Boundary-layer theory for the extremely underdamped Brownian motion in a metastable potential

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A theory for the boundary layer near the critical trajectory for the extremely underdamped Brownian motion in a metastable potential is presented. The probability distribution function in phase space near this critical trajectory, the average escape energy, and the correction terms for the zero-friction-limit escape rate are calculated.

Introduction

The decay of locally stable states due to fluctuations plays a major role in such different fields as physics, electronics, chemistry, and biology. The simplest example is the one-dimensional Brownian motion of a classical particle, either in a single metastable well or in one well of a bistable potential (see **Figure 1**). In his pioneering work Kramers [1] treated the Brownian-motion problem and, in particular, he calculated the escape rate out of a potential well. After earlier investigations by Chandrasekhar [2], Brinkman [3], and Landauer and Swanson [4], Langer [5] extended Kramers' one-dimensional treatment to multiple dimensions. In the last twenty years the problem of the escape from a metastable potential or the transition to the other well in a double-well potential has inspired many investigators. For a recent review on the escape problem, both classical and quantum-mechanical, we refer to Hänggi [6]. Further reviews on the escape problem can be found in Fonseca et al. [7], in Landauer [8], and in the monographs [9–13].

In this investigation we confine ourselves to the classical one-dimensional Brownian-motion problem, and in particular treat the extremely underdamped motion. We determine the escape rate by calculating the lowest nonzero eigenvalue of the corresponding Fokker–Planck equation. For barrier heights which are large compared to the thermal energy, the lowest nonzero eigenvalue is very well separated from the higher ones. Thus, after a small time the decay rate is given by a pure exponential function the decay constant of which is given by the lowest nonzero eigenvalue. For zero friction, the energy is a constant of motion. For the extremely underdamped motion, the energy changes very

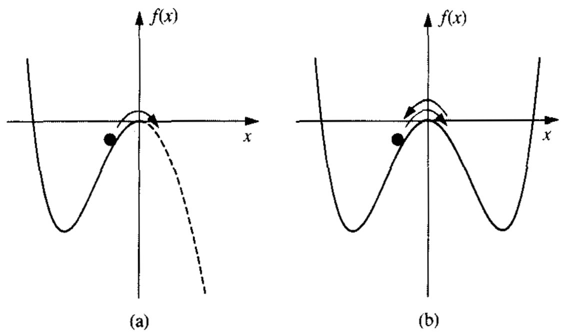

Figure 1

(a) Example of a metastable potential according to (3); (b) a quartic double-well potential. Both potentials agree for $x \leq 0$; for $x > 0$ the metastable potential in (a) is continued by an inverted parabola.

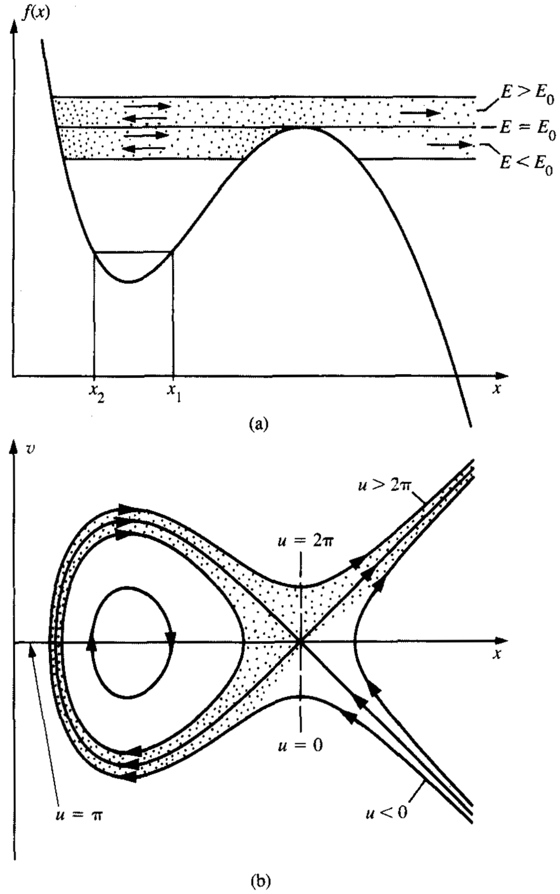

Figure 2

(a) The metastable potential (3) and (b) some of the trajectories in phase space. The dotted region is the boundary-layer region where the probability density is different from zero. The range of the variable u is also shown.

slowly in time. On a time scale in which the particle changes its energy, it performs many oscillations inside the well. Then one can average the trajectories, and the distribution function depends only on the energy variable. As seen in **Figure 2**, there is a critical energy E_0 and a corresponding critical trajectory in phase space. Above this critical trajectory particles leave the well at the top of the maximum of the potential, whereas no particle enters the well in the metastable potential case, or particles of "opposite sign" flow into it from the other well in the bistable potential case. Thus one obtains a strong space dependence of the distribution function above the critical trajectory. For continuity reasons this must also be true in a boundary layer near the critical trajectory.

The focus of this paper is on the space- and energy-dependent probability distribution function near the critical trajectory. As it turns out, the width of this boundary layer is of the order of the square root of the damping constant. At the lowest order the lowest nonzero eigenvalue is proportional to the friction constant and is the same for metastable and double-well potentials, if the potential curves coincide for $x \leqq 0$ and in the vicinity of the potential top. As an effect of the boundary layer an additional dependence of the eigenvalue on the friction constant γ of power 3/2 is obtained, i.e.,

$$\lambda = a\gamma - \kappa b \gamma^{3/2} = \gamma(a - \kappa b\sqrt{\gamma}). \tag{1}$$

The constants a and b can be expressed in terms of the specific form of the potential; see, for instance, Equation (44). For the metastable potential of Figure 1(a), which is "half" the double-well potential in Figure 1(b), the constants a and b are thus the same. The factor κ, however, is different for the two potential forms. It is later shown how κ can be calculated for the metastable potential. In this case the factor κ is about 1.7 times larger than for the double-well potential. This means that on increasing γ, the decay rate for the metastable potential becomes smaller compared with the double-well potential. For larger damping constants and the potentials in Figure 1, the decay rate for the double-well potential is twice the decay rate for the metastable potential. The ratio 2:1 occurs because one has transitions from the left to the right well and vice versa, compared to only one transition out of the well in the metastable case. For small damping constants the different κ-values in (1) for the two potentials already show the beginning of the transition to this 2:1 ratio of the eigenvalues. The expression (1) for the decay rate was given by Büttiker et al. [14], Büttiker and Landauer [15, 16], Voigtlaender and Risken [17], and Risken and Voigtlaender [18]. Also, Mel'nikov [19] and Mel'nikov and Meshkov [20] have, among other results, obtained such a dependence. For a tilted periodic-potential problem, a boundary-layer theory must also be applied, leading to a drift velocity $\langle v \rangle = (1/\gamma)(\tilde{a} + \tilde{b}\sqrt{\gamma})$ [21] similar to the second form of (1). For a detailed review on the decay rates for small friction constants and further references, the reader is referred to Büttiker [22].

In this paper we first explain the basic equations and the approximations made. In the next section, the boundary-layer theory is presented. Finally, we briefly discuss the method for obtaining decay rates.

Basic equations

The starting equation for describing the one-dimensional Brownian motion of particles in the potential $f(x)$ is the following Fokker–Planck equation (FPE) for the distribution function $W(x, v, t)$ in position-velocity space:

$$\frac{\partial W}{\partial t} = \left\{ -\frac{\partial}{\partial x} v + \frac{\partial}{\partial v} [\gamma v + f'(x)] + \gamma \Theta \frac{\partial^2}{\partial v^2} \right\} W. \tag{2}$$

In (2) γ is the friction constant and $\Theta = kT$ is the average thermal energy. [$f(x)$, Θ, γ, and, later, all energies have been divided by the mass.] The derivative of the potential $f(x)$ (i.e., the negative force) is denoted by a prime. The theory to be presented is valid for any smooth metastable potential (i.e., one without cusps or steps) with a quadratic x dependence near its minimum and maximum. As an example we use the metastable potential

$$f(x) = -\frac{d_2}{2} x^2 + \frac{d_4}{4} x^4 \text{ for } x \leqq 0;$$

$$f(x) = -\frac{d_2}{2} x^2 \text{ for } x \geqq 0; \tag{3}$$

see Figure 1(a). It is just "one half" of the quartic double-well potential. For small γ the energy

$$E = v^2/2 + f(x) \tag{4}$$

is a slow variable. We therefore use the energy as one variable [9]. By keeping the space coordinate as the second variable, the FPE (2) is transformed to

$$\frac{\partial}{\partial t} W_\pm = \mp v(x, E) \frac{\partial}{\partial x} W_\pm + \gamma v(x, E) \frac{\partial}{\partial E} v(x, E) \left(1 + \Theta \frac{\partial}{\partial E}\right) W_\pm, \tag{5}$$

where $W_\pm$ is the distribution $W(x, \pm v, t)$ expressed in terms of x, E, and t, with v given by (6). The $\pm$ signs refer to the regions $v > 0$ and $v < 0$ in phase space, and $v(x, E)$ is the velocity

$$v(x, E) = \sqrt{2[E - f(x)]}. \tag{6}$$

For small γ the particles mainly move along the trajectories E = const; see Figure 2. For energies well below the critical energy they make many oscillations inside the well in a time interval $1/\gamma$, and W thus becomes independent of the position. Taking the time integral over one period

$$\int \cdots dt = \int \cdots \frac{dx}{v(x, E)}, \tag{7}$$

we obtain for $W(x, E, t)_{\text{av}} = \tilde{W}(E, t)$ the equation

$$T(E) \frac{\partial \tilde{W}}{\partial t} = \gamma \frac{\partial}{\partial E} I(E) \left(1 + \Theta \frac{\partial}{\partial E}\right) \tilde{W}, \tag{8}$$

where the action integral $I(E)$ and the period of oscillation $T(E)$ are given by [x_1, x_2 are the turning points; see Figure 2(a)]

$$I(E) = 2 \int_{x_2(E)}^{x_1(E)} dx v(x, E),$$

$$T(E) = 2 \int_{x_2(E)}^{x_1(E)} dx/v(x, E). \tag{9}$$

For the potential (3), $I(E)$ and $T(E)$ can be expressed in terms of the complete elliptic integrals of the first and second kind; see for instance [18]. [Notice, however, that I and T defined in [18] are twice the values given by (9).] The averaging procedure just described is justified if the energy loss ΔF due to the friction in a round-trip time is small compared to the thermal energy; i.e.,

$$\Delta F = 2\gamma \int_{x_2(E)}^{x_1(E)} v(x, E) dx = \gamma I(E) \ll \Theta. \tag{10}$$

As discussed before, the distribution function must depend on x in a boundary layer near the critical trajectory $E = E_0$. As it later turns out, the energy width of the boundary is proportional to $\sqrt{\gamma}$; see (17), (18). Thus, for small γ this width also becomes small, and we may put $v(x, E) = v(x, E_0)$ in (5) for calculating the boundary layer to the lowest order. Because the distribution varies very rapidly in E in this boundary region, we neglect the first derivative with E in favor of the second one. [If the first derivative is not neglected, an additional term of the form $[\delta/(2\Theta)]\partial W/\partial \varepsilon$ appears on the right-hand side of (19).] Furthermore, if we consider a time scale for the distribution of the order of the inverse decay rate ($\sim 1/\gamma$), the time derivative need not be taken into account, because in the boundary layer of thickness $\sqrt{\gamma}$ a quasi-stationary distribution is established in a much shorter time. Therefore (5) reduces in the boundary region to

$$\frac{\partial W_\pm}{\partial x} = \pm \gamma \Theta v(x, E_0) \frac{\partial^2 W_\pm}{\partial E^2}. \tag{11}$$

Boundary-layer solution

For solving (11) we introduce the transformed space coordinate

$$u(x) = u_+(x) = \pi + \frac{2\pi}{I(E_0)} \int_{x_2(E_0)}^{x} v(x', E_0)dx' \qquad v > 0,$$

$$u(x) = u_-(x) = 2\pi - u_+(x) \qquad v < 0. \quad (12)$$

For the loop near the critical trajectory, the variable u runs from 0 to 2π; see Figure 2. For the potential (3) we have

$$I(E_0) = 4d_2^{3/2}/(3d_4) \quad (13)$$

$$u_\pm(x) = \pi[1 \pm {}^{3/2}\sqrt{1 - d_4x^2/(2d_2)}] \qquad x \leqq 0,$$

$$= \pi[1 \pm 1 \pm 3d_4x^2/(4d_2)] \qquad x \geqq 0. \quad (14)$$

Because of

$$du_\pm/dx = \pm 2\pi v(x, E_0)/I(E_0), \quad (15)$$

the boundary-layer equation (11) transforms to

$$\frac{\partial W}{\partial u} = \frac{\gamma\Theta I(E_0)}{2\pi}\frac{\partial^2 W}{\partial E^2}. \quad (16)$$

Here we no longer need the $\pm$ sign because u and E describe the particle in the phase space in an unique way. For subsequent purposes it is convenient to introduce the geometric mean between the energy loss due to friction (10) and the thermal energy Θ divided by π, i.e.,

$$\delta = \sqrt{\gamma\Theta I(E_0)/\pi} \ll \Theta. \quad (17)$$

Introducing, furthermore, a dimensionless energy ε by

$$\varepsilon = (E - E_0)/\delta, \quad (18)$$

(16) takes the form

$$\frac{\partial W}{\partial u} = \frac{1}{2}\frac{\partial^2 W}{\partial \varepsilon^2}. \quad (19)$$

The distribution function W, if it is expressed in the variables u and ε, does not depend on the special form of the potential $f(x)$.

- *Solution for $\varepsilon \leqq 0$*

Below the critical trajectory the solution $W(u, \varepsilon)$ must be periodic in u with period 2π. If $W(u, \varepsilon)$ depends on u, it must decay for large negative values of ε. The general solution which satisfies these requirements is given by the expansion

$$W(u, \varepsilon) =$$

$$w_0\left\{\kappa - \varepsilon + \sum_{n\neq 0} z_n \exp[\sqrt{|n|}(1 + in/|n|)\varepsilon + inu]\right\}, \quad (20)$$

where κ and z_n are constants to be determined from the boundary condition (23). The constant w_0 follows from the normalization of the eigenfunction.

- *Solution for $\varepsilon \geqq 0$*

Above the critical trajectory no particles are entering the well; i.e., $W(u, \varepsilon)$ is zero for $u < 0$. If u is interpreted as time, then (19) is a heat-conduction equation. The solutions to (19), with boundary conditions given by either $W(u, 0)$ or the derivative of W with respect to ε, $W_\varepsilon(u, 0)$, read [23] as follows:

$$W(u, \varepsilon) = \int_0^u \frac{\varepsilon \exp\{-\varepsilon^2/[2(u - \xi)]\}}{\sqrt{2\pi(u - \xi)^3}} W(\xi, 0)d\xi \qquad (0 < u < 2\pi), \quad (21)$$

$$W(u, \varepsilon) = -\int_0^u \frac{\exp\{-\varepsilon^2/[2(u - \xi)]\}}{\sqrt{2\pi(u - \xi)}} W_\varepsilon(\xi, 0)d\xi \qquad (0 < u < 2\pi). \quad (22)$$

The variable u should be restricted to its values inside the well. Equation (21) guarantees that the function W is continuous at $\varepsilon = 0$, whereas (22) guarantees the continuity of the derivative of W with respect to ε at $\varepsilon = 0$.

Thus, using (22) we must require that the function $W(u, \varepsilon)$ (22) itself be continuous at $\varepsilon = 0$, i.e.,

$$W(u, 0) = -\int_0^u \frac{W_\varepsilon(\xi, 0)}{\sqrt{2\pi(u - \xi)}} d\xi. \quad (23)$$

From this equation the unknown coefficients κ, z_n can be determined. The expansion (20) for $\varepsilon = 0$ converges slowly and is therefore not well suited for direct evaluation. To overcome this difficulty we write (20) for $\varepsilon = 0$ in the form

$$W(u, 0) = w_0\left\{\kappa + \sum_{\ell=0}^{\infty} \beta_\ell \zeta\left(-\frac{\ell}{2}, \frac{u}{2\pi}\right)\right\}, \quad (24)$$

where $\zeta(-\ell/2, \alpha)$ are the generalized ζ-functions [24]

$$\zeta(-\ell/2, \alpha) = \frac{2\Gamma(1 + \ell/2)}{(2\pi)^{\ell/2+1}} \sum_{n=1}^{\infty} \frac{\sin(2n\pi\alpha - \ell\pi/4)}{n^{\ell/2+1}}. \quad (25)$$

From (24), (25) the coefficients z_n are expressed in terms of β_ℓ. A dependence of the coefficients z_n proportional to n^{-1} and $n^{-3/2}$ has already been found in [18]. The form (24), (25) guarantees that the function W and all its derivatives are finite for $u \to 2\pi - 0$. For even ℓ, (25) can be reduced to Bernoulli polynomials. For $\ell = 0$ we have the "sawtooth" function $\zeta(0, \alpha) = 1/2 - \alpha$ $(0 < \alpha < 1)$. For odd ℓ, (25) can be expanded according to

$$\zeta(-\ell/2, \alpha) = \alpha^{\ell/2} + (1 + \alpha)^{\ell/2} + \sum_{k=0}^{\infty} A_{\ell k}\alpha^k \qquad (0 \leqq \alpha \leqq 1),$$

$$A_{\ell k} = (\ell/2)(\ell/2 - 1)(\ell/2 - 2) \cdots (\ell/2 - k + 1) \times [\zeta(\ell/2 - k) - 1]/k!, \quad (26)$$

where $\zeta(s)$ is the ordinary ζ-function [24]. The derivative of $W(u, \varepsilon)$ with respect to ε at $\varepsilon = 0$ can also be expressed in terms of generalized ζ-functions,

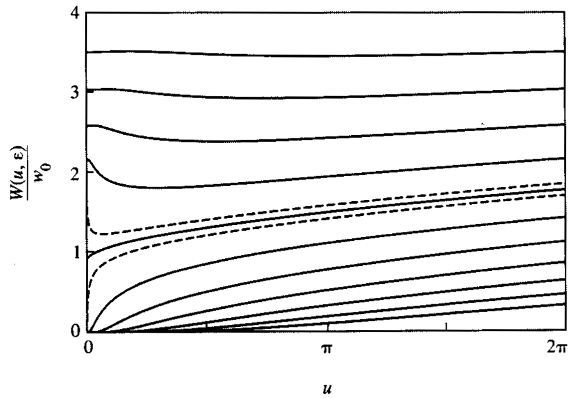

Figure 3

The distribution $W(u, \varepsilon)/w_0$ as a function of u for fixed $\varepsilon = -2, -1.5, \cdots, 3$ (from top to bottom, solid lines) and $\varepsilon = \pm 0.1$ (broken lines).

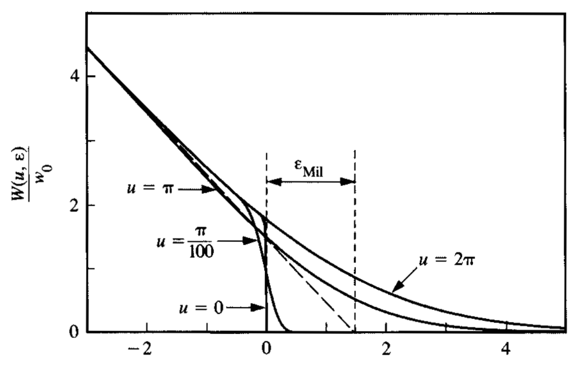

Figure 4

The distribution $W(u, \varepsilon)/w_0$ as a function of ε for the fixed u values 0, $\pi/100$, π, 2π. The Milne extrapolation length ε_{Mil} (30) is also indicated.

$$W_\varepsilon(u, 0) = w_0\left[-1 + \frac{1}{\sqrt{\pi}} \sum_{\ell=0}^{\infty} \beta_\ell \frac{\Gamma(1 + \ell/2)}{\Gamma(1/2 + \ell/2)} \zeta\left(-\frac{\ell - 1}{2}, \frac{u}{2\pi}\right)\right], \tag{27}$$

as may easily be seen by comparison with (20), (25). By inserting (24) and (27) into (23) and using (26), we can calculate the integrals analytically. Matching the expressions at L discrete x- or u-values, we obtain an inhomogeneous system of linear equations for the unknowns κ and β_ℓ, which can be solved for κ and the first $L - 1$ coefficients β_ℓ. The results for some of the lowest coefficients are given by

$$\kappa = -\zeta(1/2) = 1.4603545088 \cdots \tag{28a}$$

$$\beta_0 = -0.8862269 \cdots \quad \beta_1 = 0.736937 \cdots \quad \beta_2 = -0.24064 \cdots$$

$$\beta_3 = -0.08167 \cdots \quad \beta_4 = 0.0563 \cdots \quad \beta_5 = 0.0503 \cdots. \tag{28b}$$

The analytic expression in (28a) was obtained by Mel'nikov [19] and Mel'nikov and Meshkov [20] using a Wiener–Hopf technique. (In [18] the value $\kappa = 1.46$ was reported.) With these coefficients the distribution W/w_0 near the critical trajectory is obtained. [Because of the factors in (25) the influence of higher β coefficients decreases rapidly.] For $\varepsilon < 0$ we summed up the terms (20), whereas for $\varepsilon \geqq 0$ the best way for calculating W is seemingly to use (21). To circumvent the singularities of the integrand in (21), the integral is transformed to

$$W(u, \varepsilon)/w_0 = (2/\sqrt{\pi}) \int_0^\infty \exp\{-(\varepsilon/\sqrt{2u} + y)^2\} \times W(uy(2\varepsilon/\sqrt{2u} + y)/(\varepsilon/\sqrt{2u} + y)^2, 0)dy. \tag{29}$$

For the κ-term and for the first term ($\ell = 0$) of the expansion (24), $W(u, \varepsilon)$ can be expressed in terms of the error function.

In **Figure 3** $W(u, \varepsilon)/w_0$ is shown as a function of u keeping ε fixed, whereas in **Figure 4** $W(u, \varepsilon)/w_0$ is shown as a function of ε with u fixed. As may clearly be seen, the function becomes independent of u and therefore also of x for large negative ε, i.e., inside the well. In Figure 4 the constant slope of W/w_0 for large negative ε is clearly visible. Continuing this slope, we reach the ε-axis at

$$\varepsilon_{\text{Mil}} = \kappa. \tag{30}$$

Here the number ε_{Mil} is similar to the Milne extrapolation length in the kinetic boundary-layer theory for the Fokker–Planck equation with an absorbing wall (see for instance [25, 26]). The results for the boundary-layer distribution are similar to those for the double-well potential investigated in [18].

For $u \geqq 2\pi$ we use $W(2\pi, \varepsilon)$ as the initial condition for (19) and obtain

$$W(u, \varepsilon) = \int_0^\infty \frac{\exp\{-(\varepsilon - \varepsilon')^2/[2(u - 2\pi)]\}}{\sqrt{2\pi(u - 2\pi)}} W(2\pi, \varepsilon')d\varepsilon'. \tag{31}$$

For $u \gg 2\pi$ we have approximately

$$W(u, \varepsilon) = \sqrt{\pi/[2(u - 2\pi)]}\exp\{-\varepsilon^2/[2(u - 2\pi)]\}w_0. \tag{32}$$

[Because of (22), (23), and (26), the integral $\int W(2\pi, \varepsilon)d\varepsilon$ turns out to be πw_0.]

• *Mean energy above the critical trajectory*

An interesting quantity previously discussed by Büttiker and Landauer [15], Mel'nikov and Meshkov [20], and Büttiker [22] is the mean energy above the critical trajectory. In terms of the energy ε it is given by

$$\langle E_{ab} \rangle = E_0 + \delta \langle \varepsilon_{ab} \rangle,$$

$$\langle \varepsilon_{ab}(u) \rangle = \int_0^\infty \varepsilon W(u, \varepsilon) d\varepsilon \Big/ \int_0^\infty W(u, \varepsilon) d\varepsilon. \tag{33}$$

In **Figure 5** $\langle \varepsilon_{ab} \rangle$ is shown as a function of u (valid for every smooth metastable potential) and as a function of the length s, measured along the critical trajectory in phase space, for the potential (3). At $u = 2\pi$ the particles leave the metastable potential with an average energy

$$\langle E_{ab} \rangle = E_0 + \delta\kappa. \tag{34}$$

[Because of (21), (22), (24), and (25), the integrals in (33) can be evaluated exactly for $u = 2\pi$.] Comparing this result with that of Büttiker and Landauer [15], we obtain exact agreement in the small friction limit by using for their *ad hoc* value α

$$\alpha_{\mathrm{BHL}} = \pi/\kappa^2 = 1.4731 \cdots. \tag{35}$$

For the double-well potential Mel'nikov and Meshkov obtained the value $\kappa_{\mathrm{DW}} = (2 - \sqrt{2})\,\kappa = 0.85545586538 \cdots$. The boundary-layer theory for the periodic potential leads to the same value as for the double-well potential. Some time ago two of us obtained the value $\kappa_{\mathrm{PP}} = 0.859$ [21] (less than 0.5% off the correct value), whereas in [13, 17, 18] the value 0.8554 was given for the double-well potential.

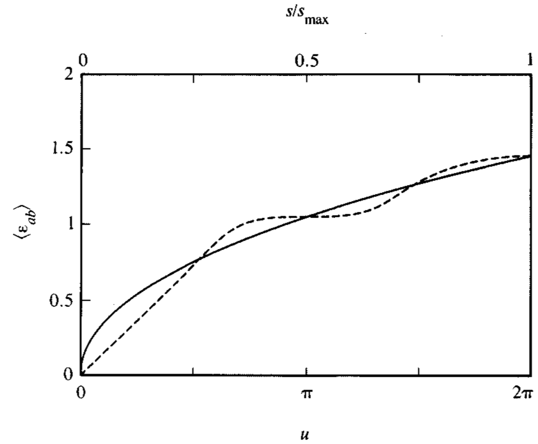

Figure 5

The mean average energy $\langle \varepsilon_{ab} \rangle$ above the critical trajectory as a function of u (full line) and as a function of the length of the trajectory in phase space divided by $s_{\max} = \sqrt{2d_2/d_4} \cdot 4.31 \cdots$. Here the differential of the length along the trajectory is defined by $ds = \sqrt{dx^2 + dv^2/d_2}$. The first result is valid for arbitrary smooth metastable potentials, the latter one for the potential (3).

Eigenvalues and eigenfunctions

The eigenvalues and eigenfunctions are determined by (8) with appropriate boundary conditions. The separation *ansatz*

$$\tilde{W}(E, t) = \phi(E) e^{-\lambda t} \tag{36}$$

leads to the eigenvalue equation

$$\left\{ \frac{d}{dE} I(E) \left(1 + \Theta \frac{d}{dE} \right) + \frac{\lambda}{\gamma} T(E) \right\} \phi(E) = 0. \tag{37}$$

This eigenvalue equation is the same as for the double-well potential in [18]. As was explained in that reference, at the bottom of the potential one has the boundary condition

$$\Theta \phi'(E_{\min}) + (1 + \lambda/\gamma)\phi(E_{\min}) = 0. \tag{38}$$

We now have to match $\phi(E)$ with the boundary solution (20). Below the critical energy (for instance, for $E < E_0 - 2\delta$), we have

$$\phi(E) = w_0[\kappa - (E - E_0)/\delta]. \tag{39}$$

If we take for $\phi(E)$ the boundary condition at $E = E_0$,

$$\phi(E_0) = -\kappa\delta\phi'(E_0), \tag{40}$$

$\phi(E)$ matches the right-hand side of (39) for energies well below the critical energy E_0 [on the scale of the eigenfunction $\phi(E)$]. For $\gamma \to 0$ we get $\delta = 0$ and (40) reduces to the boundary condition for the zero-friction limit

$$\phi(E_0) = 0. \tag{41}$$

Because δ is small compared to Θ [$\phi(E)$ changes on the scale of Θ], we can take care of the right-hand side of (40) by perturbation expansion. As was shown in [18], the eigenvalue λ/γ can then be expressed in terms of $\kappa\delta$ by

$$\lambda/\gamma = (\lambda/\gamma)|_{\gamma\to 0}\,(1 - \kappa\delta B), \tag{42}$$

with

$$B = \frac{I(E_0)\left[\Theta\left(\frac{d\phi^0}{dE}\right)\Big|_{E=E_0}\right]^2}{(\lambda/\gamma)|_{\gamma\to 0} \displaystyle\int_{E_{\min}}^{E_0} T(E)\exp\{-(E_0 - E)/\Theta\}[\phi^0(E)]^2 dE}. \tag{43}$$

The eigenfunctions $\phi^0(E)$ are the eigenfunctions for the zero-friction limit. By inserting δ we have thus arrived at the expression (1), with

$$a = (\lambda/\gamma)|_{\gamma\to 0}; \qquad b = a\kappa B\sqrt{\gamma I(E_0)/(\pi\Theta)}. \tag{44}$$

Generally the eigenfunction $\phi^0(E)$, the eigenvalue $(\lambda/\gamma)_{\gamma\to 0}$, and B must be determined by solving the eigenvalue equation (37) numerically with the boundary condition (41). Alternatively, we may solve (37) with the boundary condition (40). The eigenvalues and eigenfunctions have been calculated for the quartic double-well potential (see Figures 5–7 in [18]). Because of the different κ values, however, the friction scale $\sqrt{\gamma}$ in Figure 5 of [18] has to be multiplied by $\kappa_{DW}/\kappa = 0.58 \cdots$ and the friction constant $\gamma = 0.1$ in Figure 7 of [18] should be replaced by $\gamma = 0.1(\kappa_{DW}/\kappa)^2 \approx 0.034 \cdots$ for the metastable potential (3).

• *Weak-noise limit*

In the weak-noise limit we can solve (37) analytically for the lowest eigenvalue [18]. It turns out that in this limit $B = 1$, and we finally obtain

$$\lambda = \gamma \frac{I(E_0)}{\Theta T(E_{min})} \exp\{-(E_0 - E_{min})/\Theta\} \left\{1 - \kappa \sqrt{\frac{\gamma I(E_0)}{\pi\Theta}}\right\}. \tag{45}$$

For the model potential (3) the eigenvalue (45) specializes to

$$\lambda = \gamma \frac{2\sqrt{2} d_2^2}{3\pi d_4 \Theta} \exp\left\{-\frac{d_2^2}{4d_4\Theta}\right\} \left\{1 - \kappa\sqrt{(4\gamma d_2^{3/2})/(3\pi d_4 \Theta)}\right\}. \tag{46}$$

This form has already been obtained in [17].

The expression (45) agrees with the result of Büttiker and Landauer [15], with α given by (35) and with the result of Mel'nikov and Meshkov [20] in the small-damping limit. Thus, for small friction the boundary-layer theory presented here leads to the same results for the mean escape energy and for the decay rate as predicted by Büttiker and Landauer [14–16], provided their α parameter is chosen according to (35).

This paper is dedicated to Dr. Rolf Landauer on the occasion of his sixtieth birthday.

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