features* rigidly associated with G_{A} for a given set of tolerance assertions $\mathcal{A}$ on T_{N} is a rigid collection of geometric entities denoted as $G_{\mathrm{F}} \equiv \{F_{\mathrm{F}1}, \cdots, F_{\mathrm{F}m}\}$ such that for all i, $F_{\mathrm{F}i} \subset \mathcal{M}\partial O(H_{F_{\mathrm{N}i}}; a_i)$; $F_{\mathrm{F}i}$ is bounded and closed in the relative topology of $\mathcal{M}\partial O(H_{F_{\mathrm{N}i}}; a_i)$; and the spatial extent of $F_{\mathrm{F}i}$ is such that G_{A} satisfies $\mathcal{A}$ whenever G_{F} satisfies $\mathcal{A}$. $\square$

G_{F} is a conservative approximation to G_{A} because it is possible for G_{A} to satisfy $\mathcal{A}$ without G_{F} satisfying $\mathcal{A}$, while the converse is prohibited by the definition above. Each $F_{\mathrm{F}i}$ inherits an inside (material side) and an outside (nonmaterial side) from $H_{F_{\mathrm{A}i}}$, the fitting half-space for $F_{\mathrm{A}i}$, which means that G_{F} can be fitted (in a constrained manner) further. The definition does not prescribe any process for limiting the boundary of the fitting half-space for $F_{\mathrm{A}i}$ to derive $F_{\mathrm{F}i}$. We conjecture that for any closed and bounded subset of the boundary of the fitting half-space for a surface feature $F_{\mathrm{A}i}$ to serve as a surrogate for $F_{\mathrm{A}i}$ (i.e., to satisfy the requirement that whenever the subset satisfies $\mathcal{A}$, the actual feature satisfies $\mathcal{A}$), it is sufficient for that subset to include the projection of $F_{\mathrm{A}i}$ onto the boundary of the fitting half-space. The special cases in the next section seem to support this conjecture. However, we have not been able to prove it rigorously. There are an unlimited number of choices for surrogate features. Generally, the limiting is done in such a manner that the number of parameters necessary to characterize $F_{\mathrm{F}i}$ is minimized. Datum half-spaces and fitting half-spaces (possibly subject to additional constraints) for adjacent features are often used to limit the fitting surface for the feature of interest. We now define the notion of fitted and limited surface features for a grouping $\mathbf{G}_{\mathrm{N}}$ associated with a member T_{N} of the nominal tolerance set $\mathbf{T}_{\mathrm{N}}$.

Definition 3

Let $T_{\mathrm{N}} \equiv \{F_{\mathrm{N}1}, \cdots, F_{\mathrm{N}k}\}$ be a member of a nominal tolerance set $\mathbf{T}_{\mathrm{N}}$ and $\mathbf{G}_{\mathrm{N}} \equiv \{G_{\mathrm{N}1}, \cdots, G_{\mathrm{N}l}\}$ be a grouping associated with T_{N}. Let $T_{\mathrm{A}} \equiv \{F_{\mathrm{A}1}, \cdots, F_{\mathrm{A}k}\}$ be the corresponding member of the actual tolerance set for an actual solid S_{A} and $\mathbf{G}_{\mathrm{A}} \equiv \{G_{\mathrm{A}1}, \cdots, G_{\mathrm{A}l}\}$ be the grouping derived from $\mathbf{G}_{\mathrm{N}}$ and associated with T_{A}. A *rigid collection of surrogate surface features* $T_{\mathrm{F}} \equiv \{F_{\mathrm{F}1}, \cdots, F_{\mathrm{F}k}\}$, rigidly associated with T_{A} (subjected to G_{A}), is defined as

$$\bigcup_i G_{\mathrm{F}i},$$

where $G_{\mathrm{F}i}$ is a rigid collection of surrogate surface features rigidly associated with $G_{\mathrm{A}i}$, and the spatial extents of $F_{\mathrm{F}i}$ are such that T_{A} satisfies a given set of tolerance assertions $\mathcal{A}$ if T_{F} satisfies $\mathcal{A}$. $\square$

In this definition, since each $G_{\mathrm{N}i}$ is a member of a grouping, Definition 2 is applicable. The rigid sets of surrogate features thus associated with $G_{\mathrm{A}i}$ are collected into a single rigid set of surrogate features associated with T_{A}. For example, let $T_{\mathrm{N}} \equiv \{F_{\mathrm{N}1}, F_{\mathrm{N}2}\}$, where $F_{\mathrm{N}1}$ and $F_{\mathrm{N}2}$ are parallel planar patches. If the associated grouping $\mathbf{G}_{\mathrm{N}} \equiv \{\{F_{\mathrm{N}1}, F_{\mathrm{N}2}\}\}$, then $F_{\mathrm{F}1}$ and $F_{\mathrm{F}2}$ are parallel to each other in $T_{\mathrm{F}} \equiv \{F_{\mathrm{F}1}, F_{\mathrm{F}2}\}$. On the other hand, if $\mathbf{G}_{\mathrm{N}} \equiv \{\{F_{\mathrm{N}1}\}, \{F_{\mathrm{N}2}\}\}$, then $F_{\mathrm{F}1}$ and $F_{\mathrm{F}2}$ may not be parallel to each other.

As discussed previously, it is necessary to derive for a given VBR the ranges of values that are permissible in

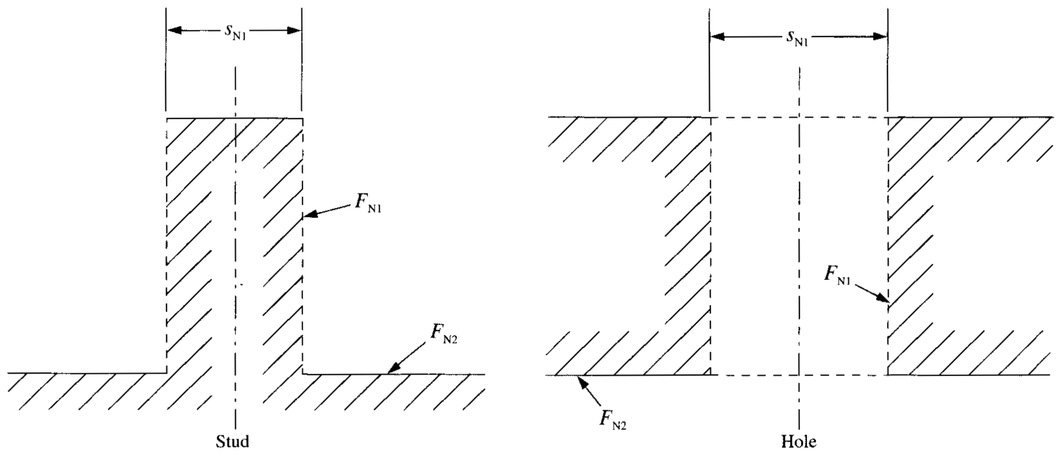

Figure 1

Nominal cylindrical features for orientation tolerancing.

certain geometric parameters characterizing the corresponding rigid collection of surrogate surface features of candidate actual solids. To define this more clearly, we note that T_F can be regarded as a member of a family of geometric entities, parameterized by a set of parameters, $(s_1, s_2, \cdots, s_m, c_1, c_2, \cdots, c_n, l_1, l_2, \cdots, l_k)$. Here, the s_i are parameters associated with the fitting half-spaces for T_A (m can be zero). They define certain intrinsic sizes of the member half-spaces and certain distances and angles among them, taking into account the grouping associated with T_N. We denote by c_i the parameters associated with the location and orientation of the rigid collection of fitting half-spaces for T_A relative to the datum system referred to in the tolerance assertions $\mathcal{A}$ on T_N (n cannot be greater than 6). If there is no datum reference, these parameters are not used to characterize T_F (i.e., $n = 0$). Finally, l_i are parameters associated with the limiting process by which F_{Fi} are derived from the fitting surfaces for F_{Ai} (k can be zero). We refer to all these parameters as *surrogate parameters.* The space spanned by the surrogate parameters is the real space R^{m+n+k}, which we refer to as the *surrogate parameter space* for T_A subjected to G_A and the specified limiting process. We can now define the *parametric tolerance zone* as follows.

Definition 4
The *parametric tolerance zone* for T_A (subjected to a specified grouping and limiting process) is a subset of the surrogate parameter space for T_A that corresponds to the set of all actual solids whose T_F satisfies $\mathcal{A}$. □

Note that the parametric tolerance zone also depends on some parameters associated with $\mathcal{A}$, but these parameters remain fixed and do not change from instance to instance of actual solids.

If the bounding surfaces of the parametric tolerance zone are hyperplanes perpendicular to the axes in the surrogate parameter space, we refer to the zone as an *unconditional tolerance zone.* Such zones imply allowable variations in each surrogate parameter which are not dependent on values of other surrogate parameters. Otherwise, the zone is referred to as a *conditional tolerance zone* and is denoted by Z. Most VBRs lead to CT zones in the surrogate parameter space.

For any given VBR, it would be preferable to derive the CT zone in the form $Z = \{\mathbf{z} : f_i(\mathbf{z}) \leq 0,\ i = 1, \cdots, j\}$, where $\mathbf{z} \equiv (s_1, \cdots, s_m, c_1, \cdots, c_n, l_1, \cdots l_k)$ is a point in the surrogate parameter space and $\mathcal{F} = \{f_1, \cdots, f_j\}$ is a set of functions. Such a representation has the attraction that by simply evaluating the functions f_i, one can determine whether a given point $\mathbf{z}$ in the surrogate parameter space is a member of Z. Further, it may facilitate the determination of the allowable range of variation in one parameter for given fixed values of other parameters.

Deriving f_i for a given VBR involves essentially finding expressions for b_i used in fitting T_F, in terms of the parameters that characterize T_F. We do not know of a general algorithm for doing this. Next, we explore the process involved by studying a number of cases that occur frequently.

Representation conversion

In practice, cylindrical features and sets of two nominally parallel planar features forming slabs or slots must frequently be considered. Here, we examine some VBRs that

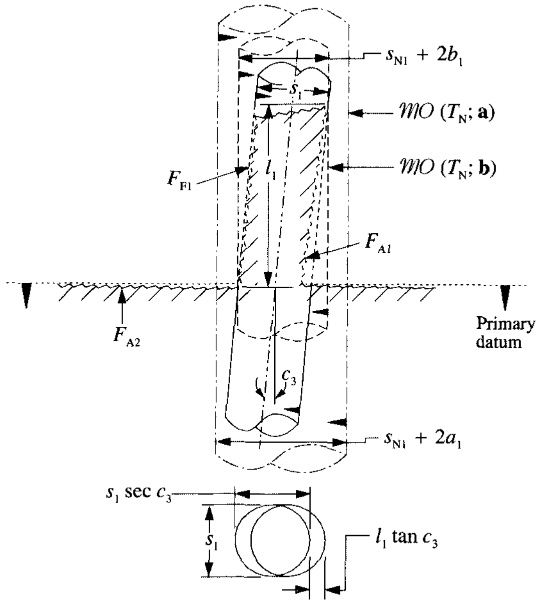

Figure 2

Orientation tolerancing: assembly requirement for a cylindrical stud.

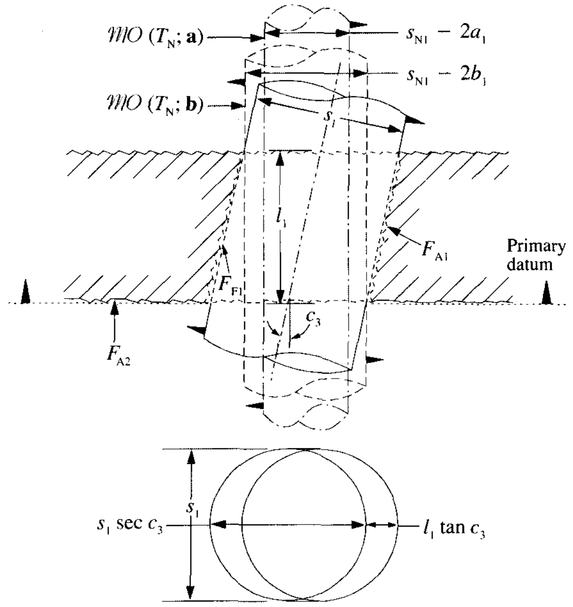

Figure 3

Orientation tolerancing: assembly requirement for a cylindrical hole.

are often invoked on such features and derive CT zones for them. All of the cases we examine refer to a datum system whose primary datum is a planar datum. We assume that the coordinate frame of reference for the actual solid is established using the primary datum surface as the xy plane. Secondary and tertiary datums, if any, are used to fix the location of the origin and the orientation of the x axis of the coordinate frame in the primary datum surface.

In all cases, we derive surrogate surface features by using a slab half-space to limit the primitive surfaces that fit the actual surface features. This slab half-space, defined as

$$H_{\text{SLAB1}} \equiv \left\{ \mathbf{p} : |p_z - l_2| \le \frac{l_1}{2} \right\},$$

where p_z is the z coordinate of $\mathbf{p}$, l_2 is the z location of the median plane of the slab, and l_1 is the thickness of the slab, is oriented such that its median plane is parallel to the primary datum surface and has the minimum thickness necessary to enclose the relevant actual surface features. Note that our limiting process introduces two parameters, l_1 and l_2, for characterizing T_F. The closeness criterion we use to estimate individual measure of closeness is the maximum distance between the candidate fitting surface and the actual feature; i.e., $\mathcal{C}(F_A, \partial H) \equiv \max\{d(\mathbf{p}, \partial H) : \mathbf{p} \in F_A\}$. We start with a single cylindrical feature.

• *Cylindrical feature*

The reader may wish to refer to Figures 10 and 11 (shown later), as a visual aid for following the description below. A cylindrical half-space is completely specified by its diameter s_1 (an intrinsic parameter) and its axis. A directed line, with the direction determined by increasing z, associated with the axis can be specified by two locational parameters c_1 and c_2 that are the coordinates of the point of intersection of the line with the xy plane, and two orientational parameters, c_3 (the *attitude* angle, defined as the acute angle between the line and the z axis) and c_4 (the *azimuth* angle, defined as the positive rotation around the z axis necessary to make the unit vector along the x axis parallel to the directed projection of the axis line onto the xy plane). Our limiting process introduces two parameters, l_1 and l_2, as stated above. Thus, the parametric space for a single cylindrical surrogate feature (derived using the limiting process described above) is the space R^7 spanned by the set of parameters $\mathbf{z} \equiv (s_1, c_1, c_2, c_3, c_4, l_1, l_2)$. A VBR may impose a limit on the range of variation of only a subset of these parameters, as is seen below.

Orientation tolerance

Consider VBRs that refer to a datum system with just a primary planar datum and are asserted on single cylindrical features. The axes of virtual cylindrical surfaces in these cases are constrained to have specified attitude angles, but their locations and azimuth angles are unconstrained. In all cases we consider, the nominal attitude angle is zero (see Example 1 in [1] for such requirements).

Figure 1 shows two nominal cylindrical features, denoted by F_{N1}, having a nominal diameter of s_{N1}. The cylindrical stud and the cylindrical hole are nominally oriented so that their axes are perpendicular to a planar feature F_{N2}. The member of the nominal tolerance set of interest here is $T_N = \{F_{N1}\}$ and the datum feature is $\{F_{N2}\}$. **Figure 2** shows an actual instance of a stud satisfying an assembly requirement. The figure for the case of a bulk requirement for a hole is very similar (see [6]). A hole with an assembly requirement is shown in **Figure 3**. Again, the case of a stud satisfying a material bulk requirement is very similar (see [6]). All of these VBRs refer to a datum system that consists of only a primary planar datum associated with the datum feature $\{F_{A2}\}$. They are characterized by a set of scalars $\mathbf{a} = \{a_1\}$. The relevant member of the actual tolerance set is $T_A = \{F_{A1}\}$, and the corresponding rigid collection of surrogate surface features is $T_F = \{F_{F1}\}$. The size parameter of F_{F1} is s_1. The top and the bottom edges of F_{F1} are ellipses parallel to the primary datum surface. The only indicated configuration parameter is the attitude angle c_3. The appropriate fitting half-spaces for T_F are indicated in the figures as $\mathcal{MO}(T_N; \mathbf{b})$, where $\mathbf{b} = \{b_1\}$. We now derive the CT zones for these VBRs.

Result 1

In the cases of the assembly requirement for a stud and the material bulk requirement for a hole, the conditional tolerance zone is given by

$$Z = \left\{ \mathbf{z} \;\middle|\; \begin{array}{l} (s_1 > 0) \wedge (l_1 > 0) \wedge \left(0 \le c_3 < \frac{\pi}{2}\right) \wedge \\ (s_1 \sec c_3 + l_1 \tan c_3 - s_{N1} - 2|a_1| \le 0) \end{array} \right\}.$$

Proof The first three inequalities are obvious from the definition of s_1, c_3, and l_1. From Theorem 1, a sufficient condition for satisfying the assembly requirement is $b_1 \le a_1$. Since $a_1 \ge 0$ for assembly, it follows that $s_{N1} + 2b_1 \le s_{N1} + 2|a_1|$. Similarly, for the material bulk requirement $a_1 \le 0$ and $b_1 \ge a_1$, which can be manipulated to read as $s_{N1} - 2b_1 \le s_{N1} + 2|a_1|$. The left-hand side of each inequality is the diameter of the fitting surface for the surrogate feature, which from Figure 2 is seen to be $s_1 \sec c_3 + l_1 \tan c_3$, leading to the inequality $s_1 \sec c_3 + l_1 \tan c_3 - s_{N1} - 2|a_1| \le 0$. □

Note that the VBRs in these cases do not constrain c_1, c_2, c_4, or l_2, and hence the CT zone spans all values of these parameters.

Result 2

In the cases of the assembly requirement for a stud and the material bulk requirement for a hole, it is necessary that $s_1 \le s_{N1} + 2|a_1|$; the conditional tolerance on c_3 is given by

$$0 \le c_3 \le 2\tan^{-1}\left[\frac{\left(\frac{s_{N1} + 2|a_1| - s_1}{l_1}\right)}{1 + \sqrt{1 + \left(\frac{s_{N1} + 2|a_1| + s_1}{l_1}\right)\left(\frac{s_{N1} + 2|a_1| - s_1}{l_1}\right)}}\right].$$

Proof From Result 1 we have $s_1 \sec c_3 + l_1 \tan c_3 \le s_{N1} + 2|a_1|$ and $0 \le c_3 < (\pi/2)$. The above inequality cannot be satisfied for any value of c_3 in this interval if $s_1 > s_{N1} + 2|a_1|$. Thus, it is necessary that $s_1 \le s_{N1} + 2|a_1|$. To obtain the upper and lower bounds for the allowable variation in c_3, note that $s_1 \sec c_3 + l_1 \tan c_3$ is a monotonically increasing function of c_3, and that the lower bound for c_3 is 0. The upper bound is obtained by solving the equation $s_1 \sec c_{3\max} + l_1 \tan c_{3\max} = s_{N1} + 2|a_1|$. □

It is common in practice to specify orientational tolerances in terms of tolerance zones in E^3 for the axis of the surrogate cylindrical surface (limited by H_{SLAB1}). If we define a cylindrical tolerance zone of diameter $d = l_1 \tan c_3$ for the axis, then a CT for d can be derived (from Result 2) as $0 \le d \le d_{\max}$, where

$$d_{\max} = \frac{(s_{N1} + 2|a_1| - s_1)(s_{N1} + 2|a_1| + s_1)}{\left[s_{N1} + 2|a_1| + s_1\sqrt{1 + \left(\frac{s_{N1} + 2|a_1| + s_1}{l_1}\right)\left(\frac{s_{N1} + 2|a_1| - s_1}{l_1}\right)}\right]}.$$

It is interesting to note that the CT for d commonly used in current practice [2] is $0 \le d \le d'_{\max} = (s_{N1} + 2|a_1| - s_1)$, which is an approximation of the expression we have derived. Note that by using the approximation it is actually possible to accept parts that should be rejected, which means that it is not a conservative approximation. The relative difference between the maximum values permitted for d by the expressions above [i.e., $(d'_{\max} - d_{\max})/d_{\max}$] is plotted as a function of the feature size (s_1) and height (l_1) in **Figure 4**

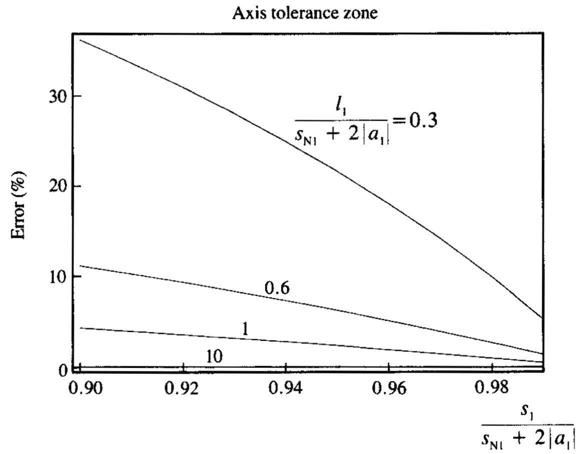

Figure 4

Error in approximating axis orientation tolerance for *loosely* toleranced cases.

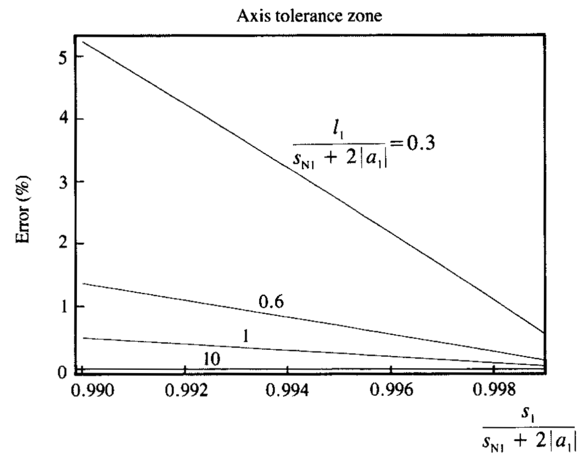

Figure 5

Error in approximating axis orientation tolerance for *tightly* toleranced cases.

(for *loosely* toleranced cases) and in **Figure 5** (for *tightly* toleranced cases). Clearly, the significance of the difference should be carefully assessed in each case before adopting the approximation.

Before we proceed further to compute the CT for Figure 3, we need some information about the symmetric axis of an ellipse.

Definition 5
A *maximal disk* for a planar object is an open circular disk that is completely contained within the object, but not in any other disk in the object. The *symmetric axis* of a planar object is the locus of the centers of all maximal disks for the object [12]. □

Property 8
The length of the symmetric axis of an ellipse, with major axis $d \sec \theta$, where $0 \le \theta < (\pi/2)$, and minor axis d, is $d \sin \theta \tan \theta$.

Proof The radius of curvature of the ellipse at the extreme points along the major axis is $(d/2) \cos \theta$. (See **Figure 6**.) Hence, the length of the symmetric axis is $d \sec \theta - d \cos \theta$, which can be reduced to $d \sin \theta \tan \theta$. □

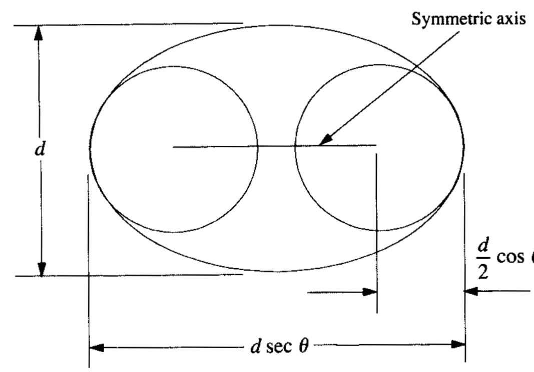

Figure 6

Symmetric axis for an ellipse.

Result 3
In the cases of the assembly requirement for a hole and the material bulk requirement for a stud, the diameter of the largest circle that inscribes both the ellipses projected onto the datum surface, denoted as d_F, is given by

$$d_F = \begin{cases} (s_1 \sec c_3 - l_1 \tan c_3) & \text{if } l_1 \ge s_1 \vee \left[l_1 < s_1 \wedge c_3 \le \sin^{-1}\left(\frac{l_1}{s_1}\right) \right], \\ \sqrt{s_1^2 - l_1^2} & \text{otherwise.} \end{cases}$$

Proof There are two possibilities (see **Figure 7**). In the first case, the symmetric axes of the two ellipses do not intersect.

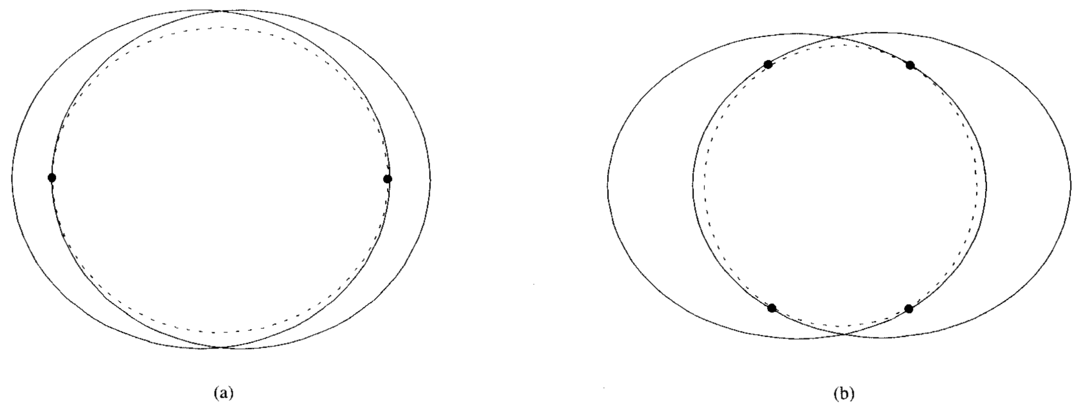

Figure 7

Inscribing circle for the projection of a fitted cylinder.

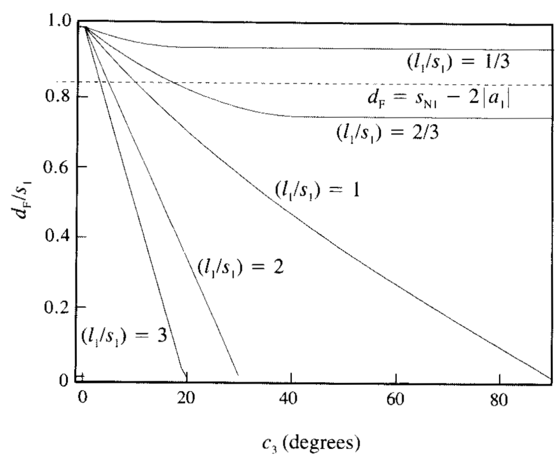

Figure 8

Variation of d_F with c_3.

Using Property 8, this condition can be stated as $l_1 \tan c_3 > s_1 \sin c_3 \tan c_3$; i.e., $\sin c_3 < (l_1/s_1)$ [which is always satisfied for $0 \le c_3 < (\pi/2)$ if $l_1 \ge s_1$]. Then the inscribed circle touches the ellipses at two points only [see Figure 7(a)], and $d_F = s_1 \sec c_3 - l_1 \tan c_3$.

In the second case, the symmetric axes of the two ellipses intersect; i.e., $\sin c_3 \ge (l_1/s_1)$. Then the inscribed circle touches the ellipses at four points [see Figure 7(b)], and $d_F = \sqrt{s_1^2 - l_1^2}$, which is, interestingly, independent of c_3. □

The variation of d_F with c_3 is shown graphically in **Figure 8**. We note that the diameter of the fitting surface for the surrogate feature in these cases is equal to d_F.

Result 4

In the cases of the assembly requirement for a hole and the material bulk requirement for a stud, the conditional tolerance zone is given by

$$Z = \left\{ \mathbf{z} \,\middle|\, \begin{array}{l} (s_1 > 0) \wedge (l_1 > 0) \wedge \left(0 \le c_3 < \dfrac{\pi}{2}\right) \wedge \\ (s_{N1} - 2|a_1| \le d_F) \end{array} \right\}.$$

Proof The first three inequalities are obvious from the definition of s_1, c_3, and l_1. From Theorem 1, a sufficient condition for satisfying the assembly requirement is $b_1 \le a_1$. Since $a_1 \ge 0$ for assembly, it follows that $s_{N1} - 2b_1 \ge s_{N1} - 2|a_1|$. Similarly, for the material bulk requirement $a_1 \le 0$ and $b_1 \ge a_1$, which can be manipulated to read as $s_{N1} + 2b_1 \ge s_{N1} - 2|a_1|$. The left-hand side of each inequality is the same as the diameter d_F of the fitting surface for the surrogate feature. Use of Result 3 in the inequalities leads to the desired result. □

The VBRs in these cases also do not constrain c_1, c_2, c_4, or l_2, and therefore the CT zone spans all values of these parameters.

Result 5

In the cases of the assembly requirement for a hole and the material bulk requirement for a stud, it is necessary that $s_1 \ge s_{N1} - 2|a_1|$, and the conditional tolerance on c_3 is given as follows: If

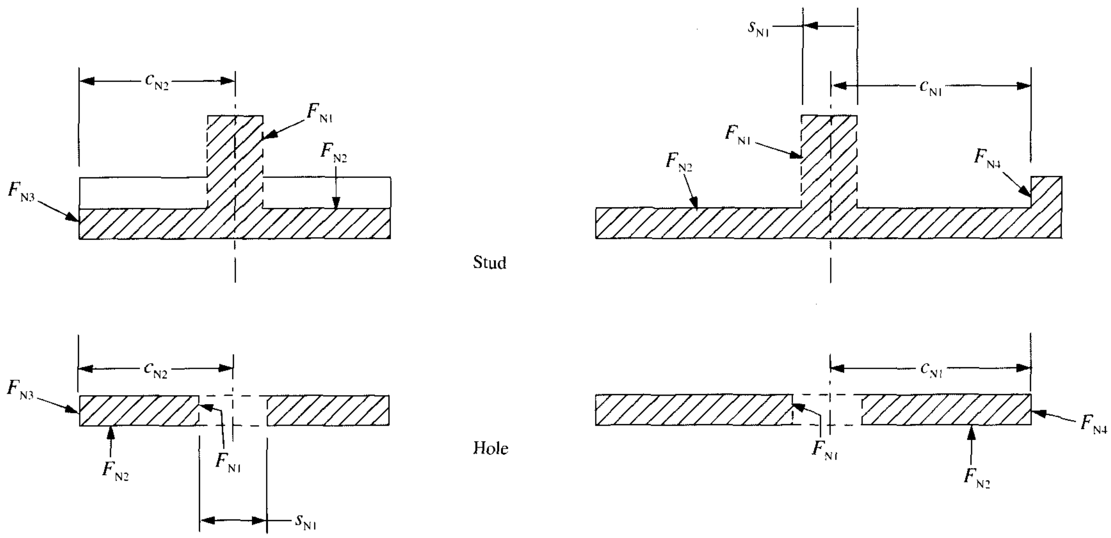

Figure 9

Nominal cylindrical features for position tolerancing.

$$l_1 \geq s_1 \vee \left[l_1 < s_1 \wedge \sqrt{s_1^2 - l_1^2} < s_{N1} - 2|a_1| \right],$$

then

$$0 \leq c_3 \leq 2 \tan^{-1} \left[\frac{\left(\frac{s_1 - s_{N1} + 2|a_1|}{l_1} \right)}{1 + \sqrt{1 - \left(\frac{s_1 + s_{N1} - 2|a_1|}{l_1} \right) \left(\frac{s_1 - s_{N1} + 2|a_1|}{l_1} \right)}} \right]$$

else $0 \leq c_3 < (\pi/2)$

Proof From Result 4 it follows that $s_{N1} - 2|a_1| \leq d_F$ and $0 \leq c_3 < (\pi/2)$. Figure 8 shows the variation of d_F as a function of c_3 for various values of l_1 and s_1 in a nondimensional form. Also shown is a *dividing* line that corresponds to $d_F = s_{N1} - 2|a_1|$. Note that any combination of parameters (s_1, l_1, c_3) that lies below this dividing line violates the VBR, whereas a combination that lies on or above the line satisfies the VBR. Since the maximum value of d_F is s_1, we conclude that it is necessary that $s_1 \geq s_{N1} - 2|a_1|$.

To derive bounds on c_3, observe that the lower bound is clearly zero. The upper bound is derived by considering the curve corresponding to a particular value of (l_1/s_1). There are two possibilities. In the first case, the curve intersects the dividing line at one point. This is true if

$$l_1 \geq s_1 \vee \left[l_1 < s_1 \wedge \sqrt{s_1^2 - l_1^2} < s_{N1} - 2|a_1| \right].$$

The upper bound is obtained by solving the equation

$$s_1 \sec c_{3\max} - l_1 \tan c_{3\max} = s_{N1} - 2|a_1|.$$

In the second case, the curve is either on or above the dividing line, and the upper bound for c_3 is clearly $(\pi/2)$. □

If we define a cylindrical tolerance zone of diameter $d = l_1 \tan c_3$ in E^3 for the axis of the surrogate cylindrical surface (limited by H_{SLAB1}), then a CT for d can be given as follows. If

$$l_1 \geq s_1 \vee \left[l_1 < s_1 \wedge \sqrt{s_1^2 - l_1^2} < s_{N1} - 2|a_1| \right],$$

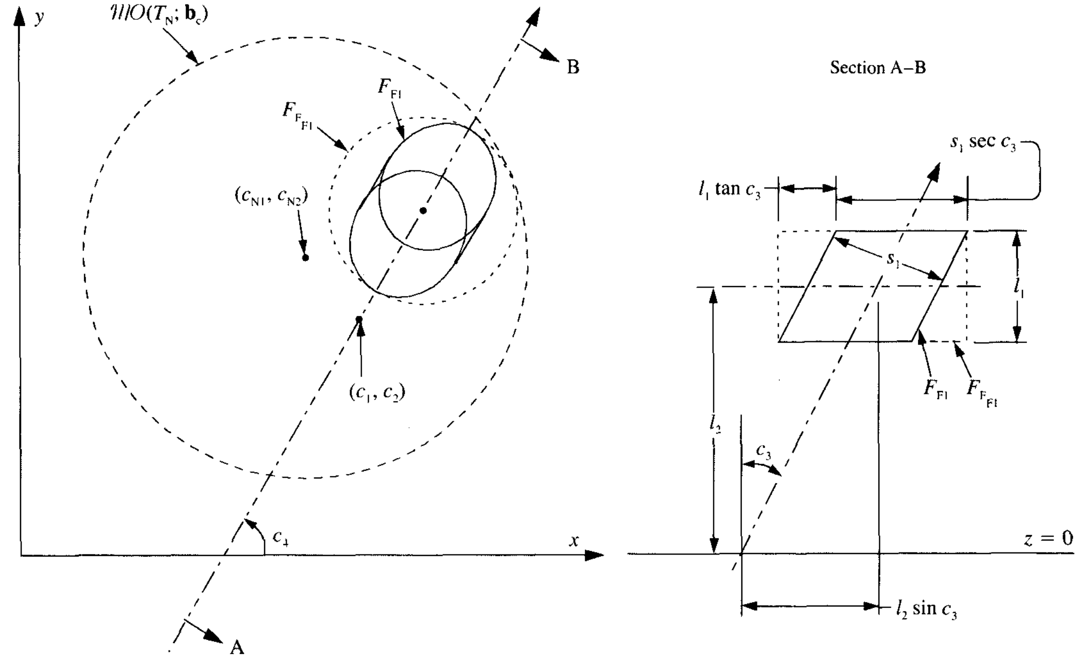

Figure 10

Position tolerancing: assembly requirement for a stud and bulk requirement for a hole.

then

$$0 \le d \le \frac{(s_1 - s_{N1} + 2|a_1|)(s_{N1} - 2|a_1| + s_1)}{\left[s_{N1} - 2|a_1| + s_1 \sqrt{1 - \left(\dfrac{s_1 + s_{N1} - 2|a_1|}{l_1}\right)\left(\dfrac{s_1 - s_{N1} + 2|a_1|}{l_1}\right)}\right]},$$

else $0 \le d < \infty$. It is once again interesting to note that the CT for d commonly used in current practice [2] is $0 \le d \le (s_1 - s_{N1} + 2|a_1|)$. In this case, however, current practice is conservative; parts that are acceptable functionally may be rejected.

Position tolerance

VBRs which refer to a datum system with three mutually orthogonal planar datums and which are asserted on single cylindrical features constrain the axes of the corresponding virtual cylindrical surfaces completely. In all such cases we consider here, the nominal attitude angle is zero (see [6] for examples of such requirements).

Figure 9 shows two nominal cylindrical features, denoted by F_{N1}, having a nominal diameter of s_{N1}. The member of the nominal tolerance set of interest here is $T_N = \{F_{N1}\}$, and F_{N2}, F_{N3}, and F_{N4} are mutually orthogonal planar features. $\{F_{N2}\}$, $\{F_{N3}\}$, and $\{F_{N4}\}$ have been designated as primary, secondary, and tertiary datum features of a complete datum system. F_{N1} is nominally oriented so that its axis is perpendicular to F_{N2} and located at distances c_{N2} from F_{N3} and c_{N1} from F_{N4}. The xyz coordinate system associated with the solid has the primary datum as the xy plane, the secondary datum as the xz plane, and the tertiary datum as the yz plane. We consider all combinations of assembly and material bulk requirements asserted on studs and holes, as previously. They are all characterized by the set of scalars $\mathbf{a} = \{a_1\}$ and refer to the datum system mentioned above. Figures illustrating actual instances of cylindrical features satisfying such VBRs are not provided here because they are too complex to draw. Instead, shown in **Figure 10** is the projection onto the xy plane and a specific cross section of the rigid collection of surrogate surface features $T_F = \{F_{F1}\}$ associated with the member of the actual tolerance set $T_A = \{F_{A1}\}$ of an actual solid. Figure 10 indicates the

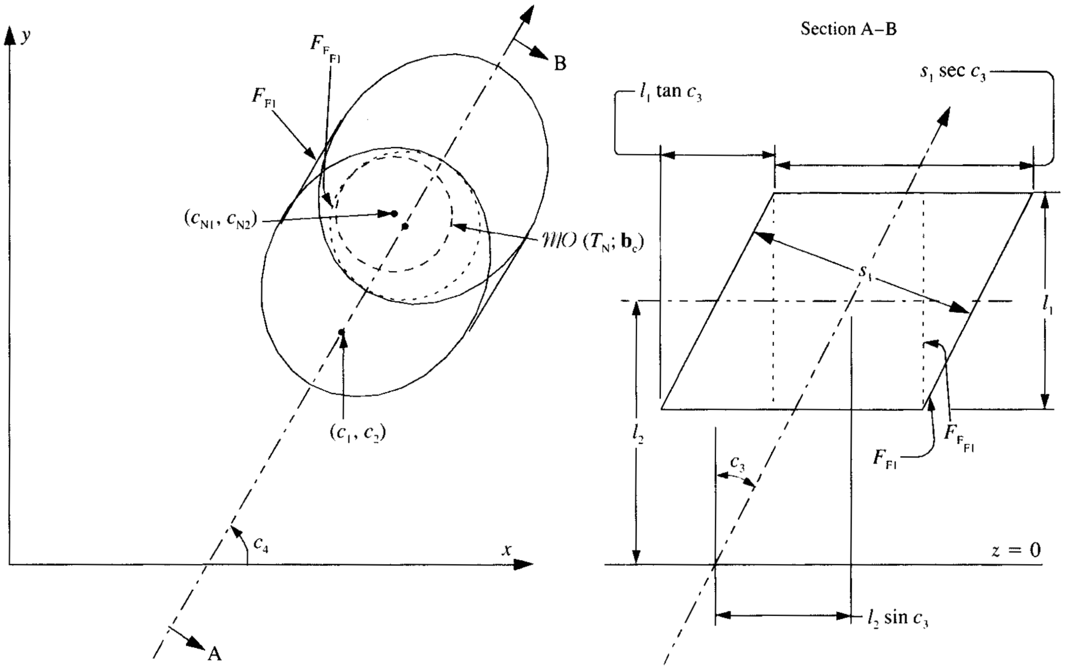

Figure 11

Position tolerancing: assembly requirement for a hole and bulk requirement for a stud.

relevant parameters and geometric constructions in the cases of the assembly requirement for a stud and the bulk requirement for a hole. Similarly, the cases of the assembly requirement for a hole and the bulk requirement for a stud are shown in **Figure 11**.

In all of these cases, finding the exact functions $\mathcal{F}$ in closed form for defining Z is a very difficult task. Essentially, in the cases depicted in Figure 10, this would require deriving the diameter of the smallest circular disk with its center at (c_{N1}, c_{N2}), and which encloses the projection of T_F; in those depicted in Figure 11, this would require the derivation of the diameter of the largest circular disk with its center at (c_{N1}, c_{N2}), and which is enclosed by the projection of T_F. Both would involve finding the zeros of fourth-order polynomials in closed form. The approach we use in these cases is to show that by using a conservative approximation to the surrogate feature, we can simplify the problem and determine relatively easily a set of functions $\mathcal{F}_C$ that characterize a subset of Z. In other words, we can obtain a conservative characterization of the CT zone as $Z \supseteq Z_C = \{\mathbf{z}: \mathcal{F}_C(\mathbf{z}) \le 0\}$. We refer to Z_C as a *conservative* CT zone.

Returning to Figure 10, we have shown a cylindrical surface with the smallest diameter that surrounds F_{F1} and is perpendicular to the primary datum. This cylindrical surface limited by H_{SLAB1} is denoted as $F_{F_{F1}}$. Note that $T_{T_F} = \{F_{F_{F1}}\}$ serves as a conservative approximation to T_F in deriving Z_C.

That is, the rigid collection of fitting half-spaces for T_F, denoted as $\mathcal{M}O(T_N; \mathbf{b})$, where $\mathbf{b} = \{b_1\}$, and the rigid collection of fitting half-spaces for T_{T_F}, denoted as $\mathcal{M}O(T_N; \mathbf{b}_C)$, where $\mathbf{b}_C = \{b_{C1}\}$, are such that $b_1 \le b_{C1}$ for assembly and $b_1 \ge b_{C1}$ for material bulk maintenance. The reader should be able to convince himself of this intuitively. Similarly, Figure 11 shows a cylindrical surface with the largest diameter that is surrounded by F_{F1} and is perpendicular to the primary datum. Here again, $b_1 \le b_{C1}$ for assembly and $b_1 \ge b_{C1}$ for material bulk maintenance.

We now derive conservative CT zones for the VBRs illustrated in these figures.

Result 6

In the cases of the assembly requirement for a stud and the material bulk requirement for a hole (Figure 10), which refer to a complete datum system composed of three mutually perpendicular planar datums, a conservative conditional tolerance zone is given by

$$Z_C = \left\{ \mathbf{z} \;\middle|\; \begin{array}{l} (s_1 > 0) \wedge (l_1 > 0) \wedge \left(0 \le c_3 < \dfrac{\pi}{2}\right) \wedge \\ (2o + s_1 \sec c_3 + l_1 \tan c_3 - s_{N1} - 2|a_1| \le 0) \end{array} \right\},$$

where

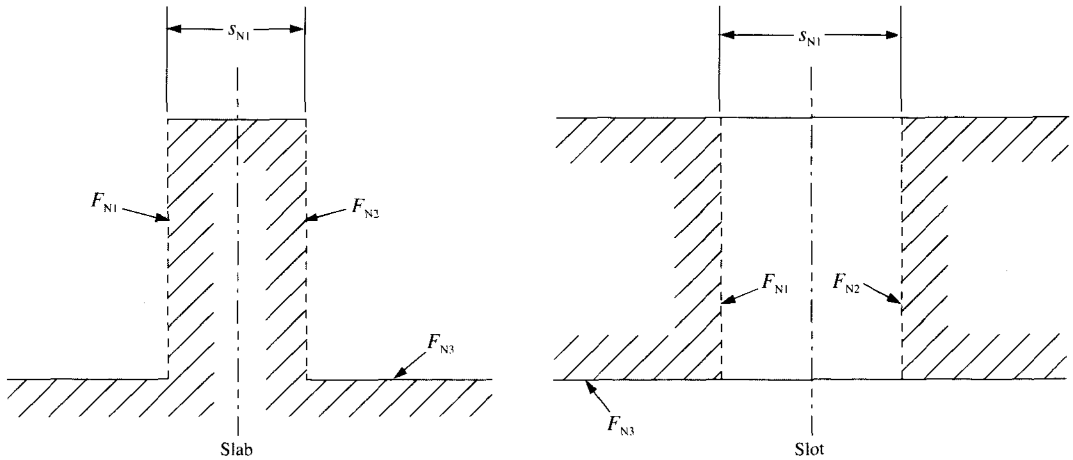

Figure 12

Nominal planar features in slab/slot combination for orientation tolerancing.

$$o = \sqrt{(c_1 + l_2 \sin c_3 \cos c_4 - c_{N1})^2 + (c_2 + l_2 \sin c_3 \sin c_4 - c_{N2})^2}.$$

Proof The first three inequalities are obvious from the definition of s_1, l_1, and c_3. From Theorem 1, and using T_{T_F} in place of T_F, we can write a sufficient condition for satisfying the VBR as $s_{N1} \pm 2b_1 \le s_{N1} \pm 2b_{C1} \le s_{N1} + 2|a_1|$, with the + sign for assembly and the − sign for material bulk. From Figure 10, we can write the distance between points (c_{N1}, c_{N2}) and the center of the projection of $F_{F_{F1}}$ as

$$o = \sqrt{(c_1 + l_2 \sin c_3 \cos c_4 - c_{N1})^2 + (c_2 + l_2 \sin c_3 \sin c_4 - c_{N2})^2}.$$

and the diameter of the fitting surface for $F_{F_{F1}}$ as $s_{N1} \pm 2b_{C1} = (2o + s_1 \sec c_3 + l_1 \tan c_3)$, which leads to the last inequality. □

Note that the VBRs in these cases constrain all of the surrogate parameters.

Result 7

In the cases of the assembly requirement for a hole and the material bulk requirement for a stud (Figure 11), which refer to a complete datum system composed of three mutually perpendicular planar datums, a conservative conditional tolerance zone is given by

$$Z_C = \left\{ \mathbf{z} \;\middle|\; \begin{array}{l} (s_1 > 0) \wedge (l_1 > 0) \wedge \left(0 \le c_3 < \frac{\pi}{2}\right) \wedge \\ (s_{N1} - 2|a_1| \le d_F - 2o) \end{array} \right\},$$

where

$$o = \sqrt{(c_1 + l_2 \sin c_3 \cos c_4 - c_{N1})^2 + (c_2 + l_2 \sin c_3 \sin c_4 - c_{N2})^2}.$$

Proof Similar to the proof of Result 6. See [6] for details. □

Note that the VBRs in these cases constrain all of the surrogate parameters. Next, we examine the VBRs for a set of two parallel planar features constituting a slab or a slot.

• *Planar features*

Consider a $T_N = \{F_{N1}, F_{N2}\}$, where F_{N1} and F_{N2} are planar features parallel to each other and forming a slab or a slot, with an associated grouping $\mathbf{G}_N = \{G_1\}$, where $G_1 = \{F_{N1}, F_{N2}\}$. The rigid collection of fitting half-spaces for the corresponding T_A is composed of a set of two parallel planar half-spaces characterized by the distance s_1 between their bounding planes. (See Figure 16 or 17, shown later.) The median plane for the fitting surfaces can be specified by one locational parameter c_1, which is the y coordinate of the intersection of the plane with the y axis and two orientational parameters, c_2 (the *azimuth* angle defined as the positive rotation around the z axis necessary to make the unit vector along the x axis parallel to $\mathbf{u}_Z \times \mathbf{u}_{ZP}$, where $\mathbf{u}_Z$ is the unit vector along the z axis and $\mathbf{u}_{ZP}$ is the projection of

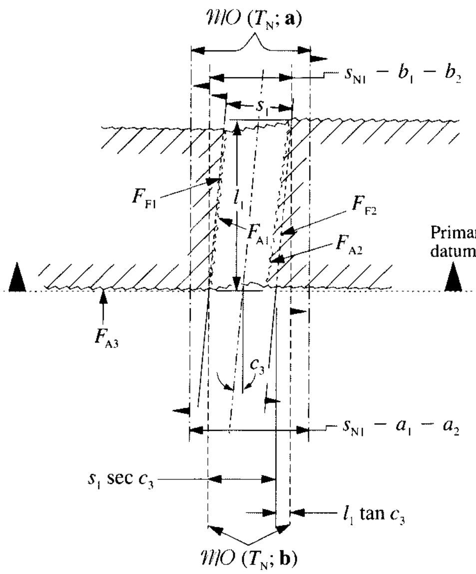

Figure 13

Orientation tolerancing: material bulk requirement for a slot.

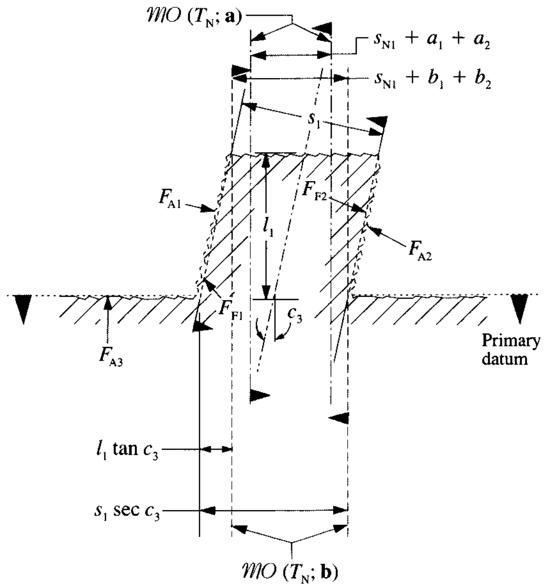

Figure 14

Orientation tolerancing: material bulk requirement for a slab.

$\mathbf{u}_Z$ onto the median plane) and c_3 (the *attitude* angle defined as the acute angle between $\mathbf{u}_Z$ and $\mathbf{u}_{ZP}$). The limiting process mentioned earlier introduces two parameters, l_1 and l_2. In addition, we further limit the fitting surfaces with another slab (H_{SLAB2}) of thickness l_3 (minimally sufficient to enclose T_A), whose median plane is perpendicular to the xy plane and further constrained such that only one parameter l_4 is needed to specify its location. Thus, the parametric space for a set of two parallel planar surrogate features (derived using the limiting process described above) is the space R^8 spanned by the set of parameters $\mathbf{z} \equiv (s_1, c_1, c_2, c_3, l_1, l_2, l_3, l_4)$. A VBR may impose a limit on the range of variation of only a subset of these parameters, as shown below.

Orientation tolerance

Consider VBRs that refer to a datum system with just a primary planar datum. The median planes for the two virtual planar surfaces in these cases are constrained to have specified attitude angles, but their locations and azimuth angles are unconstrained. In all of the cases we consider, the nominal attitude angle is zero. The median plane of H_{SLAB2} must, in addition, be perpendicular to the median plane of the fitting surfaces; the parameter l_4 is the x coordinate of the intersection of the median plane of H_{SLAB2} with the x axis.

Figure 12 shows two nominal planar features denoted by F_{N1} and F_{N2}, forming a slab and a slot. The nominal distance between the two planar features is s_{N1}. The median plane is so oriented that it is perpendicular to a planar feature F_{N3}, and $\{F_{N3}\}$ has been designated as a datum feature. The member of the nominal tolerance set of interest here is $T_N = \{F_{N1}, F_{N2}\}$. **Figures 13** and **14** show actual instances of a slot and a slab, respectively, satisfying a material bulk requirement. (See [6] for similar figures for the cases of assembly requirements for slabs and slots.) All of these VBRs refer to a datum system that consists of only a primary planar datum associated with the datum feature $\{F_{A3}\}$. They are characterized by the set of scalars $\mathbf{a} = \{a_1, a_2\}$. The member of the actual tolerance set of interest is $T_F = \{F_{A1}, F_{A2}\}$. With a grouping $\mathbf{G}_A = \{\{F_{A1}, F_{A2}\}\}$, the rigid collection of surrogate surface features $T_F = \{F_{F1}, F_{F2}\}$ is characterized by a distance parameter s_1. Other indicated parameters are the limiting parameter l_1 and the attitude angle c_3. The appropriate fitting half-spaces for T_F are indicated in the figures as $\mathcal{MO}(T_N; \mathbf{b})$, where $\mathbf{b} = \{b_1, b_2\}$.

We assume, without any loss of generality, that $b_1 = b_2$ and $a_1 = a_2$. We now derive the CT zones for the VBRs depicted in these figures.

Result 8

In the cases of the assembly requirement for a slab and the material bulk requirement for a slot, the conditional tolerance zone is given by

$$Z = \left\{ \mathbf{z} \;\middle|\; \begin{array}{l} (s_1 > 0) \wedge (l_1 > 0) \wedge (l_3 > 0) \wedge \left(0 \le c_3 < \dfrac{\pi}{2}\right) \wedge \\ (s_1 \sec c_3 + l_1 \tan c_3 - s_{N1} - 2\,|\,a_1\,| \le 0) \end{array} \right\}.$$

Proof Similar to the proof of Result 1. □

Note that the VBRs here do not constrain c_1, c_2, l_2, or l_4, and hence the CT zone spans all values of these parameters.

Result 9
In the cases of the assembly requirement for a slab and the material bulk requirement for a slot, it is necessary that $s_1 \le s_{N1} + 2|a_1|$, and the conditional tolerance on c_3 is given by

$$0 \le c_3 \le 2\tan^{-1}\left[\frac{\left(\dfrac{s_{N1} + 2|a_1| - s_1}{l_1}\right)}{1 + \sqrt{1 + \left(\dfrac{s_{N1} + 2|a_1| + s_1}{l_1}\right)\left(\dfrac{s_{N1} + 2|a_1| - s_1}{l_1}\right)}}\right].$$

Proof Similar to the proof of Result 2. □

It is common practice to specify orientational tolerances in terms of tolerance zones in E^3 for the median plane of the surrogate planar surfaces (limited by H_{SLAB1}). If we define a slab tolerance zone of width $d = l_1 \tan c_3$ for the median plane, then a CT for d can be derived as

$$0 \le d \le \frac{(s_{N1} + 2|a_1| - s_1)(s_{N1} + 2|a_1| + s_1)}{\left[s_{N1} + 2|a_1| + s_1\sqrt{1 + \left(\dfrac{s_{N1} + 2|a_1| + s_1}{l_1}\right)\left(\dfrac{s_{N1} + 2|a_1| - s_1}{l_1}\right)}\right]}.$$

Just as for the case of cylindrical features, the CT for d commonly used in current practice [2] is $0 \le d \le (s_{N1} + 2|a_1| - s_1)$, permitting the acceptance of parts that fail to meet functional requirements.

Result 10
In the cases of the assembly requirement for a slot and the material bulk requirement for a slab, the conditional tolerance zone is given by

$$Z = \left\{ \mathbf{z} \;\middle|\; \begin{array}{l} (s_1 > 0) \wedge (l_1 > 0) \wedge (l_3 > 0) \wedge \left(0 \le c_3 < \dfrac{\pi}{2}\right) \wedge \\ (s_{N1} - 2|a_1| - s_1 \sec c_3 + l_1 \tan c_3 \le 0) \end{array} \right\}.$$

Proof The first four inequalities are obvious from the definition of s_1, l_1, l_3, and c_3. From Theorem 1, we can write a sufficient condition for satisfying the VBR as $s_{N1} \mp 2b_1 \ge s_{N1} - 2|a_1|$, with the − sign for assembly and the + sign for material bulk. From Figure 14 we can write $s_{N1} \mp 2b_1 = s_1 \sec c_3 - l_1 \tan c_3$, which leads to the inequality $s_{N1} - 2|a_1| - s_1 \sec c_3 + l_1 \tan c_3 \le 0$. □

Result 11
In the cases of the assembly requirement for a slot and the material bulk requirement for a slab, it is necessary that $s_1 \ge s_{N1} - 2|a_1|$, and the conditional tolerance on c_3 is given by

$$0 \le c_3 \le 2\tan^{-1}\left[\frac{\left(\dfrac{s_1 - s_{N1} + 2|a_1|}{l_1}\right)}{1 + \sqrt{1 - \left(\dfrac{s_1 + s_{N1} - 2|a_1|}{l_1}\right)\left(\dfrac{s_1 - s_{N1} + 2|a_1|}{l_1}\right)}}\right].$$

Proof Similar to the proof of Result 5. □

If we define a slab tolerance zone of width $d = l_1 \tan c_3$ for the median plane of the surrogate planar surfaces (limited by H_{SLAB1}), then a CT for d can be given as

$$0 \le d \le \frac{(s_1 - s_{N1} + 2|a_1|)(s_{N1} - 2|a_1| + s_1)}{\left[s_{N1} - 2|a_1| + s_1\sqrt{1 - \left(\dfrac{s_1 + s_{N1} - 2|a_1|}{l_1}\right)\left(\dfrac{s_1 - s_{N1} + 2|a_1|}{l_1}\right)}\right]}.$$

It is once again interesting to note that the CT for d commonly used in current practice [2] is $0 \le d \le (s_1 - s_{N1} + 2|a_1|)$, which is conservative.

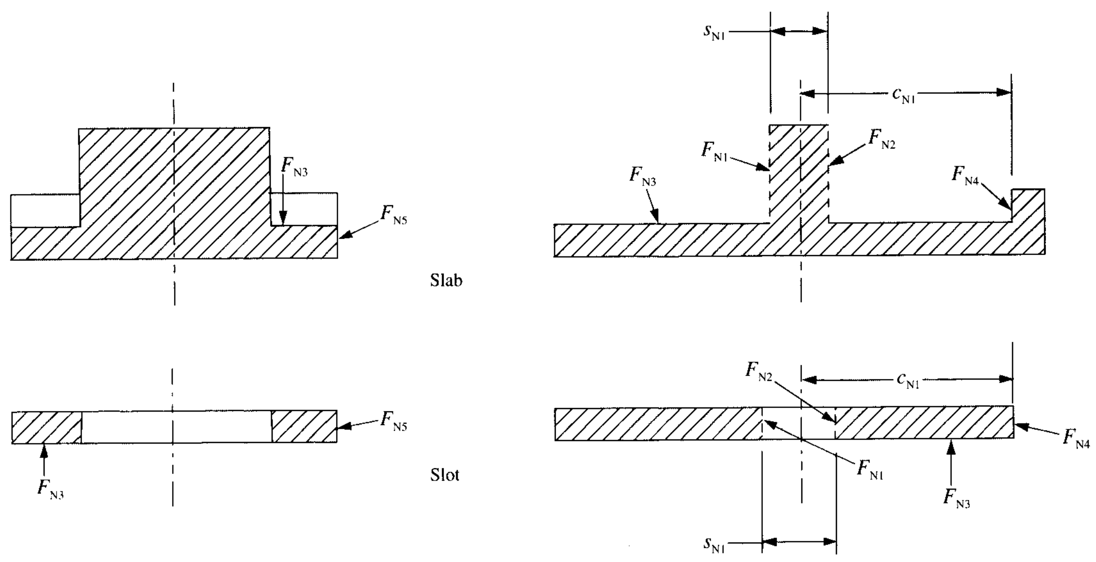

Figure 15

Nominal planar features in slab/slot combination for position tolerancing.

Position tolerance
Consider VBRs that refer to a datum system with three mutually orthogonal planar datums and are asserted on pairs of parallel planar features forming slabs or slots. The median planes of the two virtual planar surfaces in these cases are completely constrained. In all of the cases we consider, the nominal attitude angle and the nominal azimuth angle are zero. In addition, the median plane of H_{SLAB2} is required to be perpendicular to the x axis, and l_4 is the x coordinate of the intersection of this median plane with the x axis.

Figure 15 shows two nominal planar features, denoted by F_{N1} and F_{N2}, forming a slab and a slot. s_{N1} is the nominal distance between the two planar features. $T_N = \{F_{N1}, F_{N2}\}$ is the relevant member of the nominal tolerance set, and F_{N3}, F_{N4}, and F_{N5} are mutually orthogonal planar features. $\{F_{N3}\}$, $\{F_{N4}\}$, and $\{F_{N5}\}$ have been designated as primary, secondary, and tertiary datum features of a complete datum system. The median plane of F_{N1} and F_{N2} is nominally oriented such that it is perpendicular to F_{N3}, parallel to F_{N4}, and located at a distance of c_{N1} from F_{N4}. The xyz coordinate system associated with the solid has the primary datum as the xy plane, the secondary datum as the xz plane, and the tertiary datum as the yz plane. We consider all combinations of assembly and material bulk requirements for slabs and slots, as previously. They are all characterized by the set of scalars $\mathbf{a} = \{a_1, a_2\}$ and refer to the datum system mentioned above. No figures are provided to illustrate actual instances of slabs and slots satisfying such VBRs, because they would be too complex. Instead, shown in **Figure 16** is the projection onto the xy plane and a specific cross section of the rigid collection of surrogate surface features $T_F = \{F_{F1}, F_{F2}\}$ associated with the member of the actual tolerance set $T_A = \{F_{A1}, F_{A2}\}$ of an actual solid. Figure 16 is for an assembly requirement for a slab and a bulk requirement for a slot; it indicates the relevant parameters and geometric constructions. Similarly, **Figure 17** depicts an assembly requirement for a slot and a bulk requirement for a slab. The appropriate fitting half-spaces for T_F are indicated in the figures as $\mathcal{MO}(T_N; \mathbf{b})$, where $\mathbf{b} = \{b_1, b_2\}$. We now derive the CT zones for the VBRs depicted in those figures.

Result 12
In the cases of the assembly requirement for a slab and the material bulk requirement for a slot, which refer to a complete datum system composed of three mutually perpendicular planar datums (Figure 16), the conditional tolerance zone is given by

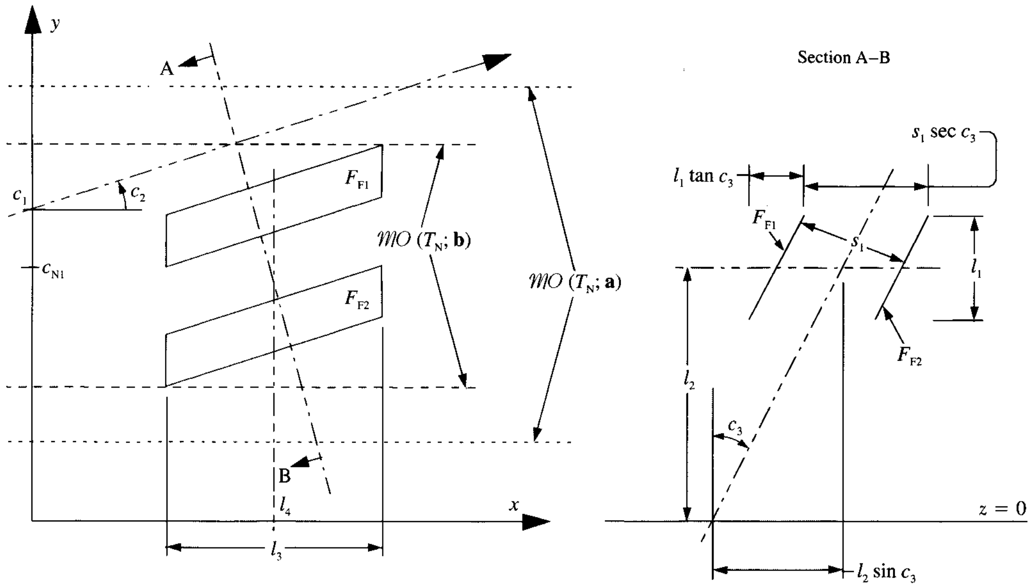

Figure 16

Position tolerancing: assembly requirement for a slab and bulk requirement for a slot.

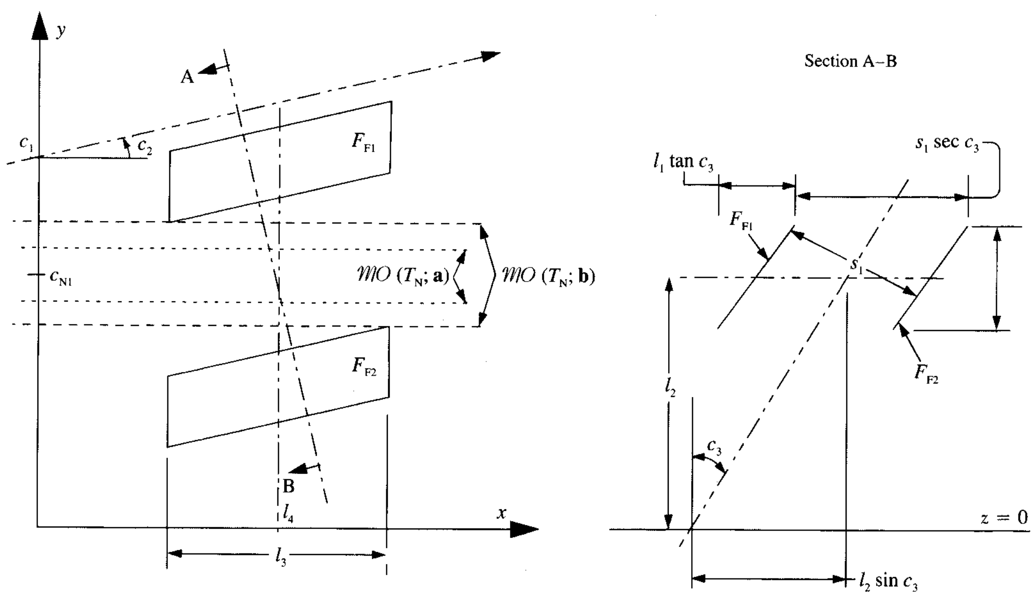

Figure 17

Position tolerancing: assembly requirement for a slot and bulk requirement for a slab.

$$Z = \left\{ \mathbf{z} \,\middle|\, \begin{array}{l} (s_1 > 0) \wedge (l_1 > 0) \wedge (l_3 > 0) \wedge \left(0 \le c_3 < \frac{\pi}{2}\right) \wedge \\ \left(y_{\max} - c_{N1} - \frac{s_{N1}}{2} - |a_1| \le 0\right) \wedge \left(c_{N1} - \frac{s_{N1}}{2} - |a_1| - y_{\min} \le 0\right) \end{array} \right\},$$

where $y_{\max} = \boldsymbol{y}(+1)$, $y_{\min} = \boldsymbol{y}(-1)$,

$$\boldsymbol{y}(k) = \left(l_4 + \frac{ikl_3}{2}\right) \tan c_2 + c_1 - \left[\left(l_2 - \frac{jkl_1}{2}\right) \tan c_3 - \frac{jks_1}{2} \sec c_3\right] \sec c_2,$$

$i = \text{sign}(\tan c_2)$, $j = \text{sign}(\cos c_2)$, and

$$\text{sign}(x) = \begin{cases} +1 & \text{if } x \ge 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Proof The first four inequalities are obvious from the definition of s_1, c_3, l_1, and l_3. From Theorem 1 we obtain a sufficient condition for satisfying the assembly VBR as $b_1 \le a_1$ and $b_2 \le a_2$, which can be rewritten as

$$y_{\max} \equiv c_{N1} + \frac{s_{N1}}{2} + b_1 \le c_{N1} + \frac{s_{N1}}{2} + |a_1|$$

and

$$c_{N1} - \frac{s_{N1}}{2} - |a_2| \le c_{N1} - \frac{s_{N1}}{2} - b_2 \equiv y_{\min}.$$

Referring to Figure 16, the expression for the y coordinate of the eight vertices of T_F can be written as

$$y_i = \left(l_4 \pm \frac{l_3}{2}\right) \tan c_2 + c_1 - \left[\left(l_2 \pm \frac{l_1}{2}\right) \tan c_3 \pm \frac{s_1}{2} \sec c_3\right] \sec c_2.$$

By using these expressions, we can obtain the maximum and the minimum values for the y coordinate of the vertices of F_{F1} and F_{F2}, respectively; the results lead to the desired inequalities. □

Note that the VBRs in these cases constrain all of the surrogate parameters.

Result 13

In the cases of the assembly requirement for a slot and the material bulk requirement for a slab, which refer to a complete datum system composed of three mutually perpendicular planar datums (Figure 17), the conditional tolerance zone is given by

$$Z = \left\{ \mathbf{z} \,\middle|\, \begin{array}{l} (s_1 > 0) \wedge (l_1 > 0) \wedge (l_3 > 0) \wedge \left(0 \le c_3 < \frac{\pi}{2}\right) \wedge \\ \left(y_{\max} - c_{N1} + \frac{s_{N1}}{2} - |a_2| \le 0\right) \wedge \left(c_{N1} + \frac{s_{N1}}{2} - |a_1| - y_{\min} \le 0\right) \end{array} \right\},$$

Similarly, for the material bulk requirement, $b_1 \ge a_1$ and $b_2 \ge a_2$, which can be rewritten as

$$y_{\max} \equiv c_{N1} + \frac{s_{N1}}{2} - b_1 \le c_{N1} + \frac{s_{N1}}{2} + |a_1|$$

and

$$c_{N1} - \frac{s_{N1}}{2} - |a_2| \le c_{N1} - \frac{s_{N1}}{2} + b_2 \equiv y_{\min}.$$

Note that $y_{\max}$ here represents the maximum value of the y coordinate of the vertices of F_{F1}, and $y_{\min}$ represents the minimum value of the y coordinate of the vertices of F_{F2}.

where $y_{\max} = \boldsymbol{y}(+1)$, $y_{\min} = \boldsymbol{y}(-1)$,

$$\boldsymbol{y}(k) = \left(l_4 + \frac{ikl_3}{2}\right) \tan c_2 + c_1 - \left[\left(l_2 - \frac{jkl_1}{2}\right) \tan c_3 + \frac{jks_1}{2} \sec c_3\right] \sec c_2,$$

$i = \text{sign}(\tan c_2)$, $j = \text{sign}(\cos c_2)$, and

$$\text{sign}(x) = \begin{cases} +1 & \text{if } x \ge 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Proof Similar to the proof of Result 12. See [6] for details. □

Note that the VBRs in these cases constrain all of the surrogate parameters.

Concluding remarks

We have examined the conversion of VBRs to CTs, which facilitate part fabrication and inspection. CTs specify allowable variations in the geometric parameters of features. A theoretical basis has been developed for deriving CTs from VBRs, and CTs have been derived for frequently occurring VBRs.

A number of issues need to be studied further. We have not provided sufficient conditions for ensuring that a given procedure to limit fitting surfaces results in surrogate features. We conjecture that inclusion of the projection of actual features onto fitting surfaces is sufficient to guarantee the surrogate property. We have not been able to prove this. Our attempts to do so have led us to believe that we must first examine the general properties of visible regions of surface features, introduced for formalizing the concept of *being on the proper material side* of actual surface features.

A general-purpose algorithm for deriving CT zones from VBRs is not known. In the case of VBRs asserted on a pattern of simple features (e.g., studs, holes, slabs, or slots), we believe that the tolerance zones can be computed as intersections of zones for each member of the pattern. Deriving the tolerance zones for a simple feature itself is often difficult, as can be appreciated from the derivation of the positional tolerance for a single cylindrical feature. In that case, the problem is reduced to one of finding the configuration space obstacle [13] for a planar object and a planar obstacle, both of which are bounded by curves. The reader is referred to [14] for some recent work on algebraic algorithms for generating the boundary of configuration space obstacles for planar objects and obstacles bounded by algebraic curves and subjected to pure translatory motions.

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