Symmetry and transport in disordered systems

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The transfer matrix for a disordered system enables averages of integer powers of the resistance to be found, R^N; application of the symmetric group generalizes this formula to fractional and negative N, providing a powerful tool for the study of transport. Consequences for fluctuations in resistance, $1/f$ noise, and frequency response are discussed, as well as a new sort of state, of fractal dimension 1/2, which is responsible for transport in localized systems.

The transfer matrix

One of the simpler problems in diffraction theory is the evolution of a set of waves as they pass through a succession of screens, scattering only in the forward direction. The evolution is handled by a succession of matrix multiplications. If $\mathbf{a}_j^+$ represents the wavefield between the jth and $(j+1)$th screens, we have

$$\mathbf{a}_L^+ = \mathbf{T}_L^{++}\mathbf{a}_0^+, \tag{1a}$$

$$\mathbf{T}_L^{++} = \prod_{j=1}^{L} \mathbf{t}_j^{++}, \tag{1b}$$

where we assume nondissipative disordered screens which scatter one set of plane waves into another. The vectors $\mathbf{a}^+$ represent the complex amplitudes of these waves. There is a direct and simple relationship between $\mathbf{a}_L^+$ and $\mathbf{a}_0^+$ which results because the wave never returns to layers from which it has previously scattered. When we introduce backscattering, waves repeatedly return to the same layer. It is this effect which gives rise to the richness of phenomena concerning Anderson localization [1] in strongly disordered systems. Under conditions of current conservation, Equation (1) can never give rise to electrical resistance because there is no mechanism for preventing electrons from continuing in a forward direction. In that sense it is a sterile equation containing very little new physics beyond the case of an ordered system.

Adding backscattering to (1) gives rise to two equations,

$$\mathbf{a}_j^+ = \mathbf{t}_j^{++}\mathbf{a}_{j-1}^+ + \mathbf{t}_j^{+-}\mathbf{a}_j^-, \tag{2a}$$

$$\mathbf{a}_{j-1}^- = \mathbf{t}_j^{-+}\mathbf{a}_{j-1}^+ + \mathbf{t}_j^{--}\mathbf{a}_j^-. \tag{2b}$$

For convenience I now specialize to the 1D case, where we have only forward and backward waves. This already contains a wealth of complexity and can be generalized to the 3D case.

These equations are used as the starting point for nearly all numerical work on disordered systems [2, 3] and for diffraction from disordered surfaces [4]. The additional structure introduced by the t^{+-} and t^{-+} terms can be appreciated from the formal solution to Equation (2):

$$a_L^+ = T_L^{++}a_0^+, \tag{3}$$

where T_L^{++} is calculated recursively,

$$T_j^{++} = t_j^{++}(1 - T_{j-1}^{+-}t_j^{-+})^{-1}T_{j-1}^{++}, \tag{4a}$$

$$T_j^{+-} = t_j^{+-} + t_j^{++}T_{j-1}^{+-}(1 - t_j^{-+}T_{j-1}^{+-})^{-1}t_j^{--}, \tag{4b}$$

with the starting condition

$$T_1^{++} = t_1^{++}, \tag{5a}$$

$$T_1^{+-} = t_1^{+-}. \tag{5b}$$

Evidently T_L^{++} is a highly nonlinear function of any given t_j. This causes problems: If we want to average Equation (1) we can do this easily, provided only that the t_j are independent (e.g., as in a random alloy); in contrast Equations (4) are highly nonlinear functions of the t_j. These functions are not even explicit functions, and averaging can only be done by expanding the expressions in power series in the t_j—i.e., the usual perturbation expansions are unproductive in providing any understanding of the localization problem.

So, although the formal solution (3) is useful for numerical calculations, when we wish to make some analytic manipulation such as taking an average, it is not of much use. However, a powerful reformulation of the problem can be made by introducing an object called the *transfer matrix.* This will regain for us the structural simplicity of (1) but with a twist that introduces the implicit complexity of the localization problem.

The idea of a transfer matrix was first introduced into localization theory by Landauer [5, 6] in his now-classic papers on the subject.

Equations (2) can be rearranged to express $a_j^{+/-}$ in terms of $a_{j-1}^{+/-}$:

$$\begin{bmatrix} 1 - t_j^{+-} & \\ 0 & t_j^{--} \end{bmatrix} \begin{bmatrix} a_j^+ \\ a_j^- \end{bmatrix} = \begin{bmatrix} t_j^{++} & 0 \\ -t_j^{--} & 1 \end{bmatrix} \begin{bmatrix} a_{j-1}^+ \\ a_{j-1}^- \end{bmatrix} \tag{6}$$

or

$$\begin{bmatrix} a_j^+ \\ a_j^- \end{bmatrix} = \begin{bmatrix} 1/t_j, & r_j/t_j \\ r_j^*/t_j^*, & 1/t_j^* \end{bmatrix} \begin{bmatrix} a_{j-1}^+ \\ a_{j-1}^- \end{bmatrix}, \tag{7}$$

where we have used time-reversal invariance to identify

$$t_j = t_j^{++}, \qquad t_j^* = t_j^{--}, \tag{8a}$$

$$r_j = r^{+-}, \qquad r_j^* = t^{-+}. \tag{8b}$$

The transfer matrix is defined to be

$$\mathbf{M}_j = \begin{bmatrix} 1/t, & r_j/t_j \\ r_j^*/t_j^*, & 1/t_j^* \end{bmatrix}, \tag{9}$$

and here we can express

$$\begin{bmatrix} a_L^+ \\ a_L^- \end{bmatrix} = \begin{bmatrix} 1/T_L, & R_L/T_L \\ R_L^*/T_L^*, & 1/T_L^* \end{bmatrix} \begin{bmatrix} a_0^+ \\ a_0^- \end{bmatrix}, \tag{10}$$

where

$$\begin{bmatrix} 1/T_L, & R_L/T_L \\ R_L^*/T_L^*, & 1/T_L^* \end{bmatrix} = \prod_{j=1}^{L} \mathbf{M}_j. \tag{11}$$

Equations (10) and (11) have the same linear structure as Equation (1), which enables averages to be taken with beautiful simplicity provided only that the $\mathbf{M}_j$ are independent. For example, we can from (11) calculate the average

$$\overline{1/T_L} = [1, 0] \prod_{j=1}^{L} \overline{\mathbf{M}}_j \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= [1, 0](\overline{\mathbf{M}})^L \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{12}$$

just as Equations (1) allow us to calculate

$$\overline{(T_L^{++})^{-1}} = \prod_{j=1}^{L} \overline{(t_j^{++})^{-1}} = [\overline{(t_j^{++})^{-1}}]^L. \tag{13}$$

Here the analogy stops, because the slightly more complex form of the transfer matrix, (9), (11), does not allow us to find a linear expression for T_L by a simple rearrangement of terms. This inability to average the transmission coefficient means that it contains a special and rich structure which we have to work much harder to find. The rest of this paper is devoted to searching for a generalization of the transfer matrix which will enable us to average T or $|T|^2$ with the same facility with which (12) gives the average of $1/T$.

2. Generalization of the transfer matrix

The transfer matrix of Equation (9) can give us $1/T$ and R/T for an arbitrary sequence of scatterers in a form that can readily be averaged. It would be useful to be able to calculate $1/|T|^2$ by the same procedure; that would enable us to calculate the average resistance, since

$$\rho_L = \frac{\pi\hbar}{e^2} \frac{1}{|T_L|^2}, \tag{14}$$

as discussed in [7].

In mathematical terms we wish to define a matrix function of $\mathbf{M}_j$, $\chi(\mathbf{M}_j)$, such that if

$$\mathbf{M}_L = \prod_{j=1}^{L} \mathbf{M}_j, \tag{15}$$

then

$$\chi(\mathbf{M}_L) = \prod_{j=1}^{L} \chi(\mathbf{M}_j). \tag{16}$$

These two equations constitute the fundamental theorem for the generalized transfer matrix χ.

The simplest example of a generalized transfer matrix is a 4×4 matrix whose elements comprise all possible pairwise products of the elements of $\mathbf{M}_j$:

$$\chi^{(2)} = \begin{bmatrix} 1/t^2, & r/t^2, & r^*/tt^*, & 1/tt^* \\ r/t^2, & r^2/t^2, & rr^*/tt^*, & r/tt^* \\ r^*/tt^*, & rr^*/tt^*, & r^{*2}/t^{*2}, & r^*/t^*t^* \\ 1/tt^*, & r/tt^*, & r^*/t^*t^*, & 1/t^{*2} \end{bmatrix}. \quad (17)$$

It can be verified that $\chi^2(\mathbf{M})$ obeys the fundamental theorem; hence we can write

$$\chi_L^{(2)} = \prod_{j=1}^{L} \chi_j^{(2)}, \quad (18)$$

which enables us to calculate

$$\overline{1/|T_L|^2} = [1,0,0,0] \prod_{j=1}^{L} \overline{\chi_j^{(2)}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (19)$$

$$= [1,0,0,0](\overline{\chi^{(2)}})^L \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \quad (20)$$

$\overline{\chi^{(2)}}$ always has a real eigenvalue >1 and we retrieve the result that the resistance of a 1D sample diverges exponentially with the length. We could also follow Landauer and make a slightly more complex definition of the resistance in terms of

$$\overline{|R_L|^2/|T_L|^2} = [0,1,0,0](\overline{\chi^{(2)}})^L \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad (21)$$

again giving exponential divergence with L but with a different pre-factor; averages of higher powers of $1/T_L$ have been considered by several authors [1, 8–10].

There is a more formal way of describing $\chi^{(2)}$ in terms of $\mathbf{M}$: It is the *direct product* of $\mathbf{M}$ with itself,

$$\chi^{(2)} = \mathbf{M} \otimes \mathbf{M} = \mathbf{M}^{\otimes 2}, \quad (22)$$

and can be thought of as a matrix whose subscripts are described by two labels:

$$\chi^{(2)} \begin{pmatrix} i \\ j \end{pmatrix}, \begin{pmatrix} i' \\ j' \end{pmatrix} = \mathbf{M}_{ii'} \times \mathbf{M}_{jj'}. \quad (23)$$

Expressed in this form, $\chi^{(2)}$ obviously satisfies the fundamental theorem.

We can generalize to $\chi^{(N)}$ for any positive integer N in the following way:

$$\chi^{(N)} = \mathbf{M}^{\otimes N} \quad (24)$$

or

$$\chi^{(N)}_{\substack{i,i' \\ j,j' \\ \cdot \\ \cdot \\ \cdot \\ k,k'}} = \mathbf{M}_{ii'} \times \mathbf{M}_{jj'} \times \cdots \mathbf{M}_{kk'}. \quad (25)$$

The elements of $\chi^{(N)}$ contain all possible Nth-order integer powers of $1/T$, $1/T^*$, R/T, and R^*/T^*.

There is an analogy we can draw with a system of N interacting particles. If instead of a single particle, several particles were diffracting at once, the total wavefunction of the system between the sth and $(s+1)$th scatterers would be the product of those for the individual particles. In fact, it would be the direct product

$$\psi_s^{(N)} = (\mathbf{a})^{\otimes N} \quad (26)$$

or

$$\psi_{s;j\cdots k} = \mathbf{a}_j \times \cdots \mathbf{a}_k, \quad (27)$$

and the evolution of $\psi^{(N)}$ is described by

$$\psi_L^{(N)} = \prod_{s=1}^{L} \chi_s^{(N)} \psi_s^{(N)}. \quad (28)$$

For any noninteracting system the transfer matrix can be decomposed into a direct product of those for the individual particles.

If we now average to give $\overline{\chi^{(N)}}$, and $\overline{\chi^{(N)}}$ is the sum over many direct products, this quantity can no longer be decomposed into a single direct product. It corresponds to a set of N particles interacting in a special way: They can only exchange momentum, not energy. We have made a transformation in the nature of our problem. Before averaging, the $\chi_j^{(N)}$s corresponded to a noninteracting but disordered system; after averaging, all the $\overline{\chi^{(N)}}$s become identical, and we have an ordered but interacting system. In fact this transformation is a bonus because of the very simple form of the interaction.

In the 2D and 3D cases, which we do not have space to discuss here, this transformation to an interacting but translationally invariant system has some important consequences. For example, the result that all states are localized in 2D can be deduced from the conservation of total momentum perpendicular to the direction of propagation [10–12].

In principle we might imagine that we have solved the problem at this stage, because we have a formalism with which any power of $1/|T_L|^2$ can be averaged. This amounts to calculating all the moments of the probability distribution, P, of the resistance. This implies that we know P itself. Two difficulties stand in the way of implementing this scheme. First, the complexity of the structure of $\chi^{(N)}$ rises rapidly with N: Its dimensions increase as 2^N and, worse still, the range of the eigenvalues becomes extreme. Second, there is a practical difficulty in reconstructing $P(R)$ even if we know the moments. The usual formula for $P(R)$ proceeds in terms of the Fourier transform

$$\int_0^\infty P(R) e^{ikR} dR = \sum_{n=0}^{\infty} \int_0^\infty P(R) \frac{(ikR)^n}{n!} dR$$

$$= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} m_n, \quad (29)$$

where

$$m_n = \int_0^\infty P(R)R^n dR. \tag{30}$$

Provided that the series in (29) converges, we can in principle find $P(R)$ directly. Unfortunately, the moments of $P(R)$ diverge more rapidly than $n!$

There are three routes open to us. We could simplify χ^N sufficiently to calculate m_n analytically and then guess the form that $P(R)$ must take; alternatively we could attempt to calculate m_n as an analytic function of n, then set $n = -1$, -2, etc. This would give us the moments of $|T|^2$ and, since $|T| \leqq 1$, would give a convergent expression for $P(|T|^2)$.

Finally, we might be able to find a way of expressing $\chi^{(N)}$ itself as an analytic function of N, then set $N = -1, -2$, etc. to calculate $1/R$. At first sight this appears to be a hopeless task, since the dimensions of $\chi^{(N)}$ are 2^N and dimensions are manifestly integer quantities. We shall show how this tricky point has been overcome with the help of some group theory.

3. Direct products and the symmetric group

The mapping of disorder into order is reminiscent of the replica trick in spin glasses. It is also suggestive of another aspect of interacting systems: the symmetry of the particles under exchange. Even though we start from a single electron, the *replicas* of the electron are not themselves electrons and their symmetry is for the moment unspecified.

Consider an arbitrary direct product of N matrices

$$\mathbf{D} = \mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C} \cdots. \tag{31}$$

The case of $\chi^{(N)}$ is different,

$$\chi^{(N)} = \mathbf{M} \otimes \mathbf{M} \otimes \cdots, \tag{32}$$

in that all the matrices in the product are identical. For $\mathbf{D}$ to have the same structure, we must have

$$\mathbf{A} = \mathbf{B} = \mathbf{C} = \cdots. \tag{33}$$

Hence, when (33) holds, $\mathbf{D}$ is symmetrical under exchange of $\mathbf{A}, \mathbf{B}, \mathbf{C} \cdots$ amongst themselves. $\chi^{(N)}$ always has this symmetry. It is analogous to a Hamiltonian matrix describing N identical particles which can be exchanged without altering the Hamiltonian. That does not tell us what sort of particles we have: They could be bosons, fermions, or any one of the intermediate types of symmetry described by the Young tableaux for the exchange of N objects.

What we can say is that $\chi^{(N)}$ will factorize according to the irreducible representations of the symmetric group.

Now we want to discover the symmetry of the particles in the system [7, 13, 15]. Looking back to Equation (12), we recall that

$$1/T_L = [1, 0] \prod_{j=1}^{L} \mathbf{M}_j \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence, taking the Nth-order direct product of this equation with itself,

$$1/T_L^N = \langle \mathbf{N} | \prod_{j=1}^{L} \chi_j^{(N)} | \mathbf{N} \rangle, \tag{34}$$

where

$$\chi_j^{(N)} = \mathbf{M}_j^{\otimes N}, \quad \langle \mathbf{N} | = [1, 0]^{\otimes N}, \quad | \mathbf{N} \rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes N}. \tag{35}$$

The vectors $\langle \mathbf{N}|$ and $|\mathbf{N}\rangle$ are totally symmetric in the exchange of their N components; therefore we are interested in the totally symmetric projection of $\chi^{(N)}$: $\chi_S^{(N)}$. We write

$$\chi^{(N)}_{\substack{i,i' \\ j,j' \\ \cdot \\ \cdot \\ \cdot \\ k,k'}} = \mathbf{M}_{ii'} \times \mathbf{M}_{jj'} \times \cdots \mathbf{M}_{kk'}, \tag{36}$$

and note that the subscripts $i \cdots k$ take the values 0 or 1. The completely symmetric Bose state is described by, say, the number of $i \cdots k$ taking the value 0 irrespective of order. Therefore there are $N + 1$ "states" of the system.

We adopt an index

$$0 < m < N \tag{37}$$

which counts the number of zeros in the subscripts; then we have for the symmetrized components [7]

$$(\chi_S^{(N)})_{mm'} = \sum_{p=0}^{\min(m,m')} {}^{m}C_p^{N-m} C_{m'-p}(t^*)^{-p}(r^*/t^*)^{m-p}(r/t)^{m'-p} \times t^{m+m'-p-N}, \tag{38}$$

where

$${}^{m}C_p = \frac{m!}{p!(m-p)!}. \tag{39}$$

In the symmetrized notation, Equation (34) becomes

$$\overline{1/T_L^N} = \left(\prod_{j=1}^{L} \overline{\chi_{Sj}^{(N)}} \right)_{m=0,m'=0} = \left(\overline{\chi_{S0}^{(N)}} \right)_{00}^{L}. \tag{40}$$

This formula is a major step forward. It reduces the dimensions of χ from 2^N for $\chi^{(N)}$ to $(N + 1)$ for $\chi_S^{(N)}$. Of greater importance, all the irrelevant junk corresponding to the other factors of $\chi^{(N)}$ is thrown out of the problem, leaving a formula that is analytically tractable. We can find the moments of $1/|T|^2$ as analytic functions of N when the disorder is weak, when the disorder is very strong, and for certain other special cases. These results are discussed separately.

Perhaps the greatest bonus is that $\chi_S^{(N)}$ can be further generalized so that it is itself an analytic function of N. We can then continue in N to calculate such quantities as

$$\overline{\ln T_L} = \frac{-d}{dN} \overline{1/T^N} \bigg|_{N=0} = \frac{-d}{dN} \left(\overline{\chi_{S0}^{(N)}} \right)_{00}^{L} \bigg|_{N=0}, \tag{41}$$

which gives the density of states (imaginary part) and the inverse localization length (real part).

There is no problem in analytically continuing the elements of $\chi_S^{(N)}$. Equations (33) and (34) contain N either in an exponent or as an argument in

$$^{N-m}C_{m'-p} = \frac{(N-m)!}{(N-m-m'+p)!(m'-p)!}. \tag{42}$$

By writing

$$(N-m)! = \Gamma(N-m+1) = \int_0^\infty x^{N-m}e^{-x}dx, \tag{43}$$

we can analytically continue in N. Provided that m and p remain as integers, $^{N-m}C_{m'-p}$ even continues to negative integer values of N because the singularities in Γ cancel.

There remains the difficulty that the dimensions of $\chi_S^{(N)}$ still depend on N as $(N+1)$. We circumvent this difficulty as follows. Suppose we ignore the bounds on the subscripts m and m' and simply substitute in the formula for $\chi_{Smm'}^{(N)}$ all integer positive values of m and m'. The result is an infinite matrix which has the form

$$\tilde{\chi}_S^{(N)} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{D} & \mathbf{C} \end{array}\right]. \tag{44}$$

For positive integer N,

$$\mathbf{A} = \chi_S^{(N)},\ \mathbf{B} = 0, \tag{45}$$

where $\chi_S^{(N)}$ is an $(N+1)\times(N+1)$ matrix. Thus,

$$(\tilde{\chi}_S^{(N)})^L = \left[\begin{array}{c|c} (\chi_S^N)^L & \mathbf{0} \\ \hline \mathbf{D}' & \mathbf{C}' \end{array}\right], \tag{46}$$

and we can still write for N = positive integer

$$\overline{1/T_L^N} = \left(\tilde{\chi}_S^{(N)}\right)^L_{m=0,m'=0} \tag{47}$$

This equation is much more amenable to analytic continuation: The dimensions of $\tilde{\chi}$ are always infinite and the matrix elements contain only analytic functions of N. The proposal is that (47) be adapted to calculate $1/T_L^N$ even when N is not an integer.

To make calculations $\tilde{\chi}_S^{(N)}$ must be truncated to a finite matrix and (47) evaluated in the limit of large dimensions. This process is a rather convergent one and extensive checks have been made that the answers obtained agree with the direct simulations to within the expected accuracy of those simulations.

Thus we have succeeded in our aim of finding a generalized transfer matrix, $\tilde{\chi}_S^{(N)}$, which is defined for any positive, negative, or fractional N, and can be used to average *any* power of $1/T$. A further twist to the story enables any power of $1/|T|^2$ to be averaged, and the reader is referred to [14] for details.

This scheme has been demonstrated for the 1D case. In higher dimensions we can find a generalization of the same methodology. There is, however, a difference: The symmetric particles we introduce in 1D are bosons, but in higher dimensions they have a different symmetry, one corresponding to a Young tableau of rectangular form. Some intuitive results are available in 2D and 3D [11, 12] but the full group-theoretical scheme is in the process of being written up.

4. Probability distributions in 1D conductors

Having outlined our formalism, we move on to describe more of the results that can be obtained.

Localized systems are characterized by extreme fluctuations of the conductance from one sample to another; therefore the probability distribution $P_1(G = 1/R)$ of the conductance is an important and nontrivial quantity. Our theory sheds new light on $P_1(G)$. Here are some results without proof.

If we consider $\ln G_L$, the fluctuations are less extreme than in G_L itself. In the limit of a specimen of long length, $P_0(\ln G_L)$ converges to a normal distribution in the sense that all the moments tend to those of this distribution, in agreement with the conclusions of many others. However, this is a rather weak statement because we are interested in $P_1(G_L)$ as well as P_0. The two are related by

$$P_1(G_L) = \frac{dN}{dG_L} = \frac{dN}{d\ln G_L}\cdot\frac{d\ln G_L}{dG_L} = \frac{P_0(\ln G_L)}{G_L}. \tag{48}$$

The problem here is that whereas $P_0(\ln G_L)$ may be relatively well behaved and convergent as a function of L, dividing by G_L disturbs this convergence. There are some samples for which G_L is very small and the convergence of P_0 may not be fast enough to ensure that $P_1(G_L)$ is log normal. In fact, only for weak disorder is P_1 a log-normal distribution.

For strong disorder this is not so, and the moments of $1/G_L$ can easily be calculated in some special cases. They show that the distribution P_1 does not in general converge to a log-normal distribution but depends in detail on the statistics of the local disorder in the sample.

One interesting aspect of $P_1(G)$ is its behavior for large values of G. It can be shown [14] that the moments of P_1 are approximated by

$$\int_0^1 P_1(G_L)G_L^N\,dG_L \approx \bar{G}_L\,\frac{\Gamma^2(N-1/2)\Gamma^2(1)}{\Gamma^2(1/2)\Gamma^2(N)}, \tag{49}$$

in the limit of large L. The factorization of the L dependence in $\bar{G}_L$ and the N dependence in the remainder of the expression tells us that the moments of G_L are dominated by a very few specimens with relatively large values of G_L of order unity. The distribution $P_1(G_L)$ near $G_L \approx 1$ has a shape that is independent of L. In fact, for large G_L

$$P_1(G_L) \approx \text{const}\ G_L^{-3/2}, \qquad 1 > G_L \gg 0. \tag{50}$$

At small G_L this distribution merges into what appears as essentially a delta function at the origin as far as the moments of G_L are concerned. This illustrates the point that a distribution that accurately reproduces the moments of G_L need not reproduce those of $\ln G_L$ or of $1/G_L$.

We can develop further this theme of the conductivity being almost always very small, but occasionally rising to be of order unity. It corresponds well to the picture of resonances proposed by Azbel [15, 16]. He shows that when the Fermi energy is degenerate with a resonant state inside the system, the transmission coefficient has a sharp peak whose width is the width of the resonance and whose height depends on how close the resonant state is to the center of the specimen.

This picture correctly predicts the existence of sharp peaks in G_L as a function of Fermi energy. However, the theory predicts that the widths of these peaks should be of the order

$$\Delta E \approx \exp(-\delta'^2 L/4) \tag{51}$$

in units of the bandwidth where δ' is a parameter describing the disorder. A more rigorous argument based on the analytic theory outlined in the previous sections [17] shows that in fact the width of peaks is

$$\Delta E \approx 2^{3/2}\delta' \frac{dE}{dk}\exp(-L^{1/2}\delta'\pi/1.816). \tag{52}$$

This turns out to be a very large discrepancy. The width ΔE dictates the maximum rate at which signals can be sent through the specimen. For typical values of δ' and L, (51) estimates a maximum signaling rate of 10^{-8} Hz, whereas the more rigorous estimate from (52) is 10^6 Hz, as shown in [17].

The resolution of this paradox [18] has to do with the statistics of rare events. If there is a localized state at the center of the sample degenerate with the Fermi energy, electrons need only tunnel halfway across the sample at a time. This probability of degeneracy is small, but the enhancement of conductivity is so great that it is still important in the average. Consider now the probability of two localized states equally spaced across the sample and degenerate in energy: That is even smaller, but in general the contribution of the conductivity will be even more massively enhanced than in the case of a single state, because electrons now need only tunnel across 1/3 of the sample at a time. It can easily be shown that the contribution to the average G_L goes on rising until the probability of finding n degenerate states just gets too small. Maximum contributions occur when

$$n \approx \delta' L^{1/2}. \tag{53}$$

Then the electron must jump a distance of only $L/(\delta' L^{1/2})$ and, given the exponential decay of the wavefunction, the tunneling rate, once the states are found to be degenerate, will be of the order

$$\exp\left(-\frac{\delta'^2 L}{\delta' L^{1/2}}\right) = \exp(-\delta' L^{1/2}), \tag{54}$$

which to logarithmic accuracy correctly reproduces the $L^{1/2}$ dependence in (53).

Thus our new theory has uncovered a new sort of state that dominates transport in 1D: In fact, it constitutes a whole band of $(\delta' L^{1/2})$ states, and since it is localized on $(\delta' L^{1/2})$ sites, it has fractal dimension of 1/2. We argue that this sort of state also dominates transport across strongly localized 3D specimens. It is easy to show that a necklace of nearly degenerate states stretching along a surface normal from one surface to the next is the optimum means of transport. These states will be extremely improbable; therefore the conduction across a localized 3D sheet of material flows almost exclusively down a few highly conducting filaments decorated by the necklace states and having a radius of approximately the localization length.

Finally there is the problem of $1/f$ noise induced by disordered surfaces [19–23]. Imagine a highly disordered layer of material such as silica in contact with a metallic conductor. It has been realized that traps in the silica can modify the thermal noise in the metallic material. It happens like this: Imagine a pulse of electrons in the metal induced by thermal fluctuations. This pulse may hit the silica surface and will be reflected by the highly disordered, essentially insulating, sample. Most of the time the pulse will be reflected almost instantaneously, but if its energy coincides with a localized state in the silica, it will be trapped for something like the lifetime of the state. The net charge trapped fluctuates with time in a way only partly dictated by the thermal noise spectrum. The main influence on the time scale of fluctuations in the trapped charge is the distribution of lifetimes of the localized states. If $\sigma(E)$ is the density of trapped states, their widths can be inferred from the correlation function

$$\overline{\sigma(E)\sigma(E + hf)}. \tag{55}$$

We also have the result that $\sigma(E)$ is related to the phase, Θ, of the transmission coefficient of the silica layer. In terms of our 1D model,

$$\sigma(E) = \frac{1}{\pi}\frac{d\Theta}{dE}, \tag{56}$$

which can be expressed as

$$\sigma(E) = \frac{1}{\pi}\frac{d^2}{dNdE}\mathrm{Im}\frac{1}{t^N}\bigg|_{N=0}, \tag{57}$$

hence

$$\overline{\sigma(E)\sigma(E' = E + hf)} = \frac{1}{\pi^2}\frac{d^4}{dNdEdN'dE'}\,\overline{\mathrm{Im}\,1/t^N \mathrm{Im}\,1/t^{N'}}\bigg|_{\substack{N=0\\N'=0}} \tag{58}$$

i.e., we have re-expressed the correlation coefficient as a product of powers of $1/t$. We have shown [23] that

$$\overline{\sigma(E)\sigma(E + hf)} \approx 1/|hf|, \tag{59}$$

and this form of the correlation coefficient is exactly what is required if the power spectrum of noise in the traps is to have the $1/f$ form. In fact, Equation (59) is not valid for arbitrarily small values of f; the theory indicates a cutoff frequency which depends on the thickness of the silica layer:

$$|hf_{\min}| \approx \pi^2\delta'^2\exp\left(\frac{-\delta'^2 L}{2}\right)\frac{dE}{dk}. \tag{60}$$

The cutoff occurs because the longest-lived traps are in the centre of the specimen, and their lifetime depends exponentially on the thickness.

Conclusion

We have shown that the transfer matrix combined with some group theory is a powerful tool for investigating the statistics of transport in disordered materials. The probability distributions take extreme and surprising forms, but the theory can produce results of the accuracy required even to average powers of the conductance. Time-dependent phenomena can be treated and the frequency response of systems calculated, as well as the spectrum of fluctuations of trapped charge, which is $1/f$ in form. Our discussion has been confined to 1D calculations, which are a good model for strongly localized systems in higher dimensions. Work is in progress on generalizing theoretical approval to 2D and 3D and will be published shortly.

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