Regenerative simulation methods for local area computer networks

by Peter J. Haas
Gerald S. Shedler

Local area computer network simulations are inherently non-Markovian in that the underlying stochastic process cannot be modeled as a Markov chain with countable state space. We restrict attention to local network simulations whose underlying stochastic process can be represented as a generalized semi-Markov process (GSMP). Using "new better than used" distributional assumptions and sample path properties of the GSMP, we provide a "geometric trials" criterion for recurrence in this setting. We also provide conditions which ensure that a GSMP is a regenerative process and that the expected time between regeneration points is finite. Steady-state estimation procedures for ring and bus network simulations follow from these results.

1. Introduction

It is difficult to establish estimation procedures for local area computer network simulations that explicitly incorporate access control algorithms. Such simulations (see, e.g., Iglehart and Shedler [1, 2], Loucks, Hamacher, and Preiss [3]) are inherently non-Markovian in the sense that the underlying stochastic process cannot be modeled as a Markov chain with countable state space. Following [1] we restrict attention to local network simulations whose underlying stochastic process can be represented as a generalized semi-Markov process (GSMP).

Although steady-state estimation for an arbitrary GSMP is a formidable problem, estimation procedures [1, 2, 4] are available for GSMPs that are regenerative processes. To establish the regenerative property for a GSMP, it is necessary to show the existence of an infinite sequence of random time points at which the process probabilistically restarts. It is often the case that a GSMP associated with a local area network simulation probabilistically restarts when a particular event triggers a transition to some fixed state. For specific models, however, it is nontrivial to determine conditions (distributional assumptions) under which the underlying GSMP returns infinitely often to the fixed state.

A "geometric trials" argument given in [1] establishes a recurrence criterion for a stochastic process $\{X(t): t \geq 0\}$ with right-continuous and piecewise-constant sample paths and countable state space, S. Let $\{T_n: n \geq 0\}$ be an increasing sequence of finite state transition times for $\{X(t): t \geq 0\}$. The process $\{X(T_n): n \geq 0\}$ hits state $s' \in S$ infinitely often with probability one provided that $P\{X(T_n) = s' \mid X(T_{n-1}), \cdots, X(T_0)\} \geq \delta$ a.s. for some $\delta > 0$. This geometric trials recurrence criterion avoids the often unrealistic "positive density" assumptions needed in arguments (cf. [4]) based on general state space Markov chain theory.

In this paper, using sample path properties of the GSMP, we provide conditions which permit application of the geometric trials recurrence criterion in the GSMP setting. Our approach is to postulate the existence of a distinguished random time in the interval $[T_{n-1}, T_n)$ and a set of distinguished events determined by the state of the system at the distinguished time such that $X(T_n) = s'$ if each of the distinguished events occurs "soon enough" before time T_n. We show that $\{X(T_n): n \geq 0\}$ hits state s' infinitely often with probability one if the clock setting distributions associated with the distinguished events have "new better than used" (NBU) distributions and satisfy a "positivity" condition. We also establish additional conditions on the building blocks of the GSMP which ensure that the successive times at which $\{X(T_n): n \geq 0\}$ hits state s' are regeneration points for the process $\{X(t): t \geq 0\}$ and that the expected time between regeneration points is finite.

2. Regenerative generalized semi-Markov processes

Heuristically, a GSMP (Matthes [5], König, Matthes, and Nawrotzki [6, 7]) moves from state to state in accordance with the occurrence of events associated with the occupied state. Each of the several possible events associated with a state competes to trigger the next transition, and each of these events has its own distribution for determining the next state. At each state transition of the GSMP, new events may be scheduled. For each of these new events, a clock indicating the time until the event is scheduled to occur is set according to an independent (stochastic) mechanism. If a scheduled event does not trigger a transition but is associated with the next state, its clock continues to run; if such an event is not associated with the next state, it is abandoned.

Let S be a finite or countable set of *states* and $E = \{e_1, e_2, \cdots, e_M\}$ be a finite set of *events*. For $s \in S$, $E(s)$ denotes the set of all events that can occur when the GSMP is in state s. When the process is in state s, the occurrence of an event $e \in E(s)$ triggers a transition to a state s'. We denote by $p(s'; s, e)$ the probability that the new state is s' given that event e triggers a transition in state s. For each $s \in S$ and $e \in E(s)$ we assume that $p(\cdot; s, e)$ is a probability mass function. The actual event $e \in E(s)$ which triggers a transition in state s depends on *clocks* associated with the events in $E(s)$ and the *speeds* at which these clocks run. Each such clock records the remaining time until the event triggers a state transition. We denote by $r_{si}(\geq 0)$ the deterministic rate at which the clock associated with event e_i runs in state s; for each $s \in S$, $r_{si} = 0$ if $e_i \notin E(s)$. We assume that $r_{si} > 0$ for some $e_i \in E(s)$. (Typically in applications, all speeds r_{si} are equal to one. There are, however, models in which speeds other than unity as well as state-dependent speeds are convenient. For example, zero speeds are needed in queueing systems with service interruptions of the preemptive-resume type; cf. Shedler and Southard [8].) At a transition from state s to state s' triggered by event e^*, new clock times are generated for each $e' \in N(s'; s, e^*) = E(s') - (E(s) - \{e^*\})$. The distribution function of such a new clock time is denoted by $F(\cdot; s', e', s, e^*)$, and we assume that $F(0; s', e', s, e^*) = 0$. For $e' \in O(s'; s, e^*) = E(s') \cap (E(s) - \{e^*\})$, the old clock reading is kept after the transition. For $e' \in (E(s) - \{e^*\}) - E(s')$, event e' ceases to be scheduled after the transition.

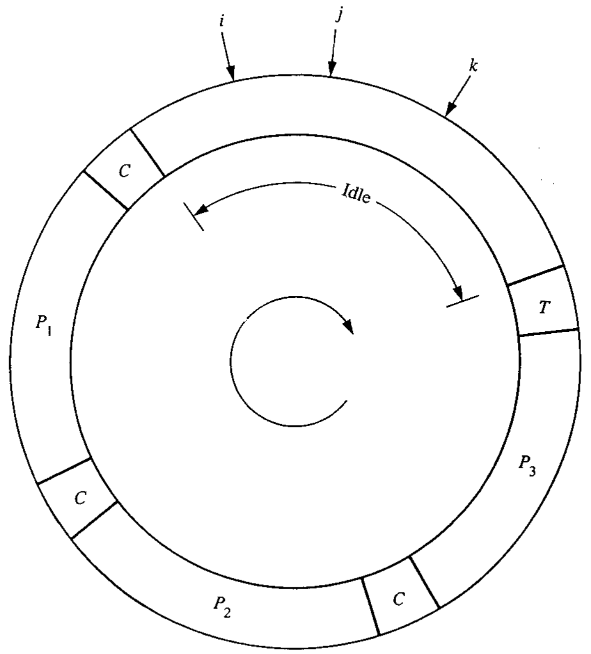

Figure 1

Token ring.

• *Example 1 (token ring)*

Consider a unidirectional ring network having a fixed number of *ports*, labeled $1, 2, \cdots, N$ in the direction of signal propagation; see **Figure 1.** (In the figure, i, j, and k denote three of the N ports.) At each port *message packets* arrive according to a random process. A single *control token* (denoted by T in Fig. 1) circulates around the ring from one port to the next. The time for the token to propagate from port $j - 1$ to port j is a positive constant, R_{j-1} ($j - 1 = N$ if $j = 1$). When a port observes the token and there is a packet queued for transmission, the port converts the token to a *connector* (C) and transmits a packet followed by the token pattern; the token continues to propagate if there is no packet queued for transmission. By destroying the connector prefix the port removes the transmitted packet when it returns around the ring. Assume that the time for port j to transmit a packet is a positive random variable, L_j, with finite mean. Also assume that packets arrive at individual ports randomly and independently of each other: The time from end of transmission by port j until the arrival of the next packet for transmission by port j is a positive random variable, A_j, with finite mean. Note that there is at most one packet queued for transmission at any time at any particular port.

For $t \geq 0$ set $N(t) = j$ if at time t port $j - 1$ is transmitting or the token is propagating to port j and set

$$X_j(t) = \begin{cases} 2 & \text{if port } j \text{ is transmitting a packet at time } t, \\ 1 & \text{if there is a packet queued for} \\ & \text{transmission at port } j \text{ at time } t, \\ 0 & \text{otherwise,} \end{cases}$$

$j = 1, 2, \cdots, N$. Then set

$$X(t) = (X_1(t), \cdots, X_N(t); N(t)). \tag{1}$$

The process $\{X(t): t \geq 0\}$ defined by Eq. (1) is a GSMP with a finite state space

$$S = \{(x_1, \cdots, x_N; n): n = 1, 2, \cdots, \text{ or } N;$$
$$x_j = 0, 1, \text{ or } 2 \text{ with at most one } x_j = 2\}$$

and event set $E = \{e_1, e_2, \cdots, e_{3N}\}$, where e_{3j-2} = "end of transmission by port j," e_{3j-1} = "observation of token by port j," and e_{3j} = "arrival of packet for transmission by port j," $j = 1, 2, \cdots, N$. For $s = (x_1, \cdots, x_N; n) \in S$, the event sets $E(s)$ are as follows. The event "end of transmission by port j" $\in E(s)$ if and only if $x_j = 2$. The event "observation of token by port j" $\in E(s)$ if and only if $x_{j-1} \neq 2$ and $n = j$. The event "arrival of packet for transmission by port j" $\in E(s)$ if and only if $x_j = 0$.

If e = "end of transmission by port j," then the state transition probability $p(s'; s, e) = 1$ when

$$s = (x_1, \cdots, x_{j-1}, 2, x_{j+1}, \cdots, x_N; j + 1) \in S$$

and

$$s' = (x_1, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_N; j + 1) \in S.$$

If e = "observation of token by port j," then $p(s'; s, e) = 1$ when

$$s = (x_1, \cdots, x_{j-1}, 1, x_{j+1}, \cdots, x_N; j) \in S$$

and

$$s' = (x_1, \cdots, x_{j-1}, 2, x_{j+1}, \cdots, x_N; j + 1)$$

and when

$$s = (x_1, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_N; j) \in S$$

and

$$s' = (x_1, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_N; j + 1).$$

If e = "arrival of packet for transmission by port j," then $p(s'; s, e) = 1$ when

$$s = (x_1, \cdots, x_{j-1}, 0, x_{j+1}, \cdots, x_N; n) \in S$$

and

$$s' = (x_1, \cdots, x_{j-1}, 1, x_{j+1}, \cdots, x_N; n).$$

(Take $j + 1 = 1$ if $j = N$.) All other state transition probabilities $p(s'; s, e)$ are equal to zero.

The distribution functions of new clock times for events $e' \in N(s'; s, e^*)$ are as follows. If e' = "end of transmission by port j," then the distribution function $F(x; s', e', s, e^*) = P\{L_j \leq x\}$. If e' = "observation of token by port j," then the distribution function $F(x; s', e', s, e^*) = 1_{[R_{j-1},\infty)}(x)$. If e' = "arrival of packet for transmission by port j," then the distribution function $F(x; s', e', s, e^*) = P\{A_j \leq x\}$.

Following Whitt [9], formal definition of a GSMP is in terms of a general state space Markov chain (GSSMC) which describes the process at successive epochs of state transition. For $s \in S$, define

$$C(s) = \{(c_1, \cdots, c_M): c_i \geq 0 \text{ and } c_i > 0$$
$$\text{if and only if } e_i \in E(s);$$
$$c_i r_{si}^{-1} \neq c_j r_{sj}^{-1} \text{ for } i \neq j \text{ with } c_i c_j r_{si} r_{sj} > 0\}. \tag{2}$$

The conditions in Eq. (2) ensure that no two events simultaneously trigger a transition (as defined below). The set $C(s)$ is the set of possible clock readings in state s. The clock with reading c_i and event e_i are said to be *active* in state s if $e_i \in E(s)$. For $s \in S$ and $c \in C(s)$, let

$$t^* = t^*(s, c) = \min_{\{i: e_i \in E(s)\}} \{c_i r_{si}^{-1}\}, \tag{3}$$

where $c_i r_{si}^{-1}$ is taken to be $+\infty$ when $r_{si} = 0$. Also set

$$c_i^* = c_i^*(s, c) = c_i - t^*(s, c) r_{si}, \; e_i \in E(s) \tag{4}$$

and

$$i^* = i^*(s, c) = i \text{ such that } e_i \in E(s) \text{ and } c_i^*(s, c) = 0. \tag{5}$$

Beginning in state s with clock vector c, $t^*(s, c)$ is the time to the next state transition and $i^*(s, c)$ is the index of the unique triggering event $e^* = e^*(s, c) = e_{i^*(s,c)}$.

Next consider a GSSMC $\{(S_n, C_n): n \geq 0\}$ having state space

$$\Sigma = \bigcup_{s \in S} (\{s\} \times C(s))$$

and representing the state (S_n) and vector (C_n) of clock readings at successive state transition epochs. (The ith coordinate of the vector C_n is denoted by $C_{n,i}$.) The transition kernel of the GSSMC $\{(S_n, C_n): n \geq 0\}$ is

$$P((s, c), A)$$
$$= p(s'; s, e^*) \prod_{e_i \in N(s')} F(a_i; s', e_i, s, e^*) \prod_{e_i \in O(s')} 1_{[0,a_i]}(c_i^*), \tag{6}$$

where $N(s') = N(s'; s, e^*)$, $O(s') = O(s'; s, e^*)$, and

$$A = \{s'\} \times \{(c_1', \cdots, c_M') \in C(s'): c_i' \leq a_i \text{ for } e_i \in E(s')\}.$$

The set A is the subset of Σ which corresponds to the GSMP entering state s' with the reading c_i' on the clock associated with event $e_i \in E(s')$ set to a value in $[0, a_i]$. [We suppose that the clock setting distributions are such that $P((s, c), \Sigma) = 1$ for all $(s, c) \in \Sigma$.]

Finally, the GSMP is a piecewise constant continuous time process constructed from the GSSMC $\{(S_n, C_n): n \geq 0\}$ in the following manner. Set $\zeta_0 = 0$ and denote by ζ_n the time of the nth state transition, $n \geq 0$. [We assume that

$$P\{\sup_{n\geq 1} \zeta_n = +\infty \mid (S_0, C_0)\} = 1 \text{ a.s.}$$

for all initial states (S_0, C_0).] Then set

$$X(t) = S_{N(t)}, \tag{7}$$

where

$$N(t) = \max\{n \geq 0: \zeta_n \leq t\}. \tag{8}$$

The process $\{X(t): t \geq 0\}$ defined by Eq. (7) is a GSMP. Henceforth we restrict attention to GSMPs in which all speeds r_{sj} are equal to 1.

The characteristic property of a regenerative stochastic process (Smith [10]) is that there exist random time points, referred to as *regeneration points* or *regeneration times*, at which the process probabilistically restarts. The essence of regeneration is that the evolution of the process in a cycle (i.e., between any two successive regeneration points) is a probabilistic replica of the process in any other cycle. In the presence of mild regularity conditions, a regenerative stochastic process $\{X(t): t \geq 0\}$ has a limiting distribution ($X(t) \Rightarrow X$ as $t \rightarrow \infty$) provided that the expected time between regeneration points is finite. Furthermore, the regenerative structure ensures that the behavior of the process in a cycle determines the expected value of a function of the limiting random variable X as a ratio of expected values. These results have important implications for simulation and are the basis for the regenerative method for simulation analysis; see Crane and Iglehart [11] and Eqs. (14)–(16) below.

• *Definition 2*

The real (possibly vector-valued) stochastic process $\{X(t): t \geq 0\}$ is a *regenerative process in continuous time* provided that

i. There exists a sequence of stopping times $\{T_k: k \geq 0\}$ such that $\{T_{k+1} - T_k: k \geq 0\}$ are independent and identically distributed;
ii. For every sequence of times $0 < t_1 < t_2 < \cdots < t_m$ $(m \geq 1)$ and $k \geq 0$, the random vectors $\{X(t_1), \cdots, X(t_m)\}$ and $\{X(T_k + t_1), \cdots, X(T_k + t_m)\}$ have the same distribution and the processes $\{X(t): t < T_k\}$ and $\{X(T_k + t): t \geq 0\}$ are independent.

[Recall that a stopping time for a stochastic process $\{X(t): t \geq 0\}$ is a random variable T (taking values in $[0, \infty)$) such that for every finite $t \geq 0$ the occurrence or nonoccurrence of the event $\{T \leq t\}$ can be determined from the history $\{X(u): u \leq t\}$ of the process up to time t.]

Recurrence properties of the underlying stochastic process of a discrete event simulation are needed to establish estimation procedures based on regenerative processes. Lemma 3 is a special case of a generalized Borel-Cantelli lemma due to Doob [12, p. 324]; see [1, Lemma 4] for an elementary proof using a "geometric trials" argument.

• *Lemma 3*

Let $\{Y_n: n \geq 0\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$ and taking on values in a set, S. Let $s' \in S$. Suppose that there exists $\delta > 0$ such that

$$P\{Y_n = s' \mid Y_{n-1}, \cdots, Y_0\} \geq \delta \text{ a.s.}$$

for all $n \geq 1$. Then $P\{Y_n = s' \text{ i.o.}\} = 1$.

Lemma 3 provides a means of showing that the underlying stochastic process of a simulation returns infinitely often to a fixed state. Specifically, let $\{X(t): t \geq 0\}$ be a stochastic process with right-continuous and piecewise-constant sample paths and countable state space, S. Let $s' \in S$ and suppose that $\{T_n: n \geq 0\}$ is an increasing sequence of finite ($T_n < \infty$ a.s.) state transition times for $\{X(t): t \geq 0\}$ such that

$$P\{X(T_n) = s' \mid X(T_{n-1}), \cdots, X(T_0)\} \geq \delta \text{ a.s.}$$

for some $\delta > 0$. Then $P\{X(T_n) = s' \text{ i.o.}\} = 1$ by Lemma 3 [with $Y_n = X(T_n)$].

Using "new better than used" distributional assumptions and the sample path structure of the process, Proposition 5 provides sufficient conditions for recurrence in the GSMP setting.

• *Definition 4*

The distribution F of a positive random variable A is *new better than used (NBU)* if

$$P\{A > x + y \mid A > y\} \leq P\{A > x\}$$

for all $x, y \geq 0$.

See Barlow and Proschan [13] for a discussion of NBU distributions. Note that every increasing failure rate (IFR) distribution is NBU. Also, if A and B are independent random variables with NBU distributions, then the distributions of $A + B$, $\min(A, B)$, and $\max(A, B)$ are NBU.

Let $\{X(t): t \geq 0\}$ be a GSMP with countable state space, S, and event set, $E = \{e_1, \cdots, e_M\}$. Suppose that $\{T_n: n \geq 0\}$ is an increasing sequence of finite ($T_n < \infty$ a.s.) state transition times such that for some $e^* \in E$ and $S^* \subseteq S$: $T_0 = 0$ and

$$T_n = \inf\{t > T_{n-1}: \text{at time } t \text{ event } e^* \text{ triggers a transition in some state } s^* \in S^*\}, \; n \geq 1. \tag{9}$$

Let $s'_0 \in S$. Proposition 5 postulates the existence of a distinguished random time (T_n^+) in the interval $[T_{n-1}, T_n)$ defined by Eq. (9), and a set $\{e_k: k \in K(s^+)\}$ of distinguished events determined by the state s^+ of the system at time T_n^+ such that $X(T_n) = s'_0$ when each of the distinguished events occurs prior to some time $T_n^+ + R_{n,k}(s^+)$ ($> T_n^+$). The proposition asserts that $\{X(T_n): n \geq 0\}$ hits state s'_0 infinitely often with probability one if the clock setting distributions associated with the distinguished events are NBU and satisfy a "positivity" condition which guarantees the existence of $\delta > 0$ as in Lemma 3.

Let $\{T_n^+: n \geq 0\}$ be a sequence of state transition times, and denote the state space of $\{X(T_n^+): n \geq 0\}$ by S^+. Set $\mathscr{A}(T_n^+) = \{(S_l, C_l): 0 \leq l \leq N(T_n^+)\}$, where $N(\cdot)$ is given by Eq. (8). Also set

$$K^+ = \bigcup_{s^+ \in S^+} K(s^+).$$

When $X(T_n^+) = s^+$ we denote by $S_{n,k}(s^+)$ the latest time less than or equal to T_n^+ at which the clock associated with event e_k $[k \in K(s^+)]$ was set, and by $A_{n,k}(s^+)$ the setting on the clock at time $S_{n,k}(s^+)$.

- *Proposition 5*

Assume that there exist state transition times $\{T_n^+: n \geq 0\}$, and for $s^+ \in S^+$, event sets $\{e_k: k \in K(s^+)\}$ and identically distributed collections of random variables $\{R_{n,k}: k \in K(s^+)\}$, independent of $\{A_{n,k}(X(T_n^+)): k \in K(T_n^+))\}$ and $\mathscr{A}(T_n^+)$, such that

i. $T_{n-1} \leq T_n^+$ a.s. and for $x_0, x_1, \cdots, x_{n-1} \in S$ and $s^+ \in S^+$,

$$\begin{aligned} P\{X(T_n) = s'_0, X(T_n^+) = s^+, X(T_{n-1}) = x_{n-1}, \cdots, \\ X(T_0) = x_0\} \\ \geq P\{S_{n,k}(s^+) + A_{n,k}(s^+) \leq T_n^+ + R_{n,k}(s^+), \\ k \in K(s^+); X(T_n^+) = s^+, \\ X(T_{n-1}) = x_{n-1}, \cdots, X(T_0) = x_0\}; \end{aligned}$$

ii. For all e_k $(k \in K^+)$, the clock setting distribution $F(\cdot; s', e_k, s, e) = F(\cdot; e_k)$ and is NBU;

iii. There exists $\delta > 0$ such that for $s^+ \in S^+$

$$\delta(s^+) = P\{A_k(s^+) \leq R_{n,k}(s^+), k \in K(s^+)\} \geq \delta,$$

where the random variable $A_k(s^+)$ has distribution $F(\cdot; e_k)$ and $\{A_k(s^+): k \in K(s^+)\}$ are mutually independent and independent of $\{R_{n,k}(s^+): k \in K(s^+)\}$.

Then

$$P\{X(T_n) = s'_0 \mid X(T_{n-1}), \cdots, X(T_0)\} \geq \delta \text{ a.s.}$$

so that $P\{X(T_n) = s'_0 \text{ i.o.}\} = 1$.

Proof Let $s^+ \in S^+$ and $x_0, \cdots, x_{n-1} \in S$. Lemma 10 of the Appendix shows that

$$\begin{aligned} P\{S_{n,k}(s^+) + A_{n,k}(s^+) \leq T_n^+ + R_{n,k}(s^+), k \in K(s^+); \\ X(T_n^+) = s^+, X(T_{n-1}) = x_{n-1}, \cdots, X(T_0) = x_0\} \\ \geq \delta\, P\{X(T_n^+) = s^+, X(T_{n-1}) = x_{n-1}, \cdots, \\ X(T_0) = x_0\}. \end{aligned} \tag{10}$$

Using Eq. (10),

$$\begin{aligned} P\{X(T_n) = s'_0, X(T_{n-1}) = x_{n-1}, \cdots, X(T_0) = x_0\} \\ = \sum_{s^+ \in S^+} P\{X(T_n) = s'_0, X(T_n^+) = s^+, \\ X(T_{n-1}) = x_{n-1}, \cdots, X(T_0) = x_0\} \\ \geq \sum_{s^+ \in S^+} P\{S_{n,k}(s^+) + A_{n,k}(s^+) \leq T_n^+ + R_{n,k}(s^+), \\ k \in K(s^+); X(T_n^+) = s^+, \\ X(T_{n-1}) = x_{n-1}, \cdots, X(T_0) = x_0\}, \\ \geq \sum_{s^+ \in S^+} \delta\, P\{X(T_n^+) = s^+, \\ X(T_{n-1}) = x_{n-1}, \cdots, X(T_0) = x_0\} \\ = \delta\, P\{X(T_{n-1}) = x_{n-1}, \cdots, X(T_0) = x_0\}. \end{aligned}$$

It follows that

$$P\{X(T_n) = s'_0 \mid X(T_{n-1}), \cdots, X(T_0)\} \geq \delta \text{ a.s.}$$

and Lemma 3 implies that $P\{X(T_n) = s'_0 \text{ i.o.}\} = 1$. □

Proposition 6 gives a set of new conditions (cf. Proposition 8 of [1]) on the building blocks of a GSMP which ensure that regeneration points exist and that the expected time between regeneration points is finite. We establish conditions on the sets of old and new events which ensure that the GSMP probabilistically restarts whenever (at some time T_n) event e^* triggers a transition to state s'_0. A geometric trials condition guarantees that this occurs infinitely often with probability one.

- *Proposition 6*

Let $\{T_n: n \geq 0\}$ be an increasing sequence of stopping times that are finite ($T_n < \infty$ a.s.) state transition times as in Eq. (9). Suppose that there exist $s, s'_0 \in S$ and $\delta > 0$ such that

$$P\{X(T_n) = s'_0 \mid X(T_{n-1}), \cdots, X(T_0)\} \geq \delta \text{ a.s.} \tag{11}$$

Also suppose that for $s^* \in S^*$, (i) the set $O(s'_0; s^*, e^*) = E(s'_0) \cap (E(s^*) - \{e^*\}) = \varnothing$, (ii) the set $N(s'_0; s^*, e^*) = E(s'_0) - (E(s^*) - \{e^*\}) = N(s'_0; s, e^*)$, and (iii) the clock setting distribution $F(\cdot; s'_0, e', s^*, e^*) = F(\cdot; s'_0, e', s, e^*)$ for all $e' \in N(s'_0; s, e^*)$. Then $\{X(t): t \geq 0\}$ is a regenerative process in continuous time. Moreover, if

$$E\{T_{n+1} - T_n\} \leq c < \infty$$

for all $n \geq 0$, then the expected time between regeneration points is finite.

Proof Using Lemma 3, Eq. (11) implies that event e^* triggers a transition to state s'_0 from some state $s^* \in S^*$ infinitely often with probability one. Furthermore, at such a time T_n, the only clocks that are active have just been set, since $O(s'_0; s^*, e^*) = \varnothing$ for all $s^* \in S^*$. The joint distribution of $X(T_n)$ and of the clocks set at time T_n depends on the past history of $\{X(t): t \geq 0\}$ only through s'_0, the previous state s^*, and the trigger event e^*. Since the new events and clock setting distributions are the same for all s^*, the process $\{X(t): t \geq 0\}$ probabilistically restarts whenever $\{X(T_n): n \geq 0\}$ hits state s'_0.

To show that the expected time between regeneration points is finite, assume for convenience that $X(T_0) = X(0) = s'_0$. Set $X_n = X(T_n)$ and $D_n = T_{n+1} - T_n$, $n \geq 0$. Observe that the random indices β_n such that $X_{\beta_n} = X(T_{\beta_n}) = s'_0$ form a sequence of regeneration points for the process $\{(X_n, D_n): n \geq 0\}$; this follows from the fact that the process $\{D_n: n \geq 1\}$ starts from scratch when $X(T_{\beta_n}) = s'_0$. Let $\tau_k = \beta_{k+1} - \beta_k$, $k \geq 1$. The τ_k are i.i.d. as τ_1 and the argument in the proof of Lemma 4 in [1] shows that

$$P\{\tau_1 > n\} \leq (1 - \delta)^n,$$

so that $E\{\tau_1\} < \infty$. Thus the expected time between regeneration points for the process $\{(X_n, D_n): n \geq 0\}$ is finite. Since $E\{\tau_1\} < \infty$ and Eq. (11) ensures that τ_1 is aperiodic, $(X_n, D_n) \Rightarrow (X, D)$ as $n \to \infty$. Using the continuous mapping theorem we have $D_n \Rightarrow D$ as $n \to \infty$ and, since $D_n \geq 0$ and $E\{D_n\} \leq c < \infty$,

$$E\{|D|\} = E\{D\} \leq \varliminf_{n\to\infty} E\{D_n\} \leq c < \infty$$

by Theorem 25.11 in [14]. Since τ_1 is aperiodic, $E\{\tau_1\} < \infty$, and $E\{|D|\} < \infty$,

$$E\{D\} = \frac{E\left\{\sum_{j=0}^{\tau_1 - 1} D_n\right\}}{E\{\tau_1\}},$$

so that

$$E\left\{\sum_{j=0}^{\tau_1 - 1} D_n\right\} < \infty,$$

and the expected time between regeneration points for $\{X(t): t \geq 0\}$ is finite. □

Note that the result of Proposition 6 also holds if condition (i) is replaced by (i′) $O(s'_0; s_0, e^*) \neq \varnothing$ and for any $e' \in O(s'_0; s_0, e^*)$ the clock setting distribution $F(\cdot; s', e', s, e)$ is exponential with mean λ^{-1} independent of s, s', and e. [Assumption (i′) ensures that no matter when the clock for event $e' \in O(s'_0; s_0, e^*)$ was set, the remaining time until event e' triggers a state transition is exponentially distributed with mean λ^{-1}.] Also note that the state transition times $\{T_n: n \geq 0\}$ defined by Eq. (9) are necessarily stopping times if

$$p(s^*; s^*, e^*) = 0 \tag{12}$$

for all $s^* \in S^*$ and

$$e = e^* \text{ whenever } p(s; s^*, e) > 0 \text{ and } p(s; s^*, e^*) > 0 \tag{13}$$

for all $s^* \in S^*$ and $s \in S$. [Conditions (12) and (13) imply that every occurrence of event e^* in a state $s^* \in S^*$, and hence every state transition time T_n, can be determined by observing the sample paths of $\{X(t): t \geq 0\}$.]

Under the conditions of Proposition 6, $X(t) \Rightarrow X$ as $t \to \infty$. Let f be a real-valued (measurable) function having domain S. From n cycles the standard regenerative method [11] provides the strongly consistent point estimate

$$\hat{r}(n) = \frac{\bar{Y}(n)}{\bar{\tau}(n)} \tag{14}$$

and the asymptotic $100(1 - 2\gamma)\%$ confidence interval

$$\hat{I}(n) = \left[\hat{r}(n) - \frac{z_{1-\gamma}s(n)}{\bar{\tau}(n)n^{1/2}},\ \hat{r}(n) + \frac{z_{1-\gamma}s(n)}{\bar{\tau}(n)n^{1/2}}\right] \tag{15}$$

for $r(f) = E\{f(X)\}$. In Eq. (14),

$$\bar{Y}(n) = n^{-1} \sum_{m=1}^{n} Y_m(f)$$

and

$$\bar{\tau}(n) = n^{-1} \sum_{m=1}^{n} \tau_m.$$

[For $m \geq 1$, τ_m is the length of the mth cycle and $Y_m(f)$ is the integral of $f(X(\cdot))$ over the mth cycle.] The quantity $s(n)$ is a strongly consistent point estimate for $\sigma(f) = \text{var}(Y_1(f) - r(f)\tau_1)$ and $z_{1-\gamma} = \Phi^{-1}(1 - \gamma)$, where Φ is the distribution function of a standardized normal random variable, $N(0, 1)$. Confidence intervals are based on the central limit theorem

$$\frac{n^{1/2}\{\hat{r}(n) - r(f)\}}{\sigma(f)/E\{\tau_1\}} \Rightarrow N(0, 1) \tag{16}$$

as $n \to \infty$. Equation (16) [and thus Eq. (15)] holds if $\sigma(f) < \infty$. It can be shown that when S is finite or f is bounded, $\sigma(f) < \infty$, provided that for some $\varepsilon > 0$

$$E\left\{(T_{n+1} - T_n)^{2+\varepsilon}\right\} \leq b < \infty$$

for all $n \geq 0$.

3. Ring and bus network models

The following examples illustrate the use of the GSMP model as a formal specification of a discrete event simulation of a local area computer network and the application of Propositions 5 and 6. These results are also applicable to the token ring and collision-free bus network models in Examples (*A*.1) and (*A*.3) of [2].

• *Example 7*

Recall the token ring model of Example 1. Following [1], set

$$X(t) = (Z_1(t), \cdots, Z_N(t); M(t); N(t)), \tag{17}$$

where

$$
Z_j(t) = \begin{cases} 1 & \text{if there is a packet queued for transmission at port } j \text{ at time } t, \\ 0 & \text{otherwise;} \end{cases}
$$

$$
M(t) = \begin{cases} j & \text{if port } j \text{ is transmitting a packet at time } t, \\ 0 & \text{if no port is transmitting a packet at time } t, \end{cases}
$$

and $N(t) = j$ if at time t port $j - 1$ is transmitting a packet or the token is propagating to port j.

The process $\{X(t): t \geq 0\}$ is a GSMP with a finite state space, S, and event set, $E = \{e_1, \cdots, e_{N+2}\}$, where $e_{N+2} =$ "observation of token," $e_{N+1} =$ "end of transmission," and $e_j =$ "arrival of packet for transmission by port j," $j = 1, 2, \cdots, N$. For $s = (z_1, \cdots, z_N; m; n) \in S$, the event sets $E(s)$ are as follows: The event "end of transmission" $\in E(s)$ if and only if $m > 0$ and "observation of token" $\in E(s)$ if and only if $m = 0$. The event "arrival of packet for transmission by port j" $\in E(s)$ if and only if $z_j = 0$ and $m \neq j$, $j = 1, 2, \cdots, N$.

As an application of Propositions 5 and 6, take $s_0' = (0, 1, \cdots, 1; 1; 2)$. Let $e^* =$ "observation of token" and $S^* = \{(z_1, \cdots, z_N; 0; 1) \in S\}$ so that T_n is the nth time at which port 1 observes the token, $n \geq 0$. Observe that $T_n < \infty$ a.s. since

$$
E\{T_n - T_{n-1}\} \leq R_1 + \cdots + R_N + \sum_{j=1}^{N} E\{L_j\} < \infty \tag{18}
$$

for all $n \geq 1$.

Let T_n^+ be the first time after T_{n-1} that the token leaves port N so that $S^+ = S^*$. Observe that $X(T_n) = s_0'$ if, while the token is propagating from port N to port 1, there is an arrival of a packet for transmission at every port that does not have a packet queued for transmission at time T_n^+. Thus, for $s^+ = (z_1^+, \cdots, z_N^+; m^+; n^+) \in S^+$, set $K(s^+) = \{k: z_k^+ = 0\}$ so that $K^+ = \{1, 2, \cdots, N\}$. Take $R_{n,k}(s^+) = R_N$ for all $k \in K(s^+)$ and $s^+ \in S^+$. Then condition (i) of Proposition 5 is satisfied. Assume that the distribution of A_j is NBU and that

$$
\delta_j = P\{A_j \leq R_n\} > 0
$$

for $j = 1, 2, \cdots, N$ so that

$$
\delta(s^+) = \prod_{j \in K(s^+)} \delta_j \geq \prod_{j=1}^{N} \delta_j = \delta > 0.
$$

Then $P\{X(T_n) = s_0' \text{ i.o}\} = 1$.

A transition of the process $\{X(t): t \geq 0\}$ defined by Eq. (17) to state s_0' can occur when event e^* is the trigger event only if e^* occurs in state $s^* = (1, \cdots, 1; 0; 1)$ and in this case the set $O(s_0'; s^*, e^*) = \varnothing$. Since Eqs. (12) and (13) hold and $P\{X(T_n) = s_0' \text{ i.o.}\} = 1$, the successive times T_n at which e^* triggers a transition (in state s^*) to state s_0' are stopping times and regeneration points for the process $\{X(t): t \geq 0\}$. The expected time between those regeneration points is finite by Eq. (18). At these time points there is a packet queued for transmission at ports 2, 3, $\cdots$, N and port 1 starts transmission of a packet. Furthermore, if $E\{L_j^{2+\varepsilon}\} < \infty$ for some $\varepsilon > 0$, then

$$
\sup_n E\{(T_{n+1} - T_n)^{2+\varepsilon}\} < \infty
$$

and the remarks following Eq. (16) apply.

- *Example 8 (collision-free bus network)*

Consider a bus network (Eswaran, Hamacher, and Shedler [15]) with N ports, numbered 1, 2, $\cdots$, N from left to right; see **Figure 2.** *Message packet* traffic on the passive bilateral bus is transmitted/received by port j at tap $\mathbf{B}(j)$. In addition to the bus, a one-way logic *control wire* also links the ports. Associated with each port j is a flip-flop, $\mathbf{S}(j)$, called the *send flip-flop*. The signal $\mathbf{P}(j)$, called the *OR-signal*, tapped at the control wire input to port j is the inclusive OR of the send flip-flops of all ports to the left of port j. Denote by T the end-to-end bus propagation delay. [For technical reasons, T actually must be the end-to-end propagation delay plus a small (fixed) quantity.] Denote the actual propagation delay along the bus between port i and port j by $T(i, j)$, $i, j = 1, 2, \cdots, N$. Thus, $T(i, j) = T(j, i) < T$ for all i, j and $T(i, j) + T(j, k) = T(i, k)$ for all $i < j < k$. Let $R(j)$ be the propagation delay (including gate delays) along the control wire from port j to port N, $j = 1, 2, \cdots, N$; thus, $R(1) \geq R(2) \geq \cdots \geq R(N) = 0$. Denote by $R(i, j)$ the propagation delay along the control wire from port i to port j. We assume that signal propagation along the control wire is slower than along the bus and that delays along shorter sections of each path scale proportionally: $R(1) > T$ and $R(i, j) > T(i, j)$ for all i, j.

Specification of distributed control scheme A1 is in terms of an algorithm for an individual port j. Packets (for transmission by port j) which arrive while an execution of the algorithm by port j is in progress queue externally. Upon completion of this execution of the algorithm, one of any such packets immediately becomes available to port j for transmission and the next execution of the algorithm begins.

Algorithm A1

- Set $\mathbf{S}(j)$ to 1.
- Wait for a time interval $R(j) + T$.
- Wait until the bus is observed (by port j) to be idle AND $\mathbf{P}(j) = 0$; then start transmission of the packet, simultaneously resetting $\mathbf{S}(j)$ to 0.

For ease of exposition we assume that $T(i, j) \neq T(k, j)$ for distinct i, k and all j. In addition we assume that there can be at most one packet in queue at each port. Specifically, suppose that the time from end of transmission by port j until the arrival of a next packet for transmission by port j is a positive random variable, A_j, with finite mean. Also suppose that the time for port j to transmit a packet is a positive random variable, L_j, with finite mean and (so that Algorithm A2 of [15] guarantees transmission of all packets) such that $P\{L_j \le R(1) + T\} = 0$.

Set

$$W(t) = (W_1(t), \cdots, W_N(t)), \tag{19}$$

where $W_j(t)$ equals 1 if at time t port j has set its flip-flop but has not yet completed the $R(j) + T$ wait, equals 2 if port j has completed the $R(j) + T$ wait but has not started transmission, equals 3 if port j is transmitting, and equals 4 otherwise. Next set

$$U(t) = (U_1(t), \cdots, U_N(t)), \tag{20}$$

where $U_j(t)$ equals k if port j observes transmission of a packet by port k on the bus at time t, and equals 0 otherwise. Also set

$$V(t) = (V_{2,1}(t), V_{3,1}(t), V_{3,2}(t), V_{4,1}(t), \cdots, V_{N,N-1}(t)), \tag{21}$$

where $V_{j,k}(t)$ equals 1 if and only if $\mathbf{S}(k) = 1$ at time $t - R(k, j)$, and equals 0 otherwise. [Port j observes $\mathbf{P}(j) = 1$ at time t if and only if $V_{j,k}(t) = 1$ for some $k < j$.] Finally, set $Z(t) = 1$ if some port is transmitting at time t and this port started transmission when it observed an end of transmission; otherwise $Z(t) = 0$. Then set

$$X(t) = (W(t); Z(t); U(t); V(t)). \tag{22}$$

The stochastic process $\{X(t): t \ge 0\}$ defined by Eq. (22) is a GSMP with a finite state space, S, and event set, E. The events in the set E are: "end of transmission by port j," "end of wait for $R(j) + T$," "setting (to 1) of flip-flop by port j," "observation by port j of start of transmission by port $k \ne j$," "observation by port j of end of transmission by port $k \ne j$," "observation by port j of end of transmission by port $k \ne j$ and start of transmission by port $l \ne j$," "observation by port j of the setting (to 1) of flip-flop by port k to the left," and "observation by port j of the resetting (to 0) of a flip-flop by port k to the left," $j = 1, 2, \cdots, N$. For $s = (w_1, \cdots, w_N; z; u_1, \cdots, u_N; v_{2,1}, \cdots, v_{N,N-1}) \in S$ the event sets $E(s)$ are as follows: The event set $E(s)$ contains "setting (to 1) of flip-flop by port j" if and only if $w_j = 4$. The event "end of transmission by port j" $\in E(s)$ if and only if $w_j = 3$. The event "end of wait for $R(j) + T$" $\in E(s)$ if and only if $w_j = 1$. The event "observation by port j of start of transmission by port k" $\in E(s)$ if and only if (i) $w_k = 3$, $z = 0$, and $u_j \ne k$ or (ii) $w_k = 3$, $z = 1$, and either $u_j = 0$ or $u_j = l$ for some l between k and j. The event "observation by port j of end of transmission by port k" $\in E(s)$ if and only if $u_j = k$ and $w_k = 1$ or 4 and either $z = 0$ or $w_l \ne 3$ for all l between j and k. The event "observation by port j of end of transmission by port $k \ne j$ and start of transmission by port $l \ne j$" $\in E(s)$ if and only if $u_j = k$, $z = 1$, and $w_l = 3$ with l between k and j. The event "observation by port j of setting

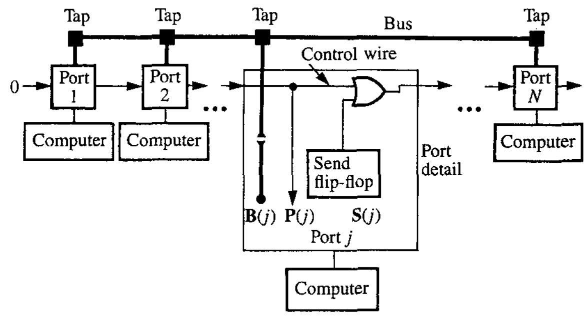

Figure 2

Collision-free bus network.

of flip-flop by port k to the left" $\in E(s)$ if and only if $w_k = 1$ and $v_{j,k} = 0$ for some $k < j$. The event "observation by port j of resetting of flip-flop by port k to the left" $\in E(s)$ if and only if $w_k = 3$ and $v_{j,k} = 1$ for some $k < j$.

Note that with this definition of the event sets $E(s)$, no "observation by port j of start of transmission by port k" and "observation by port j of end of transmission by port l" events can occur simultaneously in the GSMP model. To see this, let $k < l < j$. Suppose that port k ends transmission of a packet at time t and that port l starts transmission of a packet at time $t' = t + T(k, l)$. Then the event "observation by port j of end of transmission by port k and start of transmission by port l" is scheduled at time t' and (since $z = 1$ and $w_l = 3$ where l is between k and j) the event "observation by port j of end of transmission by port k" (which was scheduled at time t) ceases to be scheduled at time t'.

The distribution functions of new clock times for events $e' \in N(s'; s, e^*)$ are as follows. If e' = "end of transmission by port j," then the clock setting distribution function $F(x; s', e', s, e) = P\{L_j \le x\}$. If e' = "end of wait for $R(j) + T$," then the clock setting distribution function $F(x; s', e', s, e^*) = 1_{[R(j)+T,\infty)}(x)$. If e' = "setting (to 1) of flip-flop by port j," then the clock setting distribution function $F(x; s', e', s, e^*) = P\{A_j \le x\}$. If e' = "observation by port j of start of transmission by port k," then the clock setting distribution function $F(x; s', e', s, e^*) = 1_{[T(k,j),\infty)}(x)$. If e' = "observation by port j of end of transmission by port k," then the clock setting distribution function $F(x; s', e', s, e^*) = 1_{[T(k,j),\infty)}(x)$. If e' = "observation by port j of end of transmission by port k and start of transmission by port l, then the clock setting distribution $F(x; s', e', s, e^*) = 1_{[T(l,j),\infty)}(x)$. If e' = "observation by port j of setting of flip-flop by port k to the left," then the clock setting distribution function $F(x; s', e', s, e^*) = 1_{[R(k,j),\infty)}(x)$. If e' =

"observation by port j of resetting of flip-flop by port k to the left," then the clock setting distribution function $F(x; s', e', s, e^*) = 1_{[R(k,j),\infty)}(x)$.

As an application of Propositions 5 and 6, take $s_0' = (4, 2, \cdots, 2; 0; 0, 1, \cdots, 1; 0, 0, 1, \cdots, 0, 1, \cdots, 1)$. Let e^* = "end of transmission by port 1" and $S^* = \{(3, w_2, \cdots, w_N; z; 1, \cdots, 1; v) \in S: v_{j,1} = 0$ for $j = 2, 3, \cdots, N\}$ so that T_n is the nth time at which port 1 ends transmission, $n \ge 0$. Then port 1 ends transmission of a packet with every other port j having observed the resetting of port 1's flip-flop, having a packet queued for transmission, and having completed the $R(j) + T$ wait at time T_n if $X(T_n) = s_0'$. Observe that

$$T_n - T_{n-1} = A_{1n} + R(1) + T + D_n + L_{1n}, \tag{23}$$

where L_{1n} is distributed as L_1, A_{1n} is distributed as A_1, and D_n is a non-negative random variable. Provided that the distribution of L_j is NBU, it can be shown that

$$E\{D_n\} \le \sum_{j=2}^{N} E\{L_j\},$$

so that

$$E\{T_n - T_{n-1}\} \le E\{A_1\} + R(1) + T + \sum_{j=1}^{N} E\{L_j\} < \infty \tag{24}$$

and therefore $T_n < \infty$ a.s.

Let T_n^+ be the first time after T_{n-1} that port 1 begins transmission of a packet so that $S^+ = S^*$. Observe that $X(T_n) = s_0'$ if, at least $R(1) + T$ time units before the end of transmission by port 1, there is an arrival of a packet for transmission at each port that does not have a packet queued for transmission at time T_n^+. Thus, let e_j = "setting of flip-flop by port j" and for $s^+ = (w^+; z^+; u^+; v^+) \in S^+$, set $K(s^+) = \{k: w_k^+ = 4\}$ so that $K^+ = \{2, \cdots, N\}$. Take $R_{n,k}(s^+) = L_1 - (R(1) + T)$ for all $s^+ \in S^+$ and $k \in K(s^+)$. Then condition (i) of Proposition 5 is satisfied. Assume that the distribution of A_j is NBU and that

$$\delta_j = P\{A_j + R(1) + T \le L_1\} > 0,$$

$j = 2, 3, \cdots, N$. It follows that

$$\delta = P\{A_j + R(1) + T \le L_1, j = 2, 3, \cdots, N\} > 0$$

so that

$$\delta(s^+) = P\{A_j + R(1) + T \le L_1, j \in K(s^+)\} \ge \delta.$$

Then $P\{X(T_n) = s_0' \text{ i.o.}\} = 1$.

A transition of the process $\{X(t): t \ge 0\}$ defined by Eq. (22) to state s_0' can occur when event e^* is the trigger event only if e^* occurs in a state $s^* = (3, 2, \cdots, 2; z; 1, \cdots, 1; 0, 1, \cdots, 1)$ and in this case the set $O(s_0'; s^*, e^*) = \varnothing$. Since Eqs. (12) and (13) hold and $P\{X(T_n) = s_0' \text{ i.o.}\} = 1$, the successive times T_n at which e^* triggers a transition (in state s^*) to state s_0' are stopping times and regeneration points for the process $\{X(t): t \ge 0\}$. The expected time between these regeneration points is finite by Eq. (24). If, in addition, $E\{L_j^{2+\varepsilon}\} < \infty$ for some $\varepsilon > 0$, then

$$\sup_n E\{(T_{n+1} - T_n)^{2+\varepsilon}\} < \infty$$

and the remarks following Eq. (16) apply.

4. Concluding remarks

It is sometimes possible to establish recurrence results under weaker positivity assumptions than those required by hypothesis (ii) of Proposition 5. For example, in the token ring model of Example 7, $P\{X(T_n) = s_0' \text{ i.o.}\} = 1$ if the distribution of A_j is NBU and $P\{A_j \le R_j + \cdots + R_N\} > 0$.

Acknowledgment

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Appendix

Let $\{X(t): t \ge 0\}$ be a GSMP with finite state space, S, and event set, E. Recall that ζ_n is the time of the nth state transition and that $S_n = X(\zeta_n)$ is the state of the system at time ζ_n, $n \ge 0$. Also recall that C_n is the vector of clock readings at time ζ_n and that $C_{n,i}$ is the ith coordinate of the vector C_n for $e_i \in E(S_n)$. Denote by $i_n^* = i^*(S_{n-1}, C_{n-1})$ the index of the nth trigger event and let $e_n^* = e_{i_n^*}$ and $I_n = \{i: e_i \in E(S_n)\}$.

Let $s_0, s_1, \cdots, s_n \in S$ and $e_{i_1}, \cdots, e_{i_n} \in E$ with $p(s_k; s_{k-1}, e_{i_k}) > 0$. Then the joint event

$$\{X(\zeta_n) = s_n, e_n^* = e_{i_n}, X(\zeta_{n-1}) = s_{n-1},$$
$$e_{n-1}^* = e_{i_{n-1}}, \cdots, e_1^* = e_{i_1}, X(0) = s_0\} \tag{A1}$$

is equivalent to the joint event specified by the inequalities

$$C_{m,i_{m+1}} \le C_{m,i},\ i \in I_m - \{i_{m+1}\}$$
$$\text{and } m = 0, 1, \cdots, n-1 \tag{A2}$$

in conjunction with the equations

$$X(\zeta_k) = S_k = s_k,\ k = 0, 1, \cdots, n. \tag{A3}$$

If $I_m = \{i_{m+1}\}$, we write $C_{m,i_{m+1}} < \infty$.

We assume throughout that $E(s_0)$ is the set of active events at time $t = 0$ and that all active clocks are reset at time $t = 0$:

$$P\{C_{0,i} \le x\} = F(x; s', e_i, s, e)$$

for some $s, s' \in S$ and $e \in E$ (dependent on i), $e_i \in E(s_0)$. In addition, we define $N(s_m; s_{m-1}, e_m^*) = E(s_0)$ for $m = 0$.

Next observe that if $e_i \in O(s_n; s_{n-1}, e_n^*)$ so that $C_{n,i}$ is an old clock reading, then

$$C_{n,i} = C_{m,i} - \sum_{k=m}^{n-1} C_{k,i_{k+1}^*},$$

where ζ_m is the latest time prior to ζ_n at which the clock associated with event e_i was set. This implies that any old clock reading $C_{k,i}$ appearing in Eq. (A2) can be expressed in terms of one or more $C_{m,j}$ with $e_j \in N(s_m; s_{m-1}, e^*_m)$ and $m \le k$. Replacing in this manner all old clock readings appearing in Eq. (A2) with expressions which involve only new clock readings, we obtain an equivalent system of inequalities which, in conjunction with Eq. (A3), we denote by $\mathscr{E}_n$. We call $\mathscr{E}_n$ the *canonical representation* of the joint event given by Eq. (A1).

• *Lemma 9*

Let $\mathscr{E}_n$ be the canonical representation of the joint event given by Eq. (A1) and let $j_1, j_2, \cdots, j_{l(n)}$ be indices such that $1 \le j_1 \le j_2 \le \cdots \le j_{l(n)} \le n$. Suppose that $N(s_{j_k}; s_{j_k-1}, e^*_{j_k}) \ne \varnothing$, $k = 1, 2, \cdots, l(n)$. Select $e_{i_k} \in N(s_{j_k}; s_{j_k-1}, e^*_{j_k})$ and denote by $\mathscr{G}_n$ the set of inequalities

$$\zeta_{j_k} + C_{j_k, i_k} > \zeta_n, \ k = 1, 2, \cdots, l(n).$$

Either the set of inequalities $\{\mathscr{G}_n, \mathscr{E}_n\}$ has probability zero or there exists $\bar{\mathscr{E}}_n \subseteq \mathscr{E}_n$ such that (i) $\{\mathscr{G}_n, \bar{\mathscr{E}}_n\}$ and $\{\mathscr{G}_n, \mathscr{E}_n\}$ are algebraically equivalent and (ii) no random variable C_{j_k, i_k} in $\mathscr{G}_n$ appears in $\bar{\mathscr{E}}_n$.

Proof For fixed k, observe that the variable C_{j_k, i_k} appears only in those inequalities in $\mathscr{E}_n$ corresponding to state transitions at times $\zeta_{j_k}, \zeta_{j_{k+1}}, \cdots, \zeta_n$. There are two cases to consider.

Case i. For some k and $j_k \le l \le n - 1$, $\mathscr{E}_n$ contains the inequalities

$$\mathscr{E}\left(C_{j_k, i_k} - \sum_{m=j_k}^{l-1} C_{m, i^*_{m+1}}\right) < \mathscr{E}(C_{l,i}), \ i \in I_l - \{i_k\}$$

or

$$\mathscr{E}\left(C_{j_k, i_k} - \sum_{m=j_k}^{l-1} C_{m, i^*_{m+1}}\right) < \infty,$$

where $\mathscr{E}(\cdot)$ denotes an expression written in canonical form. By the structure of the GSMP this means that

$$\mathscr{E}(C_{l, i^*_{l+1}}) = \mathscr{E}\left(C_{j_k, i_k} - \sum_{m=j_k}^{l-1} C_{m, i^*_{m+1}}\right),$$

which implies that

$$\mathscr{E}(C_{j_k, i_k} + \zeta_{j_k}) = \mathscr{E}\left(\zeta_{j_k} + \sum_{m=j_k}^{l} C_{m, i^*_{m+1}}\right)$$

$$= \mathscr{E}(\zeta_{l+1}) \le \mathscr{E}(\zeta_n).$$

This contradicts the corresponding inequality in $\mathscr{G}_n$ so that $\{\mathscr{G}_n, \mathscr{E}_n\}$ has probability zero.

Case ii. For every k,

$$\mathscr{E}(C_{l, i^*_{l+1}}) < \mathscr{E}\left(C_{j_k, i_k} - \sum_{m=j_k}^{l-1} C_{m, i^*_{m+1}}\right),$$

$$l = j_k, j_k + 1, \cdots, n - 1. \qquad \text{(A4)}$$

This is equivalent to

$$\mathscr{E}(C_{j_k, i_k} + \zeta_{j_k}) > \mathscr{E}\left(\zeta_{j_k} + \sum_{m=j_k}^{l} C_{m, i^*_{m+1}}\right) = \mathscr{E}(\zeta_{l+1}),$$

$$l = j_k, j_k + 1, \cdots, n - 1.$$

But clearly, (for every k) each of these equations is implied by the inequality

$$\mathscr{E}(C_{j_k, i_k} + \zeta_{j_k}) > \mathscr{E}(\zeta_n),$$

which is an element of $\mathscr{G}_n$. Since the only inequalities in $\mathscr{E}_n$ which contain the random variable C_{j_k, i_k} are those in Eq. (A4), the required subset $\bar{\mathscr{E}}_n$ is formed by deleting (for each k) the inequalities in Eq. (A4). □

• *Lemma 10*

Let $s^+ \in S^+$ and $x_0, \cdots, x_{n-1} \in S$. Under the conditions of Proposition 5,

$$P\{S_{n,k}(s^+) + A_{n,k}(s^+) \le T^+_n + R_{n,k}(s^+), \ k \in K(s^+);$$
$$X(T^+_n) = s^+, X(T_{n-1}) = x_{n-1}, \cdots, X(T_0) = x_0\}$$
$$\ge \delta \ P\{X(T^+_n) = s^+, X(T_{n-1}) = x_{n-1}, \cdots, X(T_0) = x_0\}.$$

Proof Set

$$U_n = \{X(T^+_n) = s^+_n, X(T_{n-1}) = x_{n-1}, \cdots, X(T_0) = x_0\}$$

and let $\{V^i_n: i = 1, 2, \cdots,\}$ be the (countable) set of all joint events of the form

$$V^i_n = \{X(\zeta_{j(i,n)}) = s_{j(i,n)}, e^*_{j(i,n)} = e_{i_{j(i,n)}}, \cdots, e^*_1 = e_{i_1},$$
$$X(0) = x_0\},$$

where $\zeta_{j(i,n)} = T^+_n$ and $s_{j(i,n)} = s^+_n$, and there exist $l_1 < \cdots < l_{n-1}$ such that $\zeta_{l_j} = T_j$ and $s_{l_j} = x_j$, $j = 1, 2, \cdots, n - 1$. Also let $\mathscr{E}^i_{j(i,n)}$ be the canonical representation of the joint event V^i_n. Next consider the joint event

$$\{S_{n,k}(s^+) + A_{n,k}(s^+) \le T^+_n + R_{n,k}(s^+), \ k \in K(s^+); \ U_n\}.$$

If $S_{n,k}(s^+) = T^+_n$, then the vacuous statement $\{A_{n,k}(s^+) > 0\}$ can be written as $\{S_{n,k}(s^+) + A_{n,k}(s^+) > T^+_n\}$. If $S_{n,k}(s^+) < T^+_n$, then $S_{n,k}(s^+) + A_{n,k}(s^+) > T^+_n$ since $A_{n,k}(s^+)$ is by definition the clock reading for an event that is active at time T^+_n. Thus, the joint events

$$\{S_{n,k}(s^+) + A_{n,k}(s^+) \le T^+_n + R_{n,k}(s^+), \ k \in K(s^+); \ U_n\} \qquad \text{(A5)}$$

and

$$\{S_{n,k}(s^+) + A_{n,k}(s^+) \le T^+_n + R_{n,k}(s^+),$$
$$S_{n,k}(s^+) + A_{n,k}(s^+) > T^+_n, \ k \in K(s^+); \ U_n\} \qquad \text{(A6)}$$

are equivalent.

Now observe that for every sequence V^i_n of states and trigger events, $S_{n,k}(s^+)$ corresponds to some $\zeta_{l(i,n,k)}$ and T^+_n to some $\zeta_{j(i,n)}$. Also, $A_{n,k}(s^+)$ corresponds to some $C_{l(i,n,k), m(i,n,k)}$, where $e_{m(i,n,k)} \in N(s_{l(i,n,k)}; s_{l(i,n,k)-1}, e^*_{l(i,n,k)})$, and $R_{n,k}(s^+)$ corresponds to some $R_{q(i,n),k}(s^+)$. Since U_n is the disjoint union of the events V^i_n, we can combine the above results to obtain

$$P\{S_{n,k}(s^+) + A_{n,k}(s^+) \le T_n^+ + R_{n,k}(s^+),\ k \in K(s^+);\ U_n\}$$
$$= \sum_i P\{C_{l(i,n,k),m(i,n,k)} \le \zeta_{j(i,n)} - \zeta_{l(i,n,k)} + R_{q(i,n),k}(s^+),$$
$$C_{l(i,n,k),m(i,n,k)} > \zeta_{j(i,n)} - \zeta_{l(i,n,k)},\ k \in K(s^+);\ \mathscr{L}^i_{j(i,n)}\}, \quad \text{(A7)}$$

where all terms of probability zero are excluded from the sum. By Lemma 9, we can replace $\mathscr{L}^i_{j(i,n)}$ with $\bar{\mathscr{L}}^i_{j(i,n)}$ without altering the value of the sum.

Setting

$$\mathbf{Z}_{j(i,n),l(i,n,k)} = \mathscr{L}(\zeta_{j(i,n)} - \zeta_{l(i,n,k)})$$
$$= \mathscr{L}\left(\sum_{j=0}^{j(i,n)-1} C_{j,i^*_{j+1}} - \sum_{j=0}^{l(i,n,k)-1} C_{j,i^*_{j+1}}\right),$$

and denoting the set of random variables appearing in the canonical representation $\mathscr{L}(\zeta_{j(i,n)} - \zeta_{l(i,n,k)})$ $(k \in K(s^+))$ and in $\bar{\mathscr{L}}^i_{j(i,n)}$ by $\mathscr{V}$, it follows from Lemma 9 and the independence assumptions on $A_{n,k}(s^+)$ and $R_{n,k}(s^+)$ that we can write

$$P\{C_{l(i,n,k),m(i,n,k)} \le \mathbf{Z}_{j(i,n),l(i,n,k)} + R_{q(i,n),k}(s^+),$$
$$C_{l(i,n,k),m(i,n,k)} > \mathbf{Z}_{j(i,n),l(i,n,k)},\ k \in K(s^+);\ \bar{\mathscr{L}}^i_{j(i,n)}\},$$
$$= \int\int P\{C_{l(i,n,k),m(i,n,k)} \le \mathbf{Z}_{j(i,n),l(i,n,k)}(v) + r_{q(i,n),k},$$
$$C_{l(i,n,k),m(i,n,k)} > \mathbf{Z}_{j(i,n),l(i,n,k)}(v),$$
$$k \in K(s^+)\}dF_R(r)dF_{\mathscr{V}}(v), \quad \text{(A8)}$$

where F_R and $F_{\mathscr{V}}$ are the joint distribution functions of $\{R_{q(i,n),k}(s^+), k \in K(s^+)\}$ and $\mathscr{V}$ respectively. [Note that the outer integration is over values v which satisfy the equations and inequalities in $\bar{\mathscr{L}}^i_{j(i,n)}$.] Using hypothesis (ii) of Proposition 5, we have that

$$P\{C_{l(i,n,k),m(i,n,k)} \le \mathbf{Z}_{j(i,n),l(i,n,k)}(v) + r_{q(i,n),k},$$
$$C_{l(i,n,k),m(i,n,k)} > \mathbf{Z}_{j(i,n),l(i,n,k)}(v),\ k \in K(s^+)\}$$
$$= \prod_{k\in K(s^+)} P\{C_{l(i,n,k),m(i,n,k)} \le \mathbf{Z}_{j(i,n),l(i,n,k)}(v) + r_{q(i,n),k},$$
$$C_{l(i,n,k),m(i,n,k)} > \mathbf{Z}_{j(i,n),l(i,n,k)}(v)\}$$
$$\ge \prod_{k\in K(s^+)} [P\{C_{l(i,n,k),m(i,n,k)} \le r_{q(i,n),k}\}$$
$$\times P\{C_{l(i,n,k),m(i,n,k)} > \mathbf{Z}_{j(i,n),l(i,n,k)}(v)\}]$$
$$= P\{C_{l(i,n,k),m(i,n,k)} \le r_{q(i,n),k},\ k \in K(s^+)\}$$
$$\times P\{C_{l(i,n,k),m(i,n,k)} > \mathbf{Z}_{j(i,n),l(i,n,k)}(v),\ k \in K(s^+)\}.$$

Using hypothesis (iii), substituting the right-hand side into Eq. (A8) and integrating yields

$$P\{C_{l(i,n,k),m(i,n,k)} \le \zeta_{j(i,n)} - \zeta_{l(i,n,k)} + R_{q(i,n),k},$$
$$C_{l(i,n,k),m(i,n,k)} > \zeta_{j(i,n)} - \zeta_{l(i,n,k)},\ k \in K(s^+);\ \bar{\mathscr{L}}^i_{j(i,n)}\},$$
$$\ge \delta\, P\{C_{l(i,n,k),m(i,n,k)} > \zeta_{j(i,n)} - \zeta_{l(i,n,k)},\ k \in K(s^+);$$
$$\bar{\mathscr{L}}^i_{j(i,n)}\}. \quad \text{(A9)}$$

Substituting Eq. (A9) into Eq. (A7) and using Lemma 9,

$$P\{S_{n,k}(s^+) + A_{n,k}(s^+) \le T_n^+ + R_{n,k}(s^+), \quad k \in K(s^+);\ U_n\}$$
$$\ge \sum_i \delta\, P\{C_{l(i,n,k),m(i,n,k)} > \zeta_{j(i,n)} - \zeta_{l(i,n,k)}, \quad k \in K(s^+);\ \mathscr{V}^i_n\}$$
$$= \delta\, P\{S_{n,k}(s^+) + A_{n,k}(s^+) > T_n^+, \quad k \in K(s^+);\ U_n\}$$
$$= \delta\, P\{U_n\}.$$

The last equality follows by the same reasoning that leads to the equivalence of the events in Eqs. (A5) and (A6). □

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Peter J. Haas *Department of Operations Research, Stanford University, Stanford, California 94305.* Mr. Haas is a Ph.D. candidate in the Department of Operations Research at Stanford University. His current research is concerned with the development of discrete-event simulation methodology. In 1978, Mr. Haas received an S.B. in engineering and applied mathematics from Harvard University and an M.S. in environmental engineering from Stanford University in 1979. From 1979 to 1981 he was a staff scientist at Radian Corporation in Austin, Texas. In 1984 he received an M.S. in statistics from Stanford University. Mr. Haas is a student member of the Operations Research Society of America.

Gerald S. Shedler *IBM Research Division, 5600 Cottle Road, San Jose, California 95193.* Mr. Shedler has been a Research staff member at IBM since 1965, initially at the Thomas J. Watson Research Center, Yorktown Heights, New York, and since 1970, in the Computer Science Department at the Research laboratory in San Jose. During 1973–1974, while on sabbatical from IBM, he was associated with Stanford University as Acting Associate Professor in the Department of Operations Research, and subsequently has been Consulting Associate Professor in the same department. He has worked extensively on applications of stochastic processes, particularly to performance evaluation of computer systems. His current research is on discrete event methods for simulation of stochastic systems. Mr. Shedler is coauthor with Donald L. Iglehart of *Regenerative Simulation of Response Times in Networks of Queues*, published in 1980.