

Tunneling times and a quantum clock

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The problem of measuring tunneling times by means of a quantum clock is found to lead to difficulties which are thought to arise because the Hamiltonian of the coupled system does not separate into particle and clock parts.

Introduction

In 1980 Peres [1] described the use of a quantum clock to measure time intervals, and gave several examples. Among these was the application of the clock to measure the velocity of a free particle moving according to the Schrödinger equation, by using the clock to measure the time required for the particle to move through a given section of an axis chosen in the direction of propagation. It was emphasized that there was no need to measure two times, the time of entry and the time of leaving the chosen sector, but simply the change in time as observed on a clock which is stopped when the particle is outside the section, but which runs when it is within the section. In 1986 Davies [2] used a similar method to determine the velocity of a relativistic particle, using the Dirac and Klein-Gordon equations for the motion.

There has been a good deal of interest recently in the time taken for particles to travel through barriers [3], and it has therefore been of interest to examine some examples of this kind using the idea of a quantum clock. It has further seemed particularly appropriate to describe our work on this occasion, for the analysis has thrown up problems that we

have not been able to resolve satisfactorily, and since Rolf Landauer shares an interest in transmission times, his interest and advice would be much appreciated.

The quantum clock

The basic idea is to measure the time interval between two events, by observations on a quantum clock which only runs for the time between two events. Thus, if the clock is set to read zero at $t = -\infty$ and is then read at $t = +\infty$, it can be expected to show a nonzero value, with the difference being the time interval between the two events. It is assumed, for no observations are made, that before and after these events the clock is stopped. The events, using a one-dimensional model, are the arrival at x_1 and the departure at x_2 ($x_1 < x_2$). A simple prescription for the clock is that it consists of a large total spin, S , which produces an effective potential on the particle by means of a contribution to the Hamiltonian of

$$P(x)2\pi\hbar S_z/(2S+1)\tau,$$

where $P(x)$ is zero outside (x_1, x_2) and unity within it. The Hamiltonian of particle plus clock therefore takes the form

$$\frac{p^2}{2m} + V(x) + \frac{P(x)2\pi\hbar S_z}{(2S+1)\tau},$$

where $V(x)$ is chosen as needed. In most cases it can be expected to describe a sequence of rectangular barriers, with a region of zero potential to the left. τ is a parameter yet to be chosen.

The next step is to decide how the clock is to be used, and for this purpose it is useful to introduce an orthonormal set of clock states, defined thus:

$$|N\tau\rangle = \frac{1}{(2S+1)^{1/2}} \sum_{M=-S}^S \exp\{-2\pi iNM/(2S+1)\} |M\rangle,$$

with N taking the values 0, 1, 2, ..., $2S$.

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If the kinetic energy term in the total Hamiltonian is neglected, the remaining two terms commute, even when $P(x)$ is unity. Then any given clock state, at $t = 0$, say $|0\rangle$, becomes

$$\exp\left[\frac{-it}{\hbar} \left\{ \frac{P(x)2\pi S_z \hbar}{(2S+1)\tau} \right\}\right] |0\rangle$$

at time t . If $P(x)$ is zero, it does not change, and the clock is regarded as stopped. When $P(x)$ is unity, the states $|N\tau\rangle$ have the property that if $|0\rangle$ is the state at $t = 0$, it becomes $|1\tau\rangle$ when $t = \tau$, $|2\tau\rangle$ when $t = 2\tau$, and so on. Whenever the particle is in (x_1, x_2) the clock runs; otherwise it is stopped. It can be read at any time by examining the probability that it is in one of the states $|N\tau\rangle$, which is conveniently done by determining the expectation value of the projection operator $|N\tau\rangle\langle N\tau|$, that is, of

$$\frac{1}{(2S+1)} \sum_{M,M'} \exp\left\{\frac{-2\pi i N(M-M')}{(2S+1)}\right\} |M\rangle\langle M'|.$$

The role of the parameter τ can now be appreciated, for it determines the resolving power of the clock. It may also be noted that even with the kinetic energy in the total Hamiltonian there is no interference between the clock and the particle provided the particle is outside (x_1, x_2) . If it is within this range there is interference, and it would seem desirable that this be as small as possible.

The free particle

The case of the free particle has been studied previously, but it is useful to review its treatment briefly. The first step is to find the plane-wave-type solutions when the clock is initially in the state $|M\rangle$. The motion of the particle then becomes that of a particle moving in a potential which is zero everywhere except in the segment (x_1, x_2) , where it is

$$\frac{2\pi\hbar M}{(2S+1)\tau}.$$

It is assumed that the energy of the particle is much greater than this. Then an incident plane wave is partially reflected off the potential discontinuity and partially transmitted. For such a low barrier the amplitude of the reflected wave is very small, and the main effect, to order M , is to alter the phase of the transmitted wave by an amount proportional to $M(x_2 - x_1)$. The solution for the case when the initial state of the clock is $|0\rangle$ then readily follows, by superposition. Each $|M\rangle$ component of $|0\rangle$ has its phase changed by an amount proportional to M , so the resultant state coincides with one of the $|N\tau\rangle$ set if the time of passage is an integral multiple of τ . Under these circumstances, the time of passage is uniquely determined and is that which corresponds to the particle having its usual group velocity. If the time does not correspond to an integral number of τ 's, then there is a distribution over the expectation values of the projection operators of the various $|N\tau\rangle$ states, and there is

an uncertainty in the time of order τ . A remarkable feature of this work is that the time of passage is determined without the need for a wave packet.

General formalism of the time

We now turn to the problem of a wave packet approaching a series of rectangular barriers. The first step is to solve the problem for an incident plane wave. If the various sections, each of which has a constant potential, are labeled sequentially from the left, the solution in region I will be of the form

$$e^{i(kx-\omega t)} + R(k)e^{i(-kx-\omega t)}$$

with

$$\hbar\omega = \frac{\hbar^2 k^2}{2m},$$

that is, an incident plane wave of unit amplitude and a reflected wave in which the reflection coefficient depends on k and on the nature of the various regions II, III, etc. Successive regions have transmitted and reflected parts (though in many the appropriate k values are imaginary, corresponding to evanescent waves), except the final one, which has no reflected component. The solutions in all regions are determined by the boundary conditions at the various interfaces.

A wave-packet solution, which corresponds, at $t = 0$, to having the wave packet entirely in region I, can be found as follows. Suppose the packet is $\varphi(x)$ at $t = 0$. The Fourier transform of $\varphi(x)$, denoted by $\varphi(k)$, is

$$\varphi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) e^{-ikx} dx,$$

where the range of integration can be restricted to region I. Then

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(k) e^{ikx} dk.$$

The wave-packet solution in region I is then

$$\frac{1}{\sqrt{2\pi}} \int_C \varphi(k) [e^{+ikx} + R(k)e^{-ikx}] e^{-i\omega t} dk,$$

with

$$\hbar\omega = \frac{\hbar^2 k^2}{2m}.$$

The solutions in regions II, III, etc. can similarly be written down once the plane-wave solutions are known. The only further point which needs comment is that the integration over k is now regarded as a contour integral taken over a complex k along a contour which runs from $k = -\infty$ to $k = +\infty$. The point here is that it is necessary to ensure that at $t = 0$ there are no pulses in regions II, III, etc. This can be achieved by choosing the contour to be such that it can be

deformed into the large upper semicircle without crossing any poles of $\varphi(k)$ and $R(k)$.

If now the clock is coupled in, and it can in principle be coupled to any of the regions I, II, etc., or within a small part of any of them, the motion can be solved by first assuming that the clock is in $|M\rangle$ at $t = 0$. This produces a modified barrier somewhere, and $R(k)$ and the similar expressions in the other regions become M -dependent. If the clock at $t = 0$ is in $|0\rangle$, the solution is a superposition. In region I it is

$$\frac{1}{(2S+1)^{1/2}} \frac{1}{\sqrt{2\pi}} \int_C \sum_M \varphi(k) [e^{ikx} + R_M(k)e^{-ikx}] e^{-i\omega t} |M\rangle dk.$$

The probability that the clock is in $|N\rangle$ at time t is composed of parts from each of the segments, the part from region I being

$$\frac{1}{(2S+1)^2} \frac{1}{2\pi} \int dx \left| \sum_M e^{\frac{-2\pi i N M}{(2S+1)}} \cdot \int_C \varphi(k) [e^{+ikx} + R_M(k)e^{-ikx}] e^{-i\omega t} dk \right|^2,$$

where the integration over x is to be confined to region I.

Since the integrand is a square modulus, the clock can be expected to give a nonzero expectation value, so it will have measured something wherever it is placed. The problem is to know where to place it to measure the time to pass through, say, one of the regions.

There are various aspects of the problem. In principle the contributions of all the regions should be added, but on physical grounds it can be expected that in general an incident pulse will largely be reflected, so it should be a reasonable approximation to consider only the part coming from region I. But this consists of an incident and a reflected part, and after sufficient time the incident part should have disappeared, so only the parts containing R_M need be retained. In the free-particle problem the part analogous to the R_M of the present problem is the wave (or wave packet) which is transmitted. Here it is the part which is reflected, and from a variety of different possible barrier structures we have been unable to convince ourselves that R_M takes the form of a simple phase change which is linear in M . If it were of the form $e^{i\lambda M/\tau}$, the expectation value would peak when

$$-\frac{2\pi N M}{(2S+1)} + \frac{\lambda M}{\tau} = 0$$

or

$$N\tau = \frac{(2S+1)\lambda}{2\pi}.$$

One model which has been examined in detail is that of a single barrier of height V situated between $x = 0$ and $x = a$, with $P(x)$ only nonzero over the same range. A given $|M\rangle$

state then raises the barrier by $2\pi M\hbar/(2S+1)\tau$ and

$$R_M(k) = -\frac{(K^2 + k^2)\{1 - e^{-2Ka}\}}{(K - ik)^2 - e^{-2Ka}(K + ik)^2},$$

where

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} = \frac{-\hbar^2 K^2}{2m} + V + \frac{2\pi M\hbar}{(2S+1)\tau}.$$

If $R_M(k)$ is written as a series in $\exp(-2Ka)$, using

$$R_M(k) = -\frac{(K^2 + k^2)}{(K - ik)^2} (1 - e^{-2Ka}) \left[1 + e^{-2Ka} \frac{(K + ik)^2}{(K - ik)^2} + \dots \right],$$

the leading term, $-(K^2 + k^2)/(K - ik)^2$, describes the direct reflection from the first face of the barrier, the term in $\exp(-2Ka)$ describes the wave which has traveled to the second face of the barrier and back again before emerging, and so on. But the leading term is already M -dependent, even though the directly reflected wave has not entered the barrier, and so should not have activated the clock!

The explanation would seem to be that the coupling to the clock has already disturbed both systems sufficiently for the clock to produce a reading even if the wave packet has not entered the region to which it is connected. In the case of the free particle, the reflected wave is neglected in comparison with the transmitted wave, for it is much smaller. Here it is the main part.

Another problem which has been examined is that of a potential barrier of height V which extends from $x = 0$ to $x = \infty$. The clock is then coupled to a finite region of positive x . Again a similar result is obtained, and the conclusion seems to be that we have so far failed to find a satisfactory way of coupling a quantum-mechanical clock to a barrier so that it can be used to measure the time to cross a particular region.

This observation, if it proves to be generally true, raises again the question of what is meant by a tunneling time, and, in turn, what is meant by $\partial/\partial t$ in the Schrödinger equation [1].

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