# Exact analysis of round-robin scheduling of services

by Hideaki Takagi

**A multi-queue, cyclic-service model with a single-message buffer is considered. Each message consists of a number of characters, and one character is served at the server's each visit. Exact and explicit expressions are derived for performance measures, such as mean cycle time, mean message response time, and mean response time conditioned on the message length. The same model was previously solved by approximation by Wu and Chen [*IBM J. Res. Develop.* 19, No. 5, 486–493 (September 1975)].**

## Introduction

Some time ago, Wu and Chen [1] proposed and analyzed a multi-queue, cyclic-service model for a loop transmission system. Specifically, their model consists of $N$ queues distributed around a loop, and the queues are served in cyclic order by a traveling server. The arrivals at the queues occur as messages, where the number of characters in a message varies according to a geometric distribution with mean $1/\sigma$. (We use different notations from [1]. Correspondence between our notation and that in [1] is summarized in **Table 1**.) The server walks from one queue to the next, servicing exactly one character for each visit to a queue. (Thus, due to the memoryless property of a geometric distribution, at the completion of each character service, the message service also completes with probability $\sigma$, and does not with probability $1 - \sigma$.) The service time per character, $b$, and the walking time between adjacent queues, $r$, are both assumed to be constants. At each queue, at most one message can be stored, and the time interval from the completion of service of one message to the arrival of the next message is exponentially distributed with mean $1/\lambda$. For this model, an approximate analysis is given in [1] to compute system performance measures such as mean cycle time, mean message response time, and mean response time conditioned on message length. They are compared with simulation results.

In the light of recent developments in the analysis of cyclic queueing (or *polling*) models, however, it is possible to provide an exact, explicit, and much simpler solution to the above-mentioned problem. Such a solution is given in this paper as an extension of a series of studies on "symmetric polling systems with single message buffers." Works in this category include Mack, Murphy, and Webb [2], Mack [3], Runnenberg [4], Bharucha-Reid [5], Kaye [6], Scholl and Potier [7], Hashida and Kawashima [8], Takagi [9], and Takine, Takahashi, and Hasegawa [10]. These all analyze the case where the whole message is served at each visit. (As pointed out in [9], there are errors in [4], [5], and [8] for the analysis of the variable-length message case. The analysis in [3] is very complicated to follow.) Below, we follow the approach presented in Takagi [11] to derive exactly the mean cycle time $E[C]$, mean message response time $E[T]$, and mean response time $E[T|L]$ conditioned on message length of $L$ characters. Note that $\text{Prob}[L = n] = \sigma(1 - \sigma)^{n-1}$, $n = 1, 2, \cdots$, and $E[L] = 1/\sigma$. Our model is slightly more general than in [1] in the sense that we only assume a constant $R$ for the total walking time (the individual walking times can be different constants).

**Table 1** Notations of Wu and Chen [1] and this paper.

| | *Wu and Chen* | *This paper* |
|---|---|---|
| Number of queues | $N$ | $N$ |
| Mean message interarrival time | $1/\lambda$ | $1/\lambda$ |
| Character service time | $t_s$ | $b$ |
| Server's walking time | $t_w$ | $R/N$ |
| Mean number of characters per message | $R$ | $1/\sigma$ |
| Mean cycle time | $E[T_c]$ | $E[C]$ |
| Mean message response time | $E[T_m]$ | $E[T]$ |
| Mean walking time between successive service quanta | $E[T_w]$ | $E[T_w]$ |
| Mean waiting time until the server first visits the queue | $E[T_{wait}]$ | $E[T_{wait}]$ |
| Mean cycle time after departure from a nonempty queue | $E[T_{cl}]$ | $E[T_{cl}]$ |
| Mean response time conditioned on message length $L$ | $E[T_m \mid L]$ | $E[T \mid L]$ |
| Intervisit time, type 1 | Not used | $I_1$ |
| Intervisit time, type 2 | Not used | $I_2$ |

## Performance measures

Let us start by quoting some results from [11]. We define a polling cycle $C$ as the time interval beginning with a visit to a certain queue by the server and ending with the next visit to the same queue. Let $Q$ be the number of characters served during a polling cycle. Then we have

$$E[C] = R + bE[Q]. \tag{1}$$

The throughput $\gamma$ of the system is measured by the average number of messages served in a unit time. This is given by

$$\gamma = \frac{N}{E[T] + 1/\lambda} = \frac{\sigma E[Q]}{E[C]}, \tag{2}$$

from which we have

$$E[T] = \frac{N}{\sigma}\left(\frac{R}{E[Q]} + b\right) - \frac{1}{\lambda}. \tag{3}$$

## Solution

Let us assign indices $1, 2, \cdots, N$ to the $N$ queues in cyclic order. In order to derive the probability distribution for $Q$, we define the state $u_i$ of queue $i$ as

$$u_i = \begin{cases} 0 & \text{if queue } i \text{ does not have a message,} \\ 1 & \text{if queue } i \text{ has a message,} \quad i = 1, \cdots, N. \end{cases} \tag{4}$$

Note that $e^{-\lambda\tau}$ is the probability of no arrival at an empty queue during a time interval $\tau$. Let $P_i(u_1, \cdots, u_N)$ be the steady-state probability that the server observes a sequence of states $\{u_{i+1}, u_{i+2}, \cdots, u_N, u_1, \cdots, u_{i-1}, u_i\}$ before visiting queue $i$. This is the probability that, at a point in time when queue $i$ is polled, the server has experienced a history of states $\{u_{i+1}, u_{i+2}, \cdots, u_N, u_1, \cdots, u_{i-1}, u_i\}$ when it visited queues $i + 1, i + 2, \cdots, N, 1, \cdots, i$ (now) in the last round of polling. Note that $u_j$ is the state of queue $j$ when it was polled. Considering events that occur during this cycle, we have the steady-state equation

$$\begin{aligned} P_i(u_1, \cdots, u_{i-1}, 0, u_{i+1}, \cdots, u_N) &= \exp\left[-\lambda\left(R + b\sum_{\substack{k=1\\(k\neq i)}}^{N} u_k\right)\right] \\ &\quad \cdot P_{i-1}(u_1, \cdots, u_{i-1}, 0, u_{i+1}, \cdots, u_N) \\ &\quad + \sigma \exp\left[-\lambda\left(R + b\sum_{\substack{k=1\\(k\neq i)}}^{N} u_k\right)\right] \\ &\quad \cdot P_{i-1}(u_1, \cdots, u_{i-1}, 1, u_{i+1}, \cdots, u_N); \end{aligned} \tag{5}$$

$$\begin{aligned} P_i(u_1, \cdots, u_{i-1}, 1, u_{i+1}, \cdots, u_N) &= \left\{1 - \exp\left[-\lambda\left(R + b\sum_{\substack{k=1\\(k\neq i)}}^{N} u_k\right)\right]\right\} \\ &\quad \cdot P_{i-1}(u_1, \cdots, u_{i-1}, 0, u_{i+1}, \cdots, u_N) \\ &\quad + \left\{1 - \sigma \exp\left[-\lambda\left(R + b\sum_{\substack{k=1\\(k\neq i)}}^{N} u_k\right)\right]\right\} \\ &\quad \cdot P_{i-1}(u_1, \cdots, u_{i-1}, 1, u_{i+1}, \cdots, u_N). \end{aligned} \tag{6}$$

These equations are concerned with two time points: One is the instant when queue $i$ is polled (on the left-hand side), and the other is the instant when queue $i - 1$ is polled (on the right-hand side). Since $\{u_{i+1}, u_{i+2}, \cdots, u_N, u_1, \cdots, u_{i-1}\}$ is a history, they do not change between the two time points. Equations (5) and (6) are satisfied by

$$P_i(u_1, \cdots, u_N) = \begin{cases} K, & u_1 = \cdots = u_N = 0, \\ K \displaystyle\prod_{j=0}^{\sum_{k=1}^{N} u_k - 1} \sigma^{-1}\{\exp[\lambda(R + bj)] - 1\}, & \displaystyle\sum_{k=1}^{N} u_k > 0 \end{cases} \tag{7}$$

(note that the right-hand side is independent of $i$), where $K$ is a normalization constant to be determined shortly. Since

$$Q \triangleq \sum_{k=1}^{N} u_k, \tag{8}$$

we obtain

$$P(n) \triangleq \text{Prob}[Q = n] = \begin{cases} K, & n = 0, \\ K \dbinom{N}{n} \sigma^{-n} \displaystyle\prod_{j=0}^{n-1} (e^{\lambda\tau_j} - 1), & 1 \le n \le N, \end{cases} \tag{9}$$

where

$$\tau_j \triangleq R + jb, \qquad j = 0, 1, \cdots, \tag{10}$$

$$K^{-1} = 1 + \sum_{n=1}^{N} \binom{N}{n} \sigma^{-n} \prod_{j=0}^{n-1} (e^{\lambda\tau_j} - 1), \tag{11}$$

and so

$$E[Q] = \sum_{n=1}^{N} nP(n)$$

$$= KN \sum_{n=0}^{N-1} \binom{N-1}{n} \sigma^{-(n+1)} \prod_{j=0}^{n} (e^{\lambda\tau_j} - 1). \tag{12}$$

From (12) all performance measures in (1)–(3) can be computed.

## Comparison with Wu and Chen

Besides $E[C]$ and $E[T]$, Wu and Chen [1] evaluate (in their notation) $E[T_w]$, the mean walking time between successive service quanta, $E[T_{wait}]$, the mean waiting time from the arrival of a message until the server reaches that queue for the first time, and $E[T_{c1}]$, the mean cycle time beginning with the server's departure from a nonempty queue. Then it is shown that the mean response time conditioned on a message length of $L$ characters is given by

$$E[T|L] = E[T_{wait}] + b + (L - 1)E[T_{c1}]. \tag{13}$$

Let us derive Wu and Chen's measures $E[T_w]$, $E[T_{wait}]$, and $E[T_{c1}]$ based on the above solution. First, $E[T_w]$ is simply given by

$$E[T_w] = \frac{R}{E[Q]}. \tag{14}$$

To evaluate $E[T_{wait}]$ and $E[T_{c1}]$, we introduce an *intervisit time* as the time interval which begins with the server's departure from a certain queue and ends with the next visit to the same queue. Consider two types of intervisit time. Type 1, whose duration is denoted by $I_1$, is one in which a message can arrive. Such a case occurs either when the server leaves without service because there was no message at the visit, or when the server completes the service of the last character of a message. Type 2, whose duration is denoted by $I_2$, is one in which a new message cannot arrive because the queue is full. This case occurs when the server completes a character service and there still remain some characters in a message. These two types of intervisit time occur with

$$\text{Prob[type 1]} = 1 - \alpha + \alpha\sigma;$$

$$\text{Prob[type 2]} = \alpha(1 - \sigma), \tag{15}$$

where

$$\alpha \triangleq \text{Prob}[u_i = 1] = \frac{E[Q]}{N} \tag{16}$$

is the probability that a message is found at the server's visit to a certain queue. The probability that the server observes a sequence of states $\{u_{i+1}, u_{i+2}, \cdots, u_N, u_1, \cdots, u_{i-1}\}$ during each type of intervisit time is given by

$$\text{Type 1: } \frac{P_i(u_1, \cdots, u_{i-1}, 0, u_{i+1}, \cdots, u_N) + \sigma P_i(u_1, \cdots, u_{i-1}, 1, u_{i+1}, \cdots, u_N)}{1 - \alpha + \alpha\sigma}$$

$$= \frac{K}{1 - \alpha + \alpha\sigma} \exp\left[\lambda\left(R + b \sum_{\substack{k=1 \\ (k\neq i)}}^{N} u_k\right)\right] \cdot \prod_{j=0}^{\sum_{k=1, (k\neq i)}^{N} u_k - 1} \sigma^{-1}(e^{\lambda\tau_j} - 1); \tag{17}$$

$$\text{Type 2: } \frac{(1 - \sigma)P_i(u_1, \cdots, u_{i-1}, 1, u_{i+1}, \cdots, u_N)}{\alpha(1 - \sigma)}$$

$$= \frac{K}{\alpha} \prod_{j=0}^{\sum_{k=1, (k\neq i)}^{N} u_k} \sigma^{-1}(e^{\lambda\tau_j} - 1). \tag{18}$$

It follows that the LST (Laplace-Stieltjes transform) of the DF (distribution function) for the intervisit time, $I_1(s)$ and $I_2(s)$ for types 1 and 2, respectively, is given by

$$I_1(s) = \frac{K}{1 - \alpha + \alpha\sigma} \cdot \left[e^{(\lambda-s)R} + \sum_{n=1}^{N-1} \binom{N-1}{n} e^{(\lambda-s)\tau_n} \sigma^{-n} \prod_{j=0}^{n-1} (e^{\lambda\tau_j} - 1)\right], \tag{19}$$

$$I_2(s) = \frac{K}{\alpha} \sum_{n=0}^{N-1} \binom{N-1}{n} e^{-s\tau_n} \sigma^{-(n+1)} \prod_{j=0}^{n} (e^{\lambda\tau_j} - 1). \tag{20}$$

The corresponding means are given by

$$E[I_1] = \frac{K}{1 - \alpha + \alpha\sigma} \cdot \left[Re^{\lambda R} + \sum_{n=1}^{N-1} \binom{N-1}{n} \tau_n e^{\lambda\tau_n} \sigma^{-n} \prod_{j=0}^{n-1} (e^{\lambda\tau_j} - 1)\right], \tag{21}$$

$$E[I_2] = \frac{K}{\alpha} \sum_{n=0}^{N-1} \binom{N-1}{n} \tau_n \sigma^{-(n+1)} \prod_{j=0}^{n} (e^{\lambda\tau_j} - 1), \tag{22}$$

from which we can readily compute

$$E[T_{c1}] = E[I_2] + b. \tag{23}$$

The LST of DF for $T_{wait}$, denoted by $W(s)$, is given by [see Equation (2.56b) of [11] or Equation (37) of [8] for derivation]

$$W(s) = \frac{\lambda[I_1(s) - I_1(\lambda)]}{(\lambda - s)[1 - I_1(\lambda)]}, \tag{24}$$

from which we obtain

$$E[T_{wait}] = \frac{E[I_1]}{1 - I_1(\lambda)} - \frac{1}{\lambda}. \tag{25}$$

Using (23) and (25), we can compute the conditional response time $E[T \mid L]$ by (13). Unconditioning (13) on $L$,

**Table 2** Comparison of results in Wu and Chen [1] and this paper. In each triplet, the first value is the approximate result of [1], the second value is the simulation result of [1] (using 2000 messages), and the third value is the exact result of this paper. It is assumed that $1/\sigma = 10$ and $R/(Nb) = 0.05$.

| *Cases* | $N = 5$ $\lambda b = 0.0066$ | $N = 7$ $\lambda b = 0.0046$ | $N = 5$ $\lambda b = 0.0505$ | $N = 7$ $\lambda b = 0.0316$ |
|---|---|---|---|---|
| $\lambda E[T_m]$ | 0.1046 | 0.08045 | 1.805 | 1.457 |
| | 0.1032 | 0.08181 | 1.749 | 1.505 |
| $\lambda E[T]$ | 0.10448 | 0.080363 | 1.7999 | 1.4514 |
| $\lambda E[T_w]$ | 0.01549 | 0.01083 | 0.005592 | 0.003497 |
| | 0.01575 | 0.01015 | 0.005555 | 0.003479 |
| | 0.015490 | 0.010834 | 0.0054982 | 0.0034194 |
| $\lambda E[T_c]$ | 0.002353 | 0.002294 | 0.1266 | 0.1110 |
| | 0.002336 | 0.002337 | 0.1273 | 0.1115 |
| $\lambda E[C]$ | 0.0023531 | 0.0022936 | 0.12858 | 0.11327 |
| $\lambda E[T_{cl}]$ | 0.01065 | 0.008219 | 0.1866 | 0.1511 |
| | 0.01061 | 0.008243 | 0.1844 | 0.1504 |
| | 0.010650 | 0.0082170 | 0.18662 | 0.15097 |

we have a relationship

$$E[T] = E[T_{\text{wait}}] + b + \left(\frac{1}{\sigma} - 1\right)E[T_{\text{cl}}], \tag{26}$$

which can also be proved algebraically using the above equations.

In our **Table 2**, we compare our exact results with the approximation and simulation results given in Table 1 of [1]. We see that the approximation results of [1] and our exact results are very close. However, our solution is much simpler than that of [1].

## Remarks

The model we have solved can also be viewed as a polling system with feedback. Namely, a message departs with probability $\sigma$ and does not with probability $1 - \sigma$ after service completion. Such a case occurs in the error-prone transmission channel, as pointed out in Kuehn [12]. The model is also applicable to a time-sharing system with $N$ multiprogramming levels where a processor gives service quanta in a round-robin fashion. In another paper [13], we have analyzed a similar polling system with feedback, where there is infinite queueing capacity at each queue. An application suggested in [13] is one to the token-ring local-area network where a long message (e.g., a file or program) is transmitted by segments. The user's interest is his mean time until the whole message is transmitted. We finally note an application to a model of a crossbar switching machine in [14].

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