A maximum-energy-concentration spectral window

by Lineu C. Barbosa

A time-discrete solution method is proposed for realization of a spectral window which is ideal from an energy concentration viewpoint. This window is one that concentrates the maximum amount of energy in a specified bandwidth and hence provides optimal spectral resolution.

Introduction

In computing the spectrum of a truncated data sequence, it is frequently desirable to multiply the data by a set of weights. This is motivated by the fact that the truncation itself corresponds to a set of weights all equal to one within the truncation range and to zero outside it, which is equivalent to a convolution of the untruncated data spectrum with a sin x/x type of spectrum. This, in turn, can lead to a spectral result with the well-known Gibbs oscillations if the original data have discontinuities or sharp variations in their spectrum. A similar phenomenon can also occur in the design of finite impulse response (FIR) digital filters.

A number of spectral windows (i.e., the Fourier transform of the set of weights) have been proposed for use in both spectral analysis and design of FIR filters. One of these, an ideal window from an energy concentration viewpoint as noted in [1], is the prolate spheroidal function introduced in [2]. This function is the one with maximum concentration of energy in the frequency domain among all time-limited functions. It is appealing to use such a window function when high spectral resolution is desired: The maximum energy concentration within a frequency band implies minimum total energy spill outside the desired band, hence minimum total interference from side frequencies. However, the prolate spheroidal function, besides being difficult to compute, is a time-continuous function and, as such, it cannot be used with time-discrete data. The time-discretization of such a function certainly implies a loss in its ideal property. In [1] and [3], the authors propose an approximation to the prolate spheroidal function, namely, the I_0 − sinh window, also known as the Kaiser window.

Here we propose as an alternative a discrete-time realization having the same objectives as the prolate spheroidal function. This approach has already been used successfully by the author in connection with pulse slimming in magnetic recording [4]. It entails maximizing energy concentration directly in terms of the weights, without resorting to a discretized version of the continuous problem. This has the clear advantage of achieving the best realizable energy concentration for a particular vector size at the expense of not having a "closed" solution. The computation of the weights involves the search for the eigenvector corresponding to the maximum eigenvalue of a certain N by N symmetric matrix, N being the size of the data segment. Programs to compute such eigenvectors, however, are readily available.

It should be mentioned, moreover, that other windows exhibiting an equal ripple property have been reported in the literature, for example, in [5, 6]. These windows are superior when the *first* sidelobe attenuation is of concern. However, they also spread more overall energy in the sidelobes, and

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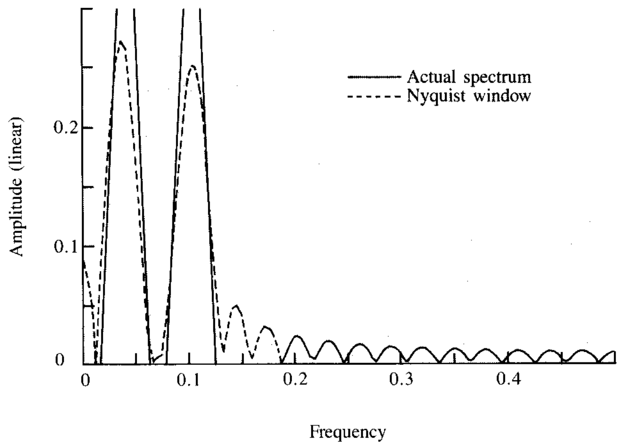

Figure 1

Spectra of the untruncated time sequence and the truncated time sequence using the Nyquist window.

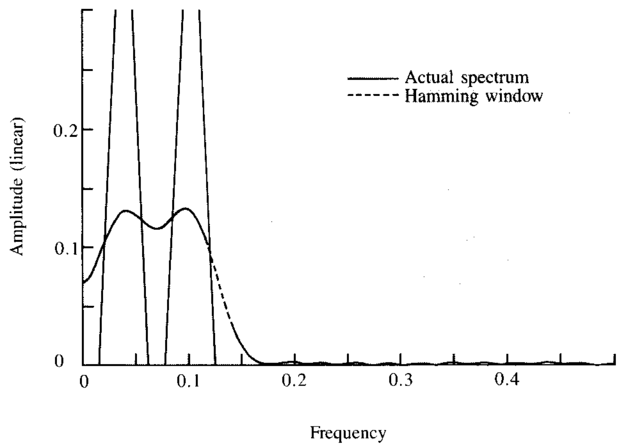

Figure 2

Spectra of the untruncated time sequence and the truncated time sequence using the Hamming window.

the calculation of these windows requires special computer programs to implement Remez-like algorithms. We stress the fact that the energy distribution of the window proposed here has a superior concentration in the main lobe, which implies that the overall side interference is minimal.

The maximum energy concentration window

Let the N data points (or "taps") be taken at the time intervals $t = k\Delta$, $k = 0, 1, \cdots, N-1$, and let the weights be the vector $\mathbf{w} = (w_0, w_1, \cdots, w_{N-1}) \in E^N$. The spectrum of $\mathbf{w}$, denoted by $F\mathbf{w}$, is then

$$H(f) \triangleq F\mathbf{w} = \sum_{k=0}^{N-1} w_k e^{-i2\pi f k\Delta}, \tag{1}$$

where the operator F maps E^N into $\mathcal{L}$, the spectral space of periodic functions with period $1/\Delta$.

Moreover, since F defines a unitary transformation, there also exists an inverse F^{-1} mapping $\mathcal{L}$ into E^N:

$$(F^{-1}H(f))_k = (F^*H(f))_k = \int_{-\frac{1}{2\Delta}}^{\frac{1}{2\Delta}} H(f)e^{i2\pi f k\Delta}df, \tag{2}$$

where F^* is the adjoint of F.

Select a frequency f_w, the edge of a frequency band $[-f_w, f_w]$ within which it is desired to concentrate the energy of the spectrum of the weights. Let P be a truncation function in $\mathcal{L}$ defined by

$$(PG)(f) = \begin{cases} G(f) & \text{for} \quad |f| \le f_w, \\ 0 & \text{elsewhere,} \end{cases} \tag{3}$$

and, for any $G(f) \in \mathcal{L}$, let the norm of G, $\|G\|$, be

$$\|G\|^2 = \int_{-\frac{1}{2\Delta}}^{\frac{1}{2\Delta}} |G(f)|^2 df.$$

Then, $PF\mathbf{w}$ is the portion of the spectrum of $\mathbf{w}$ within the above frequency band and the desired weights $\mathbf{w}$ can be computed by solving the following problem:

$$\max_{\mathbf{w}} \|PH\|^2 = \|PF\mathbf{w}\|^2 \quad \text{subject to} \quad \|F\mathbf{w}\|^2 = 1. \tag{4}$$

Let the scalar or dot products in E^N and in $\mathcal{L}$, respectively, be defined as follows: For $\mathbf{u}$ and $\mathbf{w}$ in E^N,

$$(\mathbf{u}, \mathbf{w}) = \sum_{k=0}^{N-1} u_k w_k,$$

and for H and G in $\mathcal{L}$,

$$\langle H, G \rangle = \int_{-\frac{1}{2\Delta}}^{\frac{1}{2\Delta}} H(f)G^*(f)df.$$

Note that $P^* = P = P^2$; hence,

$$\|PF\mathbf{w}\|^2 = \langle PF\mathbf{w}, PF\mathbf{w} \rangle = (\mathbf{w}, F^*PF\mathbf{w})$$

and

$$\|F\mathbf{w}\|^2 = (\mathbf{w}, F^*F\mathbf{w}) = (\mathbf{w}, I\mathbf{w}) = \|\mathbf{w}\|^2 \triangleq 1.$$

Then, it is well known from the theory of symmetric matrices that

$$(\mathbf{w}, F^*PF\mathbf{w}) \le \lambda \|\mathbf{w}\|^2$$

for some λ, the maximum eigenvalue of F^*PF, with equality if and only if

$$\lambda \mathbf{w} = F^*PF\mathbf{w}, \tag{5}$$

that is, $\mathbf{w}$ is an eigenvector of the matrix $\mathbf{M} = F^*PF$ corresponding to its maximum eigenvalue. By taking the dot product of the eigenequation (5) with $\mathbf{w}$ and considering the constraint $(\mathbf{w}, \mathbf{w}) = 1$,

$$\lambda = \| PF\mathbf{w} \|^2.$$

Hence, λ, the maximum eigenvalue of F^*PF, corresponds to the percentage of the spectral energy confined in the frequency band $[-f_w, f_w]$.

Let us now compute the matrix $\mathbf{M} = F^*PF$. First recall from (2) that

$$(F^*G)_k = \int_{-\frac{1}{2\Delta}}^{\frac{1}{2\Delta}} G(f) e^{i2\pi f k \Delta} df.$$

Hence, from (1)–(3), assuming $f_w < 1/2\Delta$,

$$(F^*PFw)_k = \int_{-f_w}^{f_w} \sum_{\ell} w_\ell e^{-i2\pi f \Delta(\ell - k)} df$$

$$= 2f_w \sum_{\ell} w_\ell \frac{\sin 2\pi f_w \Delta(\ell - k)}{2\pi f_w \Delta(\ell - k)} = (\mathbf{Mw})_k .$$

In turn, this implies that $\mathbf{w}$ is the eigenvector corresponding to the maximum eigenvalue of the matrix $\mathbf{M}$ with entries

$$m_{\ell k} = 2f_w \frac{\sin 2\pi f_w \Delta(\ell - k)}{2\pi f_w \Delta(\ell - k)} .$$

A variety of scientific software packages can be employed to obtain the eigenvector solution for matrix $\mathbf{M}$.

Results

In this section, results for the proposed window realization are compared with some for other well-known windows. Such comparison is difficult, and the final choice is inevitably dictated by subjective preferences. The first comparison here takes into account the resolution and the capability of making trade-offs on resolution (i.e., first lobe bandwidth) versus side interference (i.e., sidelobes). The original spectrum used for this comparison presents two peaks close together, as shown in **Figure 1.** The time domain data were truncated so as to have only 33 points, and the resulting spectrum was computed. This corresponds to the Nyquist window, as shown in Figure 1. Notice the high resolution of this window and the characteristic "ringing" in the baseline. **Figure 2** again shows the actual spectrum compared with the one obtained using the 33-point Hamming window. The only parameter in the Hamming window is the number of points, and, therefore, this window does not allow trade-offs. The result is a relatively broad main lobe, with the first "zero" at a frequency slightly higher than twice the Nyquist frequency. It hardly resolves the two peaks contained in the data. However, the baseline is smooth due to the sidelobes' high attenuation of about 42 dB. In **Figure 3,** the spectrum obtained by the use of the proposed window realization is compared with the actual spectrum. This window allows trade-offs and was computed for maximum energy within the Nyquist bandwidth. Its first "zero," however, occurs at a frequency about 38% higher. Notice the resolution of the peaks and the smooth baseline. The highest sidelobe for this window occurs at −24.5 dB.

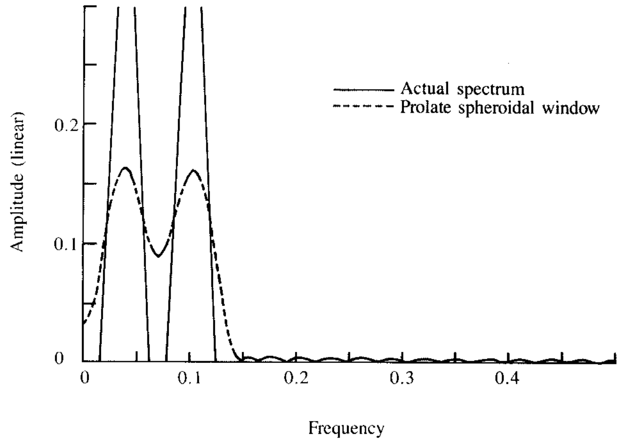

Figure 3

Spectra of the untruncated time sequence and the truncated time sequence using the discrete prolate spheroidal window.

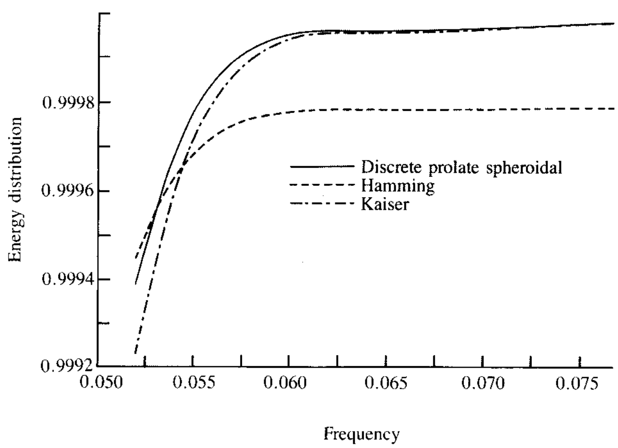

Figure 4

Spectral energy distribution comparison for the discrete prolate spheroidal window, the Hamming window, and the Kaiser window.

A second comparison was made on the basis of energy distribution. The energy concentration of the discrete prolate spheroidal window was compared with the energy concentration of the Hamming window, the Kaiser window, and the FIR window as proposed in [5]. The sample size in

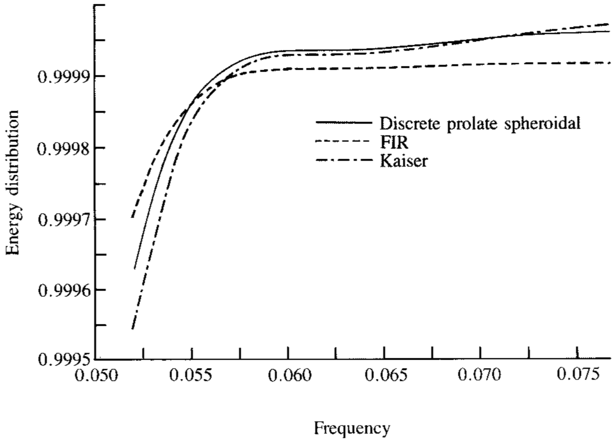

Figure 5

Spectral energy distribution comparison for the discrete prolate spheroidal window, the FIR window, and the Kaiser window.

these comparisons is $N = 33$. Due to the fact that the sidelobes are over 40 dB below the attenuation at zero frequency, the energy spills outside the main lobe are quite small. **Figure 4** shows the energy distribution (for each frequency f, the percentage of the total energy below f) for the discrete prolate spheroidal window, the Kaiser window, and the Hamming window; these windows have the first "zero" at the frequency 0.065, and the discrete prolate spheroidal window was computed for a maximum energy concentration at the frequency of 0.06. Notice the similarity between the results for the proposed ideal window realization and the Kaiser window. Notice also that the residual energy outside the main lobe is very small. **Figure 5** shows the comparison of the discrete prolate spheroidal window, the Kaiser window, and the FIR window as proposed in [5]. They all have the first "zero" at $f = 0.0624$. The discrete prolate spheroidal window, in this case, was computed to maximize the energy concentration at $f = 0.057$.

The third comparison was in terms of sidelobe attenuation. This comparison was made with windows having the main lobe (i.e., the first "zero") as similar as possible to the Hamming window corresponding to 33 data points as above, with the first "zero" at about $f = 0.065$. **Figures 6, 7, 8,** and **9** show, respectively, the spectra of the Hamming, the Kaiser, the discrete prolate spheroidal, and the FIR window of [5]. The discrete prolate spheroidal window has a sidelobe rejection of 44.5 dB as compared with 41.8 dB for the Hamming, 42.3 dB for the Kaiser, and 47.1 for the FIR. Notice that the main lobe of the FIR window is slightly narrower than the others, giving it an additional advantage in sidelobe rejection.

Conclusions

The discrete prolate spheroidal window is the one that maximizes the percentage of its energy within a specified frequency band. The maximization is done directly in terms of the window weights, and hence it is not a discrete approximation to the continuous version of the window problem, like the Kaiser window. The price to pay for having the optimal energy concentration is the lack of a closed solution. However, the numerical computation involves a standard eigenvector problem for an N by N symmetric matrix. For this study, such an eigenvector was found by iterating the matrix ten times, starting with an initial weight of one at the center and zero elsewhere. This contrasts with the Kaiser window, which is not optimal but has a closed solution and involves the computation of a Bessel function for $N + 1$ arguments. It was observed that the Kaiser window is indistinguishable from the discrete prolate spheroidal when the main lobe is near the Nyquist bandwidth. However, they differ at broader bandwidths, as seen in Figures 7 and 8. The FIR window reported in [5] is an equal-ripple-type window and, as such, has a better *first* sidelobe rejection instead of maximum energy concentration in the main lobe. Therefore, this window trades off close interference for resolution and total interference. The Hamming window is by far the simplest one, and its overall performance should be suitable for most applications. Except for the fact that the Hamming window does not allow trade-offs to be made, all other windows considered here are quite comparable for practical purposes. Figures 1, 2, and 3 show a requirement for resolution. At narrow bandwidth, when the energy concentration is important because the sidelobes are higher, it was verified that the Kaiser window performs as well as the discrete prolate spheroidal. However, all windows discussed here are indistinguishable when they have a broader main lobe, similar to that of the Hamming window.

Acknowledgments

The author wishes to thank J. W. Cooley for providing the weights of the FIR window as reported in [5]. Thanks are due to Sylvia Fujii for skillfully handling the manuscript. Finally, the author wishes to thank the referees for extensive reviews and many constructive suggestions that helped to make this paper more readable.

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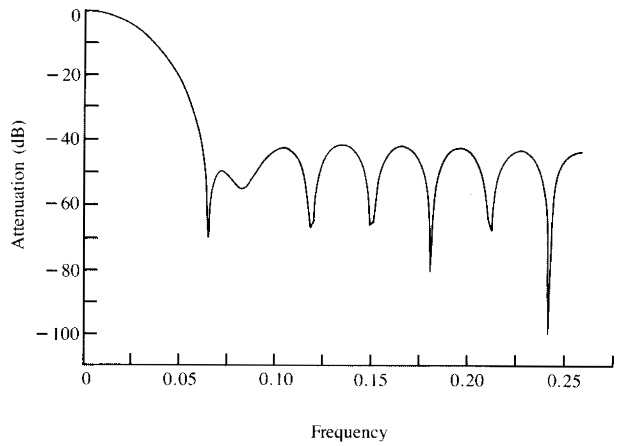

Figure 6

Spectral attenuation of the Hamming window.

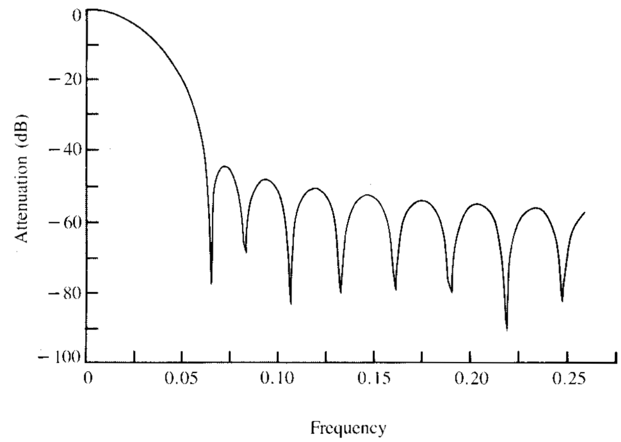

Figure 8

Spectral attenuation of the discrete prolate spheroidal window.

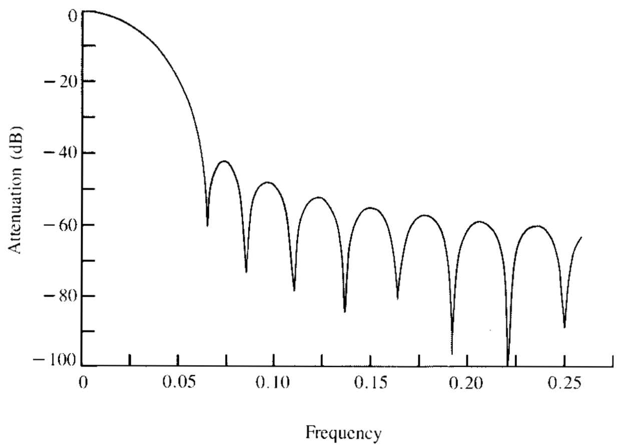

Figure 7

Spectral attenuation of the Kaiser window.

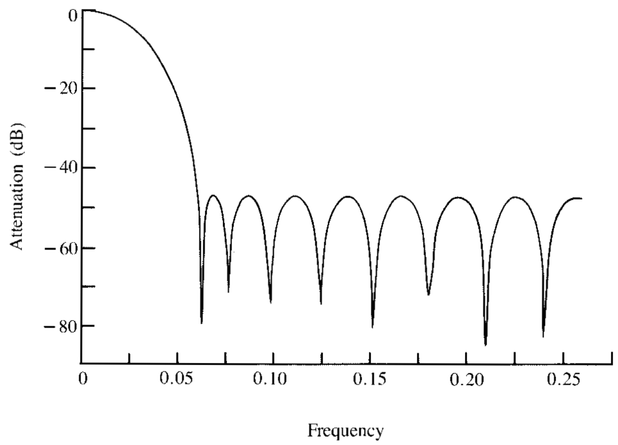

Figure 9

Spectral attenuation of the FIR window.

5. J. W. Cooley and S. Winograd, "On the Use of Filter Design Programs for Generating Spectral Windows," *Research Report RC-8728*, IBM Thomas J. Watson Research Center, Yorktown Heights, NY, January 15, 1981.
6. L. R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing*, Prentice Hall, Inc., Englewood Cliffs, NJ, 1975.

Received October 8, 1985; accepted for publication December 12, 1985

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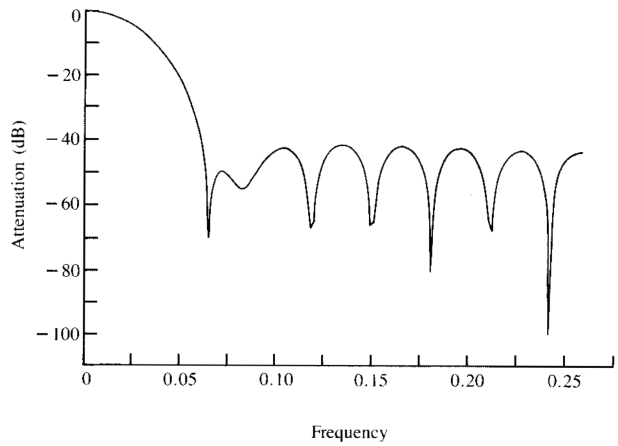

Figure 6

Spectral attenuation of the Hamming window.

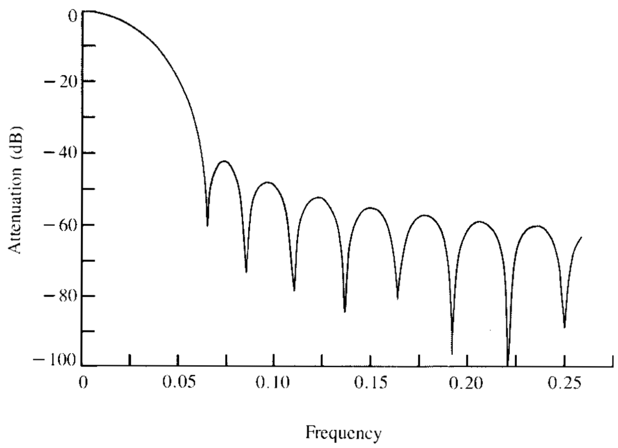

Figure 7

Spectral attenuation of the Kaiser window.

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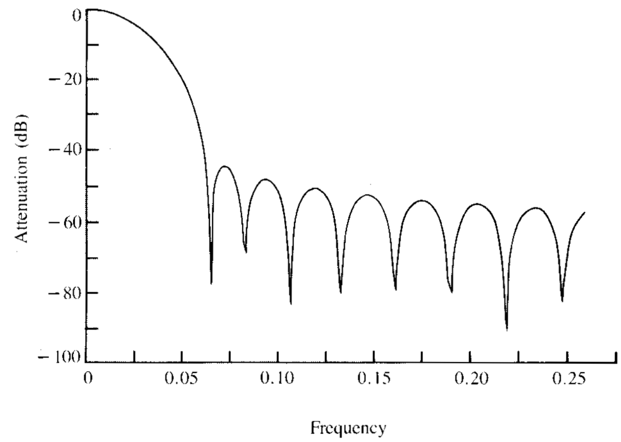

Figure 8

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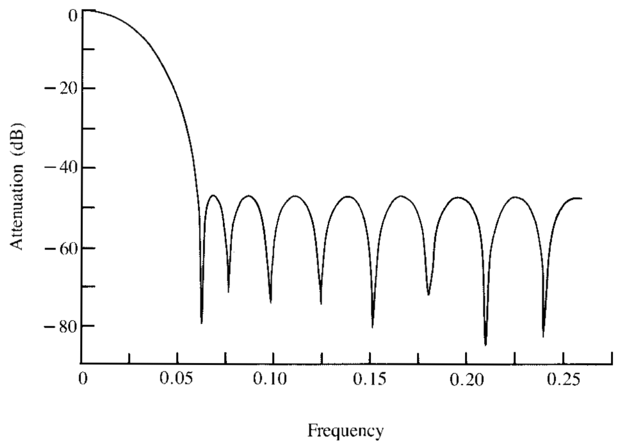

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