Information transport obeying the continuity equation

by Tommaso Toffoli

We analyze nontrivial dynamical systems in which information flows as an additive conserved quantity—and thus takes on a strikingly tangible aspect. To arrive at this result, we first give an explicit characterization of equilibria for a family of lattice gases.

1. Introduction

In many spatially extended dynamical systems governed by short-range interactions—such as an ordinary fluid—one encounters additive conserved quantities (e.g., energy, electric charge). As the system evolves, these quantities continually redistribute themselves. Though the details of this shuffle depend on the specific dynamics, the flow of each quantity obeys a continuity equation: Any amount that disappears from one place at one moment must reappear somewhere in the immediate vicinity at the next moment.

In an invertible system, information (or "fine-grained entropy") is always conserved.[1] However, information is not, in general, additive. During the evolution of a system, correlations almost invariably arise between initially uncorrelated variables. Because of spatial correlations, if the volume occupied by the system is partitioned into several pieces and the information from all the individual pieces is added up, the result is not the total information of the current macroscopic state—which does not change with time as the distribution evolves—but an overestimate which may fluctuate widely with the passage of time.

In other words, information, though conserved in a global sense, is not localized, and one cannot write for it transport equations of the kind that are familiar for energy, momentum, etc.

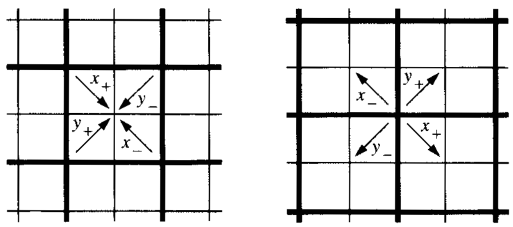
In this paper we present and analyze a situation where information flow strictly obeys the continuity equation, and thus takes on a strikingly tangible aspect. This situation is not limited to trivial systems; on the contrary, it arises in a wide class of systems of concrete interest, including some that support a full-featured hydrodynamics and others that are known to be computation-universal.

The situation we have in mind is characterized by the following four conditions:

a. Small perturbations
b. from an equilibrium state
c. of particle-conserving
d. invertible cellular automata.

Though additivity of information may well be a more widespread phenomenon, the above conditions allow us to arrive at an exact proof in a number of interesting cases (and we conjecture that additivity follows in general from these conditions). In particular, we take advantage of the fact that a wide class of equilibrium states for systems obeying conditions (c) and (d) are explicitly computable, as is shown in the next section.

[1] Note that information is a quantity associated with a distribution of states rather than with an individual state. (The "macroscopic states" or "statistical ensembles" considered in statistical mechanics are examples of distribution.) For a large system, it may be meaningful to apply the term "information" to an individual microstate, treated as a representative of a distribution; this approach is formalized by algorithmic information theory. Our discussion is consistent with this extended meaning, but does not rely on it.



(a) (b)

Figure 1

Even grid (a) and odd grid (b) in a cellular-automaton implementation of the HPP lattice gas. The squares represent cells, the solid lines, 2×2 blocks. Each arrow denotes the direction in which a particle contained in the corresponding cell is moving.

We assume some familiarity with the concepts of information and correlation, as presented, for instance, in [1].

If X is an arbitrary set of objects, called "microstates," a (probability) distribution over this set is an assignment of a nonnegative weight $P(x)$ to each microstate x such that the sum of the weights equals unity. The information of the distribution P is defined as

$$s(P) = -\sum_{x \in X} P(x) \log P(x). \tag{1}$$

In the special case where X consists of just two microstates, 0 and 1, the distribution P is completely determined by the number $p = P(0)$ [since $P(1) = 1 - p$], and can be identified (by convention) with that number. In this case, (1) reduces to the well-known information function

$$s(p) = -(p \log p + \bar{p} \log \bar{p}) \tag{2}$$

(where $\bar{p}$ denotes $1 - p$). In what follows, we also make use of the first two derivatives of the information function, namely

$$r(p) = \frac{ds}{dp} = -\log \frac{p}{\bar{p}}, \tag{3}$$

$$u(p) = \frac{d^2 s}{dp^2} = -\frac{1}{p\bar{p}}. \tag{4}$$

2. Explicitly known equilibrium distributions

Many dynamical systems arise as a stylization of a physical problem, where experience or intuition suggests the existence of one or more equilibrium states, i.e., time-invariant distributions of microstates. Usually, however, only some of the equilibrium properties (say, the energy distribution) can be explicitly calculated; a complete and explicit characterization of the entire distribution of microstates is seldom available. The Bernoulli shifts and similar "toy" systems are given particular stress in the teaching of ergodic dynamics precisely because such a characterization is known for them.

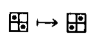
We prove that a wide class of equilibrium states can be completely determined in the case of particle-conserving, invertible cellular automata—of which invertible lattice gases are a special case.

As a preliminary, let us contrast the way equilibrium is reached in two simple, well-understood systems—namely a deterministic Ising spin system and a lattice gas (both briefly described below)—which are defined by laws having very similar formats: In both cases we have a regular array of binary variables governed by a time-discrete, local, and uniform dynamics (thus, we are dealing with cellular automata). Moreover, both systems are deterministic and invertible (i.e., microscopically reversible). For each of these systems we intend to study the evolution of the corresponding microcanonical ensemble, started from a known initial distribution of microstates.[2]

In the Ising system we are considering, called Q2R [2, 3], 0 and 1 represent the two possible spin orientations. Using a well-known technique (cf. [3]), the even and odd "checkerboard" subarrays are updated on alternating steps. Each spin changes state if and only if it is in an indifferent energetic situation with respect to its four nearest neighbors, i.e., if exactly two of its neighbors are up and two down.

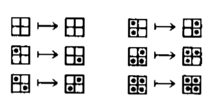
In the lattice gas we are considering, called HPP [3, 4], 1 represents the presence and 0 the absence of a particle. Using another well-known technique [3, 5], the even and odd "grid partitions" of the array are updated on alternating steps. We recall that in a grid partition cells are grouped into 2×2 blocks, the four cells of each block representing the four possible directions of travel, all pointing toward the center of the block, as illustrated in **Figure 1**. For instance, a particle in the upper-left cell of a block [labeled x_+ in part (a)] moves diagonally down and right, and is found after one step in the opposite corner of the block unless a collision occurs. The grid used on odd steps straddles that used on even steps, so that at the beginning of the next step the above particle again appears in the upper-left cell of a block [labeled x_+ in part (b)].

Particles travel straight, with one exception: When exactly two particles coming from opposite directions collide, they bounce off in the other two directions:



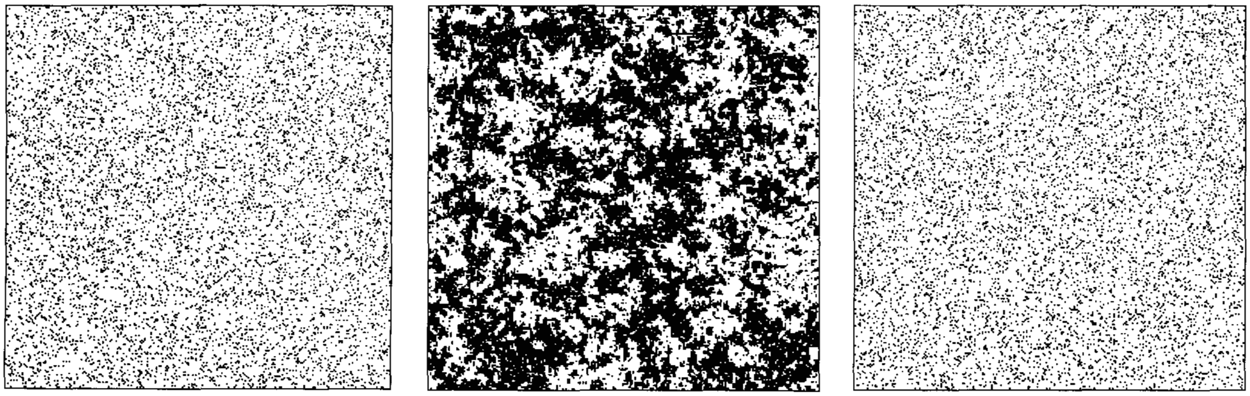
(5)

The following table lists (up to a rotation) all possible cases:



(6)

[2] It should be clear that a microstate is a state of the entire array, not of an individual cell.



(a) (b) (c)

Figure 2

(a) Initial uniform configuration. Final configuration in (b) Ising spin model and (c) lattice gas.

Note that this interaction is particle-conserving as well as deterministic and invertible.

The initial distribution we consider—which we shall call U_ρ—is simply the product of identical independent distributions for each cell, with probability $\rho_i = \rho$ for cell i to be in state 1. That is, 1's are uniformly distributed in space with an expectation ρ per site and no spatial correlations. The expected value for the total number of particles is $N\rho$, where N is the number of sites. The (fine-grained) entropy for this distribution is $S = s(U_\rho) = Ns(\rho)$.

A typical configuration of U_ρ—with $\rho = 1/8$ and $N = 256 \times 256$—is shown in **Figure 2(a)** (the system is laid out on a torus; i.e., the left and right edges of the figure coincide, and so do the top and bottom edges). After a long time, the spin system will have evolved as in **Figure 2(b)**, with obvious spatial correlations over a wide range of distances ("ordering"). On the other hand, the lattice gas will have evolved as in **Figure 2(c)**, where no spatial correlations appear to the eye.

Incidentally, the usual explanation for this difference in behavior is that the clumping of 1's and 0's in the spin system is due to the presence of attractive forces—which are lacking in the lattice gas. This "explanation" has little predictive power;[3] the following discussion provides a more productive explanation, based on the concept of particle conservation.

Note that in the lattice-gas case we cannot *a priori* rule out the presence of subtle correlations in the equilibrium distribution. To convince ourselves of this, let us start the two systems from a spatially nonuniform distribution [**Figure 3(a)**] having the same expectation $N\rho$ as U_ρ. After a long time, the Ising model and the lattice gas will have evolved as in **Figures 3(b)** and **3(c)**, respectively. Visually, the results are similar to those of Figure 2; in particular, the distribution of Figure 3(c) is again spatially uniform, with $\rho_i = \rho$. However, in this case we know that there are hidden correlations in the lattice gas. In fact, since particles are conserved, the final expectation per site is still ρ; on the other hand, since the system is invertible, the fine-grained entropy is the same as that of Figure 3(a)—which is certainly less than $Ns(\rho)$.

We prove that there are no spatial correlations in Figure 2(c). The entropy at time t can always be written in the form

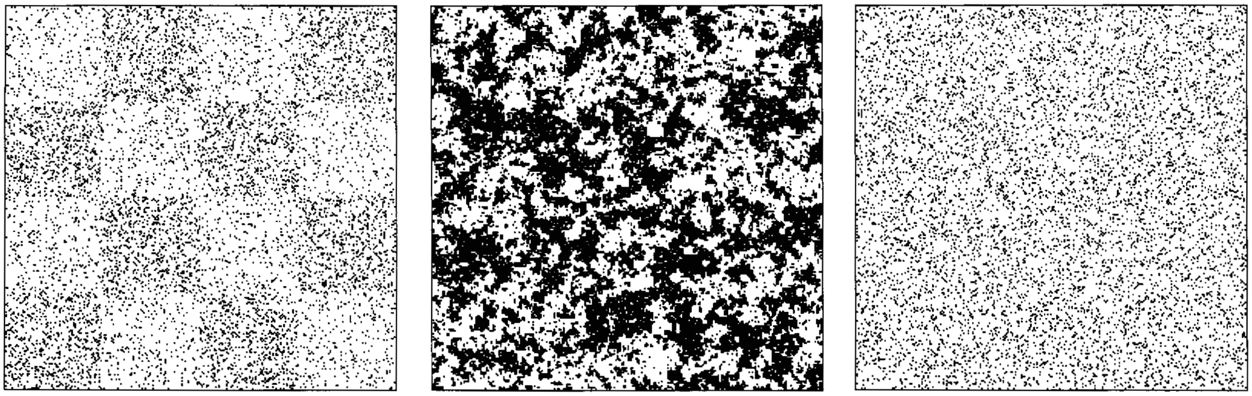
$$S^t = \sum_{i=1}^{N} s(\rho_i^t) - S^t_{\text{correl.}}, \tag{7}$$

where $S^t_{\text{correl.}}$ is whatever correction is necessary to give the correct entropy S^t from the entropy of the marginal distributions ρ_i^t. Let us explicitly rewrite relation (7) for times $t = 0$ (when U_ρ is given as an initial distribution) and $t = 1$ (i.e., after one step):

$$S^0 = \sum_{i=1}^{N} s(\rho_i^0) - S^0_{\text{correl.}},$$

$$S^1 = \sum_{i=1}^{N} s(\rho_i^1) - S^1_{\text{correl.}}. \tag{8}$$

[3] (a) The term "attractive forces" is hard to rigorously define for discrete-state systems. (b) There are lattice gases in which attractive forces are definitely present, and in which, however, this clumping of 0's and 1's does not occur. (c) The mechanism by which attractive forces tend to increase the order of a system is reasonably clear when the system is in contact with a low-temperature reservoir—and thus the effective dynamics is dissipative. In fact, when work is performed by the attractive forces, energy is made available to the thermalizing processes; as the resulting heat is carried away, the system relaxes to a state of lower energy and lower entropy. On the other hand, here we have an isolated, invertible system—whose fine-grained entropy is constant and whose coarse-grained entropy cannot decrease. The less-than-average disorder contained in the large domains of 0's and 1's that appear in the final configuration is compensated by greater-than-average disorder on the boundaries of the domain themselves.



(a)   (b)   (c)

Figure 3

(a) Initial nonuniform configuration. Final configuration in (b) Ising spin model and (c) lattice gas.

At time 0, $\rho_i^0 = \rho$ and $S^0_{\text{correl.}} = 0$ by construction. At time 1, $\rho_i^1 = \rho$ because of particle conservation.[4] Thus,

$$S^0 = N\rho,$$
$$S^1 = N\rho - S^1_{\text{correl.}}. \qquad (9)$$

On the other hand, $S^0 = S^1$ because of invertibility; thence

$$S^1_{\text{correl.}} = 0. \qquad (10)$$

In other words, after one step the probability at each cell remains the same and no correlations are introduced. Thus,

Theorem 1 For any ρ, U_ρ is a time-invariant distribution of the lattice gas.

Hereinafter, we call separable a distribution that is the product of its marginal ("cell-by-cell" or "singlet") distributions.

In the lattice-gas equilibrium states considered above, particles are found with equal probability in each of the four directions of travel. We extend Theorem 1 to macrostates in which the gas as a whole undergoes a steady drift.

Let us consider a separable distribution U_{drift} in which particles traveling in the four directions (cf. Figure 1) occur with different probabilities $\rho_{x_+}, \cdots, \rho_{y_-}$; the distribution is otherwise spatially uniform.

A necessary condition for equilibrium is that for each direction of travel particles be created and annihilated at the same rate. For the specific dynamics we are considering, which is characterized by the reversible "reaction" [cf. (5)]

$$e_{x_+} + e_{x_-} \rightleftharpoons e_{y_+} + e_{y_-} \qquad (11)$$

(where e_{x_+} denotes a particle traveling in the x_+ direction, etc.), this constraint is expressed by the balance relation

$$\sigma_x = \sigma_y \equiv \sigma, \qquad (12)$$

where

$$\sigma_x = \rho_{x_+}\rho_{x_-}\bar{\rho}_{y_+}\bar{\rho}_{y_-}, \qquad (13)$$

$$\sigma_y = \bar{\rho}_{x_+}\bar{\rho}_{x_-}\rho_{y_+}\rho_{y_-}; \qquad (14)$$

in fact, (13) is the rate for the forward reaction and (14) that for the reverse reaction in (11). Note that the equilibrium parameter σ is invariant under the exchange of 0's and 1's.

An argument strictly analogous to that used for Theorem 1 shows that condition (12) is also sufficient for equilibrium; i.e., after one step the marginal distributions not only retain the initial probabilities ρ_{x_+}, ρ_{x_-}, ρ_{y_+}, and ρ_{y_-}, but also remain uncorrelated. Thus,

Theorem 2 For any ρ's satisfying the balance relation (12), the distribution U_{drift} is time-invariant.

Thus, a separable distribution is an equilibrium one if and only if it satisfies the balance relation; moreover, this relation can be explicitly solved in terms of the four ρ's. Therefore we have a complete knowledge of the separable equilibria. Do these exhaust the range of possible equilibria?

Clearly there exist nonseparable equilibria; for example, the distribution containing with equal weights exactly those configurations where only one particle is present.[5] On the other hand, the family of separable equilibria discussed above is large enough to contain all the "traditional" equilibria, i.e., those considered by naive kinetic theory. By (12), the four ρ's can be expressed in terms of the more familiar-looking variables

[4] Since both the dynamics and the initial distribution are spatially uniform, all the ρ_i^1 must be equal.

[5] Note that, from a physical viewpoint, this is an extremely degenerate case, since for these initial conditions the system decomposes into a collection of independent one-dimensional systems with no interactions at all.

$$\rho = \rho_{x_+} + \rho_{x_-} + \rho_{y_+} + \rho_{y_-}, \tag{15}$$

$$j_x = \frac{\rho_{x_+} - \rho_{x_-}}{\rho}, \tag{16}$$

$$j_y = \frac{\rho_{y_+} - \rho_{y_-}}{\rho}. \tag{17}$$

That is, the family of equilibrium states described by (12) can be parametrized by three quantities, namely mass density and the two components of momentum density. This is in agreement with the case of an ideal gas, where energy and momentum are the only additive conserved quantities and (assuming the ergodic hypothesis) completely characterize equilibrium.

A balance relation (or set of relations) analogous to (12) can be explicitly derived for any cellular automaton. If the marginal distributions of a spatially uniform, separable distribution obey this relation at a given moment, they will obey it again after one step of the dynamics—after which they may have become correlated, so that in general the argument cannot be iterated. However, as we have seen, if this relation holds, then relation (10) automatically follows if the automaton is invertible and particle-conserving, and thus the argument can be iterated. Therefore

Lemma 3 If a cellular automaton is invertible and particle-conserving, all of its separable, spatially homogeneous distributions that obey the balance relations are time-invariant. These distributions can be explicitly computed from the automaton's local law.

3. Small perturbations from equilibrium

Armed with Lemma 3, we proceed now to study the evolution of the information associated with small perturbations from equilibrium.

A remark is in order. Consider an arbitrary distribution U^0 of microstates and the corresponding distribution U^t obtained by letting each microstate follow its natural evolution for a time t. For cellular automata, U^t is a continuous function of U^0; in other words, a small perturbation in the initial distribution results in a small perturbation in the final one.

Let U be an equilibrium distribution of an invertible, particle-conserving cellular automaton, consisting of the product of independent marginal distributions ρ_i $(i = 1, \cdots, N)$ as in Lemma 3; and consider a variation δU of this distribution such that the marginal distributions $\rho_i + \delta\rho_i$ of $U + \delta U$ are also independent. By definition, one step of the dynamics transforms U into itself. On the other hand, in general $U + \delta U$ is transformed into a different distribution $U + \delta U'$; moreover, the new marginal distributions $\rho_i + \delta\rho_i'$ may not be independent.

Because of particle conservation,

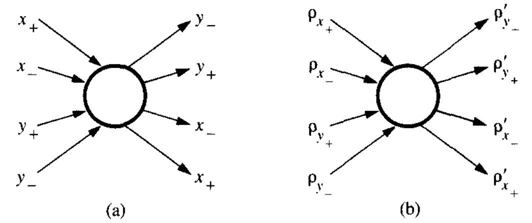


Figure 4

Elementary interaction site in the HPP lattice gas.

$$\sum_{i=1}^{N} \delta\rho_i' = \sum_{i=1}^{N} \delta\rho_i \tag{18}$$

for variations of any size (i.e., not necessarily small); thus, a mass-density perturbation, collectively represented by the $\delta\rho_i$ $(i = 1, \cdots, N)$, evolves as an additive conserved quantity.

Because of invertibility,

$$s(U + \delta U') = s(U + \delta U). \tag{19}$$

On the other hand, in general

$$\frac{\sum \delta s_i'}{\sum \delta s_i} \geq 1, \tag{20}$$

because of the correlations that may be introduced by the system's evolution.

In the next section we show that relation (20) becomes an equality in the limit as $\delta\rho_1, \cdots, \delta\rho_N \rightarrow 0$; as a consequence, in this limit the marginal distributions $\rho_i + \delta\rho_i'$ are spatially uncorrelated. Thus, in this limit also the information-density perturbation, collectively represented by the δs_i, evolves as an additive conserved quantity.

4. Evolution of mass and information perturbations

We study in detail the evolution of small perturbations in the HPP lattice gas introduced in Section 2.

With reference to Figure 1(a), a block represents the potential locus of interaction between particles at a given step, and thus can be depicted as a "computing node" with four input lines and four output lines, as in **Figure 4**. The inputs represent the states of the four cells of the block before the interaction, and the outputs the states of the same cells after the interaction; at the next step, these outputs are fed as inputs to the four blocks that, in the new partition, straddle the given block [Figure 1(b)]. In Figure 4(b), the ρ's represent the probabilities of 1's on the input lines, and the ρ''s those on the output lines. We use h and k as indices ranging over the four directions x_+, x_-, y_+, y_-.

Let us consider a spatially uncorrelated distribution representing a uniformly drifting gas. That is, the occupation probabilities ρ_{x_+}, ρ_{x_-}, ρ_{y_+}, and ρ_{y_-} for the four cells that make up a block may be different, but each of the four probabilities has the same value in every block; moreover, all the cell-by-cell marginal distributions are independent.

With respect to the marginal distributions, the operation of the node of Figure 4 is described as follows:

$$\rho'_{x_+} = \rho_{x_+} - (\sigma_x - \sigma_y),$$

$$\rho'_{x_-} = \rho_{x_-} - (\sigma_x - \sigma_y),$$

$$\rho'_{y_+} = \rho_{x_+} + (\sigma_x - \sigma_y),$$

$$\rho'_{y_-} = \rho_{x_-} + (\sigma_x - \sigma_y); \tag{21}$$

note that here we are not yet assuming equilibrium, and thus the ρ's are completely arbitrary.

The response of the ρ''s to small perturbations of the ρ's (first-order response) is given by the Jacobian matrix

$$\mathbf{J}^{\text{mass}} = \left[\frac{\partial \rho'_h}{\partial \rho_k}\right], \tag{22}$$

of which, for clarity, we explicitly write down two elements:

$$\frac{\partial \rho'_{x_+}}{\partial \rho_{x_+}} = 1 - \frac{d}{d\rho_{x_+}}(\sigma_x - \sigma_y), \tag{23}$$

$$\frac{\partial \rho'_{x_-}}{\partial \rho_{x_+}} = -\frac{d}{d\rho_{x_+}}(\sigma_x - \sigma_y). \tag{24}$$

At equilibrium [cf. (12)] one has

$$-\frac{d}{d\rho_{x_+}}(\sigma_x - \sigma_y) = \sigma u_{x_+}, \tag{25}$$

where

$$u_h = -\frac{1}{\rho_h \bar{\rho}_h} \tag{26}$$

[cf. (4)] and the Jacobian matrix for mass-density perturbations becomes simply

$$\mathbf{J}^{\text{mass}} = \mathbf{I} + \sigma \mathbf{A}, \tag{27}$$

where $\mathbf{I}$ is the identity matrix and

$$\mathbf{A} = [A_{hk}] = \begin{bmatrix} u_{x_+} & u_{x_-} & -u_{y_+} & -u_{y_-} \\ u_{x_+} & u_{x_-} & -u_{y_+} & -u_{y_-} \\ -u_{x_+} & -u_{x_-} & u_{y_+} & u_{y_-} \\ -u_{x_+} & -u_{x_-} & u_{y_+} & u_{y_-} \end{bmatrix}. \tag{28}$$

Intuitively, σ determines the overall intensity of mass-perturbation scattering, while $\mathbf{A}$ (the normalized scattering matrix) gives, for each of the four input directions, the relative amounts of scattering into the four output directions.

Matrix $\mathbf{A}$ can be written, more compactly, as the outer product of two vectors:

$$\mathbf{A}^T = \begin{bmatrix} u_{x_+} \\ u_{x_-} \\ -u_{y_+} \\ -u_{y_-} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}^T. \tag{29}$$

Note that in (29) the sum of the elements of the second vector equals zero—which implies that the sum of mass-density perturbations is time-invariant. This is no surprise; in fact, it is an immediate consequence of particle conservation, and remains true for arbitrary distributions (e.g., not necessarily equilibrium, homogeneous, or spatially uncorrelated) and arbitrary perturbations. In other words, the mass surplus or deficit introduced by a perturbation always evolves as an additive conserved quantity.

A mass-density perturbation (with respect to the equilibrium value) at a given cell yields a corresponding information-density perturbation; the ratio between these two perturbations is

$$\frac{ds_h}{d\rho_h} = r_h \tag{30}$$

[cf. (4)]. We can thus write a Jacobian matrix for the propagation of information-density perturbations (always in the context of small perturbations from equilibrium):

$$\mathbf{J}^{\text{info}} = [J^{\text{info}}_{hk}] = \left[\frac{\partial s'_k}{\partial s_h}\right] = \left[\frac{r_h}{r_k} J^{\text{mass}}_{hk}\right] = \mathbf{I} + \sigma \mathbf{B}, \tag{31}$$

where

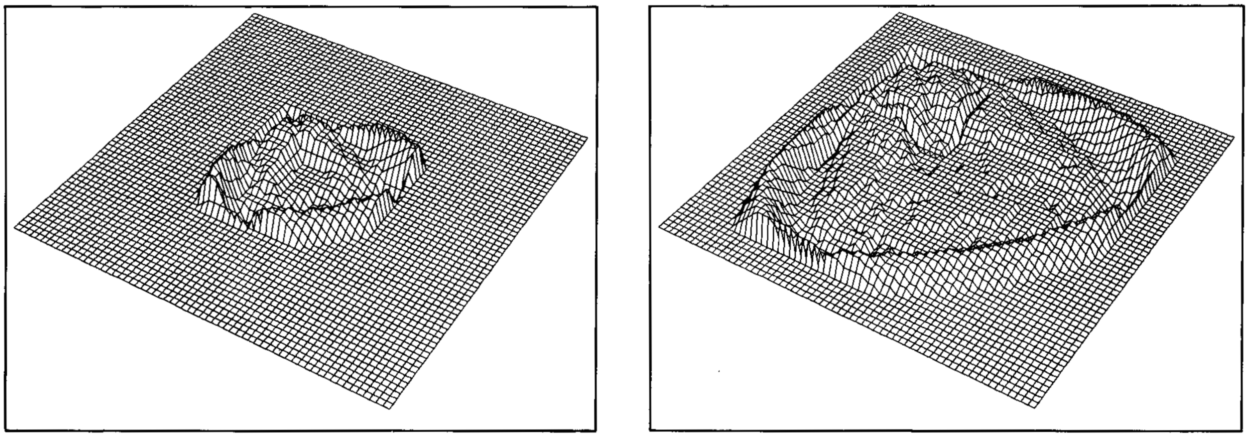
$$\mathbf{B}^T = \begin{bmatrix} u_{x_+}/r_{x_+} \\ u_{x_-}/r_{x_-} \\ -u_{y_+}/r_{y_+} \\ -u_{y_-}/r_{y_-} \end{bmatrix} \begin{bmatrix} r_{x_+} \\ r_{x_-} \\ -r_{y_+} \\ -r_{y_-} \end{bmatrix}^T. \tag{32}$$

Note that here the elements of the second vector are not equal in magnitude as in (29); however, their sum still equals zero, because of (12). Thus, by substantially the same argument as that leading to Theorem 1, we conclude that the marginal distributions remain uncorrelated as the perturbed system evolves:

Theorem 4 In the HPP lattice gas, the information surplus (or deficit) associated with small, spatially uncorrelated perturbations of a separable equilibrium distribution is transported as an additive conserved quantity.

The HPP gas is far from trivial, in that it displays a full-featured hydrodynamics reminiscent of the Navier–Stokes equation.[6]

[6] A closer approximation to this equation is given by a lattice gas similar to HPP in all respects, but with particles running in six (rather than four) directions [6].



(a) (b)

Figure 5

Evolution of mass-density for a small perturbation from equilibrium in the HPP gas, shown at times $t = 16$ (a) and $t = 128$ (b).

A theorem analogous to Theorem 4 holds for many other lattice gases; in particular, for the BBM cellular automaton [3, 5], which is known to be computation-universal.[7]

From the above considerations, we advance the following.

Conjecture 5 *In all invertible, particle-conserving cellular automata, the information associated with small perturbations of a separable equilibrium distribution is an additive constant of the motion.*

5. Conclusions

We have exhibited nontrivial systems in which information is transported as an additive conserved quantity. An important property of these systems is that the time evolution of near-equilibrium distributions can be explicitly computed using modest resources; in fact, the amount of computation is proportional to the number N of cells (rather than to the number of microstates, which increases exponentially with N). Thus, one has direct computational access to the dynamics of distributions as well as to that of microstates. For example, **Figure 5** shows two stages in the evolution of a localized perturbation in the HPP gas (with $\rho_{x_+} = 0.933$, $\rho_{x_-} = 0.500$, and $\rho_{y_+} = \rho_{y_-} = 0.789$); this figure plots the deviations of the ρ's from their equilibrium values. The corresponding deviations of the s's are proportional to those of the ρ's—with different proportionality coefficients for the four directions of travel, as given by (30)—and yield a qualitatively similar plot.

[7] This gas is not momentum-conserving, and consequently the family of equilibria is spanned by a single real parameter $p = \rho_{x_+} = \rho_{x_-} = \rho_{y_+} = \rho_{y_-}$. In addition, there are some degenerate equilibrium distributions, which we disregard.

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