

The group problem and integer programming duality

by Ellis L. Johnson

Some duality results for integer programming based on subadditive functions are presented first for linear programs and then for the group problem. A similar result for the knapsack problem is given and then a relationship between facets for the group relaxation and facets of the knapsack problem is given. The mixed integer cyclic group problem is then considered and a dual problem given. A common theme is to try to characterize the strongest possible dual problem or equivalently the smallest possible cone of subadditive functions.

1. Introduction

A shortcoming of integer programming theory is that it has not provided a dual problem to use algorithmically as the dual linear programming problem is used. Dual variables in linear programming are also used as prices and for local sensitivity analysis when the right-hand side is perturbed by very small amounts. We ignore this use of the dual problem for the simple reason that we do not know, in general, how to give a price interpretation to the dual problem.

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The dual problem of linear programming provides another major benefit, however: It gives an optimality criterion toward which the simplex algorithm moves. That is to say, if a primal solution and a dual solution can be found so that their objective values are equal, then the solutions are both optimal to their respective problems. Even more, the complementary slackness conditions are sufficient conditions for the solutions to be optimal without knowing the objective values, and it is these conditions that the simplex method, in its usual form, maintains while working toward dual feasibility. We do not discuss solution methods here, but it is our hope that the above uses of the linear programming dual will carry over, to some extent, to the integer programming problem.

Linear programming duality is presented in Section 2 in this way. A different dual problem for linear programs is presented in order to introduce the idea of a dual problem involving functions and in order to show two duals, one of which is weaker than the other. Section 3 gives a more general primal problem for which this dual is still a correct dual problem.

One way to get a dual problem to integer programming problems is to form the convex hull of integer solutions and then use linear programming duality on the resulting linear program. The difficulty with this statement is that a system of defining inequalities is required in order to have the linear program in a form that dual variables can be assigned to linear equations and inequalities. From a pricing point of view, the meaning of dual variables assigned to inequalities not originally present presents major difficulties. From an algorithmic point of view, it is frequently not possible.

2. Linear programming duality

The standard form of the linear programming (primal) problem [1] is to minimize cx subject to

$$Ax = b \text{ and } x \geq 0, \quad (1)$$

and the dual problem is then to maximize πb subject to

$$\pi A \leq c. \quad (2)$$

The weak duality theorem says that $cx \geq \pi b$ for all x satisfying (1) and π satisfying (2). The strong duality theorem asserts that there exists an x satisfying (1) and a π satisfying (2) such that $cx = \pi b$. Such an x and π will obviously be optimal solutions to the primal and dual problems, respectively.

The complementary slackness theorem ([1], p. 136) implies that an x satisfying (1) is optimal whenever there is a dual solution π satisfying (2) and the additional condition

$$x_j > 0 \text{ implies } \pi A^j = c_j.$$

There are other dual problems that also satisfy weak and strong duality. In order to introduce the notion of a dual problem over a cone of functions and the notion of stronger dual classes, we give another dual problem [2]. For a fixed m (number of rows in A), let C be the cone of functions

$$C = \{f: R^m \rightarrow R \mid \lambda f(x) = f(\lambda x) \text{ for any } x \in R^m \text{ and } \lambda \geq 0, \\ \text{and } f(x + y) \leq f(y), \text{ all } x, y \in R^m\}.$$

The alternative dual problem is

maximize $\pi(b)$ subject to

$$\pi(A^j) \leq c_j, \text{ and}$$

$$\pi \in C.$$

The normal dual is for functions that are, in fact, linear over R^m . Such linear functions are a subspace contained in C . The functions in C are the positively homogeneous and convex functions on R^m and are called gauge functions when they are nonnegative [3].

The proofs of both weak and strong duality are easy for this alternative dual problem. We can say that it is a weaker dual in that strong duality continues to hold for a proper subset of C , namely the linear functions. The linear functions are the smallest subset of C for which strong duality holds for all problems on m rows because for any given vector π in R^m we can find a linear program so that the dual is uniquely optimized over linear functions by π . We conjecture that C is the largest convex cone of functions for which weak duality holds.

3. Linear programs with multiple right-hand sides

The (primal) problem considered in this section is to minimize cx subject to

$$x \geq 0, \text{ and } Ax \in R, \quad (3)$$

where R is a finite set of acceptable right-hand-side values.

The dual problem we claim is similar to the alternative dual from the preceding section:

maximize $\{\min \pi(b) \mid b \in R\}$ subject to

$$\pi(A^j) \leq C_j, \text{ and}$$

$$\pi \in C,$$

where C is the cone of functions in Section 2. Weak duality is again easy to prove:

$$\min_{b \in R} \pi(b) \leq \pi(Ax) \leq \sum_j \pi(A^j)x_j \leq \sum_j c_j x_j,$$

for any $\pi \in C$ and x satisfying (3). In [2], the problem is considered in inequality form, and both weak and strong duality are proven. The proofs are not difficult. The result is also given there that a linear dual suffices if and only if the primal problem is equivalent to

$$Ax \in \text{conv}(R), x \geq 0,$$

where $\text{conv}(R)$ denotes the convex hull of the vectors in R . An example given where a linear dual works is when A is a Leontief substitution matrix.

In order to get a stronger dual problem for the general problem, the cone C of dual functions can be characterized by linear inequalities (Theorem 11 of [3]). One way to view the necessary inequalities is by means of support functions. Positively homogeneous convex functions f are in fact support functions; i.e., there is some convex set S such that

$$f(x) = \min \{x \cdot y \mid y \in S\}.$$

The question considered in [3] is: What are the convex sets S such that the resulting support function f gives a facet of the convex hull of solutions x to (3)? Just as in the proof in the next section, linear programming duality and a facet characterization give a strongest dual problem in our sense. Without going into details, a stronger dual problem for a given matrix A is given by requiring f to be in the cone

$$C_A = \{f: R^m \rightarrow R \mid f(x) = \max \{x \cdot x^* \mid x^* \in P^*\}\},$$

where P^* can be any polyhedron of the form

$$P^* = \{x^* \mid x^* \cdot A^j \leq \mu_j\}.$$

4. The group problem

The group problem is to minimize

$$\sum_{g \in G_+} c(g)t(g)$$

subject to

$$\sum_{g \in G_+} g t(g) = b, t \geq 0 \text{ and integer,}$$

where G is a finite Abelian group and G_+ denotes the nonzero elements of G . By $gt(g)$, where $t(g)$ is a nonnegative integer, we mean g added to itself $t(g)$ times. The problem as stated is sometimes called a *master problem* because there is a variable $t(g)$ for every nonzero group element. A problem with only a subset of these elements present is called a *subproblem*. Gomory showed relationships between master problems and subproblems ([4], Theorems 12 and 13), and in this section we restrict ourselves to master problems.

Gomory introduced this problem as a relaxation of a pure integer programming problem. He also proved an "asymptotic" theorem that gives sufficient conditions for the integer programming problem to be solved by solving the group relaxation. Here we focus on another of Gomory's results: his subadditive characterization of facets ([4], Theorem 18), which leads to a duality result.

The problem we present as the dual problem to the group problem is to maximize $\pi(b)$ subject to

$$\pi(g) \leq c(g), \text{ and}$$

$$\pi \in S_b = \{f: G \rightarrow R \mid f(g+h) \leq f(g) + f(h), \text{ and}$$

$$f(b) = f(g) + f(b-g)\}.$$

The set of functions S_b is clearly a polyhedral cone that we will call the *b-complementary subadditive cone*.

We still have the weak duality result $\pi(b) \leq cx$ and the strong duality theorem $\pi(b) = cx$ for some solution to the two problems. The proof of weak duality is easy:

$$\begin{aligned} \pi(b) &= \pi\left(\sum_g gt(g)\right) \\ &\leq \sum_g \pi(gt(g)) \leq \sum_g \pi(g)t(g) \leq \sum_g c(g)t(g). \end{aligned}$$

This proof also shows what is needed for equality to hold: $t(g) \geq 1$ implies $\pi(g) = c(g)$; and $x(g) \geq 1$ and $x(h) \geq 1$ [or $x(g) \geq 2$, i.e., $g = h$] implies $\pi(g+h) = \pi(g) + \pi(h)$. The first of these conditions is complementary slackness, and we refer to the second as complementary linearities.

The strong duality result is based on linear programming duality and Gomory's subadditive characterization of facets [4]. That characterization implies that the group problem is equivalent to the linear program

$$\text{minimize } \sum c(g)t(g) \text{ subject to}$$

$$t(g) \geq 0,$$

$$\sum_{g \in G_+} \pi(g)t(g) \geq \pi(b), \text{ for all } \pi \in S_b,$$

in that there is an optimal answer to the above linear program that is integer for any objective function c for which the problem has an optimum; i.e., $c(g) \geq 0$, all $g \in G_+$. As stated, the above problem is not exactly a linear program in that there is a constraint for every $\pi \in S_b$; i.e., there are an infinite number of constraints. However, S_b is a polyhedral

cone, as is clear from its definition, so the infinite number of constraints can be reduced to one constraint for a π on each extreme ray of S_b ; i.e.,

$$\text{minimize } c(g)t(g) \text{ subject to}$$

$$x(g) > 0,$$

$$\sum_{g \in G_+} \pi^k(g)t(g) \geq \pi^k(b), \quad k = 1, \dots, K,$$

where π^k , $k = 1, \dots, K$, are vectors, one on each extreme ray of S_b . Gomory's result ([4], Theorem 18), says that all of these inequalities are required for master problems. That is, if any one of the inequalities corresponding to some extreme ray π^k of S is dropped from the inequality system, then there will be some objective function $c(g)$, $g \in G_+$, such that the linear program has only noninteger optimal solutions. In other words, Gomory gave the smallest linear inequality system for master group problems. Of course, there may be a huge number of these inequalities even for moderate size groups. Also, the extreme rays are not very well characterized, so they cannot be easily produced or identified. However, they are, in a sense, characterized as extreme rays of S_b , and to that extent we have a linear programming equivalent of the group problem.

Gomory's characterization of this linear programming equivalent of the group problem together with linear programming duality gives the dual problem previously introduced. To see this result, let λ_k be the linear programming dual variable corresponding to the constraint k in the above linear program. Then, the dual linear programming problem is

$$\text{maximize } \sum_{k=1}^K \lambda_k \pi^k(b) \text{ subject to}$$

$$\lambda_k \geq 0, \quad k = 1, \dots, K, \text{ and}$$

$$\sum_{k=1}^K \lambda_k \pi^k(g) \leq c(g).$$

Since S_b is a convex cone and since $\lambda_k \geq 0$, $k = 1, \dots, K$, the function

$$\pi(g) = \sum_{k=1}^K \lambda_k \pi^k(g)$$

is in S_b and is the desired dual function. Thus, an optimal solution λ_k to the linear programming dual will produce an optimal dual function π in S_b , and strong duality is proven. Gomory's result that all of the inequalities π^k are needed implies that S_b is the smallest convex cone for which strong duality holds for master group problems. In other words, it is the smallest cone that works for any subproblem of a given master problem. We remark that weak and strong duality both hold for the larger cone S_b obtained by dropping the

conditions that $f(b) = f(g) + f(b - g)$. In the proof of weak duality, that condition was not used, so weak duality still holds. Strong duality obviously holds because we have just expanded the cone S_b . However, expanding S_b gives a weaker dual in the sense already discussed in Section 2.

5. Knapsack problem

The master knapsack problem is

minimize $\sum c_j x_j$ subject to

$x_j \geq 0$ and integer, $j = 1, \dots, n$, and

$$\sum_{j=1}^n jx_j = n,$$

where n is any positive integer and $c \in R^n$. The dual problem for the knapsack problem is

maximize $\pi(n)$ subject to

$\pi(j) \leq c_j$, $j = 1, \dots, n$, and

$\pi \in T_n = \{f: \mathbb{Z}_n \rightarrow R \mid f(n) = f(j) + f(n-j), \text{ and}$

$$f(i+j) \leq f(i) + f(j)$$

whenever $i+j \leq n\}$,

where $\mathbb{Z}_n = \{1, \dots, n\}$, the set of integers from 1 to n .

This result first appeared in Aroaz [5] (see also Johnson [6] and Aroaz and Johnson [7]). A subadditive lifting method [8] based on this dual has been given.

One way to view the cone T_n as a subadditive cone is to introduce a semi-group addition $\hat{+}$ over $\mathbb{Z}_n$ defined to be

$$i \hat{+} j = \begin{cases} i+j, & \text{if } i+j \leq n, \\ \emptyset, & \text{otherwise.} \end{cases}$$

In [6] this unifying approach was presented, and there an infeasible element was used in place of $\emptyset$. In [9], the convention that the sum of two elements may simply be an empty set was adopted and has the same effect in defining the subadditive cone T_n : no inequality is imposed on the pair $f(i), f(j)$ whenever $i+j > n$.

For example, letting $n = 3$ gives the problem $x_1 + 2x_2 + 3x_3 = 3$. There are only three solutions: $x = (3, 0, 0)$, $(1, 1, 0)$, and $(0, 0, 1)$. The convex hull of these three points is shown in Figure 1 and has the inequality description

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0,$$

$$x_1 + 2x_2 + 3x_3 = 3,$$

$$x_1 + x_3 \geq 1.$$

The addition table here is

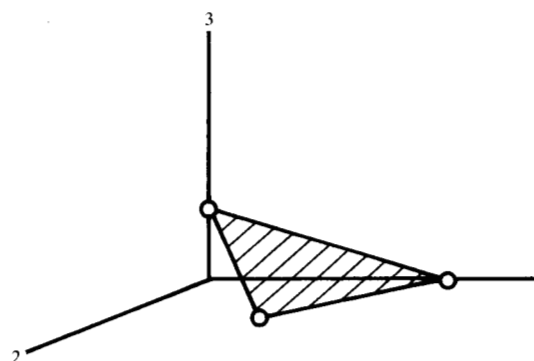


Figure 1

Convex hull of knapsack solutions.

+	1	2	3
1	2	3	0
2	3	0	0
3	0	0	0

The subadditive cone T_n is defined by the two inequalities

$$f(3) = f(1) + f(2),$$

$$f(2) \leq 2f(1),$$

and is drawn in Figure 2. It has lineality generated by $(1, 2, 3)$, giving the equation $x_1 + 2x_2 + 3x_3 = 3$, and generating ray $(1, 0, 1)$, giving the inequality $x_1 + x_3 \geq 1$.

The dual problem here is to maximize $\pi(3)$ subject to the above equation and inequality; i.e., $\pi(3) = \pi(1) + \pi(2)$ and $\pi(2) \leq 2\pi(1)$, and satisfying $\pi(j) \leq c_j$, $j = 1, \dots, 3$, where c_1, c_2, c_3 are the coefficients in the primal objective function.

The dual problem can also be stated in terms of the linear programming dual as

maximize $3\lambda_1 + \lambda_2$ subject to $\lambda_2 \geq 0$ and

$$\lambda_1 + \lambda_2 \leq c_1,$$

$$2\lambda_1 \leq c_2,$$

$$3\lambda_1 + \lambda_2 \leq c_3.$$

The relationship between this dual problem and the dual function is that the function π is given by

$$\pi = \lambda_1(1, 2, 3) + \lambda_2(1, 0, 1).$$

Conversely, if a dual function π in T_b is given, then π is a linear combination of $(1, 2, 3)$ and $(1, 0, 1)$, where the

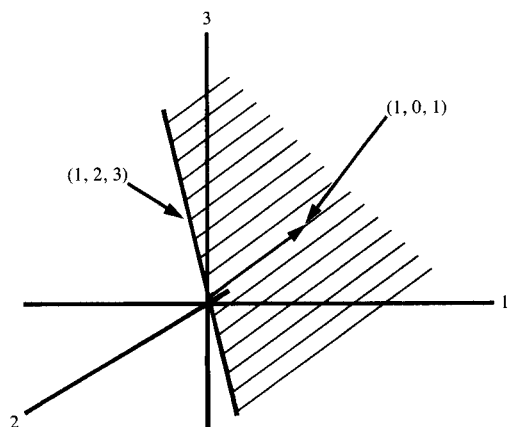


Figure 2

Subadditive cone.

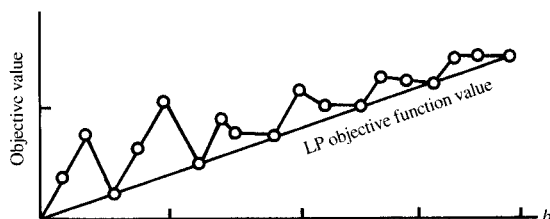


Figure 3

IP and LP objective function values.

coefficient λ_2 of $(1, 0, 1)$ must be nonnegative. These multipliers give the values of λ_1 and λ_2 for the linear programming dual.

6. Cyclic group relaxations of the knapsack problem

Gilmore and Gomory [10] considered methods for solving knapsack problems in order to generate columns for the linear programming formulation of the stock cutting problem. In the process they showed a periodicity of the solutions. We give an example. Consider the knapsack problem (not a master problem)

minimize $x_1 + 2x_2 + 3x_3$ subject to

$x_j \geq 0$ and integer, $j = 1, 2, 3$, and

$3x_1 + x_2 + 8x_3 = 10$.

For a moment, consider that the right-hand side is not 10 but a parameter that is varied while the remainder of the problem is fixed. Then the solutions are given in Table 1.

The periodicity in the solution begins at $b = 15$. From that point on, $x_2 = 0$, $x_3 = 0, 2, 1$, and then repeats, whereas $x_1 = 5, 0, 3$, then 6, 1, 4, then 7, 2, 5, etc. The general pattern is that from some point on there is some one variable x_j , $j = j^*$, that increases by one every time b increases by a_j , and the other variables simply repeat with period a_j . The objective function increases by c_j every period (see Figure 3).

In general, a knapsack problem (not a master problem) has the form

minimize $c_1x_1 + c_2x_2 + \dots + c_nx_n$ subject to

$x_j \geq 0$ and integer, $j = 1, \dots, n$, and

$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$,

where a_j is a positive integer less than or equal to b , a positive integer, and c_j is a real number that can be assumed to be positive, without loss of generality, by adding the equation to the objective function. Let us assume that the x have been ordered so that

$c_1/a_1 \leq c_j/a_j$, $j = 2, \dots, n$.

Then, the optimal linear programming solution is $x_1 = b/a_1$, with cost c_1b/a_1 . If the nonnegativity condition on x_1 is relaxed, then the problem becomes

minimize $(c_2 - a_{2a_1}^c)x_2 + \dots + (c_n - a_{na_1}^c)x_n$ subject to

$a_2x_2 + \dots + a_nx_n \equiv b(\text{mod } a_1)$,

$x_j \geq 0$ and integer, $j = 2, \dots, n$,

which is a special case of the group problem from Section 4 but is not generally a master group problem. The answer to the group problem gives values $(x_2^*, \dots, x_n^*)$ for the nonbasic variables which are optimal to the original knapsack problem provided x_1 given by

$x_1 = (b - (a_2x_2^* + \dots + a_nx_n^*)) / a_1$

is nonnegative. This x_1 is integer because $a_2x_2^* + \dots + a_nx_n^*$ is congruent to b modulo a_1 .

In the example given, the group problem obtained by relaxing nonnegativity of x_1 is

minimize $\frac{1}{3}x_2 + \frac{1}{3}x_3$ subject to

$x_2 + 10x_3 \equiv b(\text{mod } 3)$ and $x_j \geq 0$ and integer, $j = 2, 3$.

There are three different optimal solutions for different values of the right-hand side b : $(x_2, x_3) = (0, 0)$ for b

Table 1 Knapsack solutions.

b	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
x_1	0	0	1	1	1	2	2	0	3	3	1	4	4	2	5	0	3	6	1	4	7
x_2	1	2	0	1	2	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0
x_3	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	2	1	0	2	1	0
obj	2	4	1	3	5	2	4	3	3	5	4	4	6	5	5	6	6	6	7	7	7

congruent to 0 modulo 3, $(x_2, x_3) = (0, 2)$ for b congruent to 1 modulo 3, and $(x_2, x_3) = (0, 1)$ for b congruent to 2 modulo 3. These three solutions are optimal to the knapsack problem when $b \geq 0$, $b \geq 16$, and $b \geq 8$, respectively, for the three different values of b modulo 3.

Aroaz and Johnson [9] have shown that very many facets of the knapsack problem can be obtained from facets for the group relaxation. This result can be contrasted with the above result on getting solutions to the knapsack problem from solutions for the group relaxation. Consider the master knapsack problem

$$1x_1 + 2x_2 + \dots + nx_n = n, x_j \geq 0 \text{ and integer,}$$

and its group relaxation, which can be written

$$1t_1 + 2t_2 + \dots + (d-1)t_{d-1} \equiv n(\text{mod } d), t_j \geq 0 \text{ and integer,}$$

where d is a positive integer $d < n$ and $n(\text{mod } d)$ is not equal to zero. A valid inequality for the group relaxation

$$\pi_1 t_1 + \pi_2 t_2 + \dots + \pi_{d-1} t_{d-1} \geq \pi_0$$

can be expanded to a valid inequality for the knapsack problem

$$\pi_1 t_1 + \pi_2 t_2 + \dots + \pi_{d-1} t_{d-1} + \pi_1 t_{d+1} + \pi_2 t_{d+2} + \dots + \pi_{d-1} t_{2d-1} + \dots \geq \pi_0$$

in the periodic way indicated above. The result of Aroaz and Johnson is that if we start with a facet for the group relaxation of the master knapsack problem and if $d < n/2 + 1$, then the resulting valid inequality for the knapsack problem is also a facet for that problem. Thus the facets of group relaxations may give facets for knapsack problems and always do under the condition above.

7. Dual functions for mixed integer cyclic group problems

In order to introduce the mixed integer programming dual, we begin with a special case: the cyclic group problem with continuous variables. This problem was introduced by Gomory and Johnson [11], and the results given here appear there except that the dual problem is not explicitly stated there but is a consequence of their results.

The cyclic group problem with continuous variables is

minimize $ct + c^+ s^+ + c^- s^-$ subject to

$$t_j \geq 0 \text{ and integer, } j = 1, \dots, n; s^+ \geq 0; s^- \geq 0;$$

$$\sum_{j=1}^n a_j t_j + s^+ - s^- \equiv b(\text{mod } 1).$$

Thus, s^+ and s^- are continuous variables. For such a problem, it only makes sense to have at most two continuous variables since there is only one constraint and any other variables could be scaled to have a coefficient of plus or minus one. The a_j are real numbers but can be assumed to be in the interval $(0, 1)$. Since we can have any vector $(a_1, \dots, a_n)$, for any $n \geq 1$, the solutions t can be considered to be functions on the unit interval having finite skeleton; i.e., $t(a) = 0$ except on a finite set.

The dual problem is

maximize $\pi(b)$ subject to

$$\pi(a_j) \leq c_j, \pi^+ \leq c^+, \pi^- \leq c^-, \quad \pi \in C,$$

where C is the cone

$$C = \{f: [0, 1] \rightarrow R \mid f(b) = f(a) + f(b-a),$$

$$f(a+a') \leq f(a) + f(a'), \text{ all } a, a' \in [0, 1]\},$$

and f has direction derivatives at 0 and 1. The π^+ and π^- are defined to be the directional derivatives

$$\pi^+ = \lim_{a \rightarrow 0^+} f \frac{(a)}{a},$$

$$\pi^- = \lim_{a \rightarrow 1^-} f \frac{(1-a)}{1-a}.$$

Although the convex hull of all solutions t is not polyhedral (not even closed), if we fix $a_1, \dots, a_n$ to be rational numbers in the unit interval $(0, 1)$, then the restriction of the solutions to this subproblem does have a polyhedral convex hull, and the facets of it are all given by functions f in C . In [11], some of these functions are identified as always giving facets for every rational subproblem.

References

1. G. B. Dantzig, *Linear Programming and Extensions*, Princeton University Press, Princeton, NJ, 1963.
2. D. Granot, F. Granot, and E. L. Johnson, "Duality and Pricing in Multiple Right-Hand Choice Linear Programming Problems," *Math. Oper. Res.* 7, 545-556 (1982).
3. E. L. Johnson, "Characterization of Facets for Multiple Right-Hand Choice Linear Programs," *Math. Program. Study* 14, 112-142 (1981).
4. R. E. Gomory, "Some Polyhedra Related to Combinatorial Problems," *Linear Alg. Appl.* 2, 451-558 (1969).
5. J. Aroaz, *Polyhedral Neopolarities*, Ph.D. Thesis, Department of Computer Science and Applied Analysis, University of Waterloo, Waterloo, Ontario, Canada, 1973.
6. E. L. Johnson, "Integer Programming: Facets, Subadditivity and Duality for Group and Semigroup Problems," *CBMS-NSF Regional Conference Series in Applied Mathematics*, Vol. 32, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1980.
7. J. Aroaz and E. L. Johnson, "Some Results on Polyhedra of Semigroup Problems," *SIAM J. Alg. Disc. Math.* 2, 244-258 (1981).
8. E. L. Johnson, "Subadditive Lifting Methods for Partitioning and Knapsack Problems," *J. Algorithms* 1, 75-96 (1980).
9. J. Aroaz and E. L. Johnson, "Mappings and Liftings for Group and Semigroup Problems," in preparation.
10. P. C. Gilmore and R. E. Gomory, "The Theory and Computation of Knapsack Fractions," *Oper. Res.* 14, 1045-1074 (1966).
11. R. E. Gomory and E. L. Johnson, "Some Continuous Functions Related to Corner Polyhedra," *Math. Program.* 3, 23-85 and 359-389 (1972).

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