The dynamical equations governing a lubricating film consisting of a gas film overlying a liquid film

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The dynamical equations governing a two-phase lubricating configuration are derived. It is assumed that a gas film overlies a liquid film, both thin enough that the lubrication approximation may be used. The analysis leads to coupled Reynolds equations governing the pressure and the relative thickness of the two films. The coupling, which is determined by continuity of tangential stress across the gas-liquid interface, is considerably simplified if the shear viscosity of the liquid greatly exceeds that of the gas.

1. Introduction

Virtually all work on hydrodynamic lubrication deals with configurations where the load is carried by a single fluid, which may be either a liquid or a gas. The basic theory is encompassed by Reynolds' lubrication equation, which is derived from the hydrodynamic equations by using the thinness of the film, relative to its lateral extent, to provide a perturbation parameter. Fluid-film bearings may either be externally pressurized or self-acting. That is, the load-carrying pressure may be generated either by pumping fluid through the bearing or by relative motion of its surfaces. Both types have been the subject of extensive theoretical and experimental literatures.

By contrast, there has been almost no work on bearings that involve multiple-fluid films. An exception is a recent experimental paper [1] dealing with an externally pressurized gas bearing whose load-carrying capacity and stiffness are enhanced by an oil film under the gas layer. However, one design for computer memory disks that is currently in wide use involves coating the disk with a film of very viscous liquid. Although it is exceedingly thin, this film is of a thickness that is comparable with the air film beneath the self-acting slider bearing that carries the read-write circuitry. It is to be expected that the gliding characteristics of the slider are modified by the presence of a liquid substrate in place of the conventional rigid driving surface.

The actual situation is even more complicated. For example, it is not certain that the liquid layer behaves as a

classical film. However, even the ideal situation of two fluid layers, gas overlying liquid, appears to lie beyond prior theoretical investigations.

The present study derives the dynamical equations governing such a two-phase lubricating configuration. We assume that the total thickness H of the combined films is a specified function of time and space. In practice, the equations derived here may be coupled to the external dynamics of the slider suspension system, so that H becomes a dependent variable in a more general problem. We take

$$H = L + G, \tag{1}$$

where L and G denote, respectively, the thicknesses of the liquid and gas films. Since these are related through (1), one or the other of them is a dependent variable that must be determined as part of the lubrication problem.

In the next section, we use the lubrication approximation to examine those relationships which are common to both phases. More analysis can be carried out in this joint fashion if we allow for partial slip in both phases and, later, set the slip coefficients to zero for the liquid. Since the gas is bounded from above by a solid and from below by the liquid, we permit the slip coefficients on the bounding surfaces to be different.

Section 3 specializes the result to the gas layer, which is assumed to be formed by an isothermal ideal gas. The corresponding Reynolds equation extends a result due to Burgdorfer [2], who dealt with a steady-state gas film with slip terms corresponding to perfectly diffuse reflection of gas molecules from both boundaries. Section 4 carries out the corresponding specialization for the liquid layer. The coupled Reynolds equations developed in these two sections govern the film pressure and the thickness L (or, equivalently, G) as dependent variables. They also involve the interface velocity. This is determined in Section 5 by imposing continuity of tangential stress across the interface. When the shear viscosity of the liquid is much larger than that of the gas, their ratio can be introduced as a perturbation parameter, thus simplifying the results considerably. This reduction is carried out in Section 6.

Under the circumstances described in Section 6, the gas and liquid flows become substantially decoupled. At least on short time scales, the gas film can be investigated with the liquid film regarded as stationary. Nevertheless, the presence of the liquid still makes a difference if the slip coefficients of the solid and liquid boundaries are not the same. The consequence of this is that the Reynolds equation includes two rather awkward terms that are absent when the slip coefficients are identical. One of these is qualitatively new, in that it involves the difference, rather than the sum, of the lateral surface velocities. Section 7 examines a somewhat artificial configuration in which the effect of this term is isolated. Without the differential slip, lubricating pressure in this configuration would be totally absent.

2. Dynamical relationships which hold in both fluid layers

Assume that a thin film of fluid lies between the surfaces

$$x_3 = \mathscr{H}(x_1, x_2, t),$$

$$x_3 = \mathscr{H}'(x_1, x_2, t). \tag{2}$$

The film thickness h, defined by

$$h(x_1, x_2, t) = \mathscr{H}'(x_1, x_2, t) - \mathscr{H}(x_1, x_2, t), \tag{3}$$

is assumed positive for all values of x_1, x_2, t. The bounding surfaces move with local velocity components V_κ, V'_κ, which are subject to the kinematic constraints

$$V_3 = \partial\mathscr{H}/\partial t + V_i \partial\mathscr{H}/\partial x_i,$$

$$V'_3 = \partial\mathscr{H}'/\partial t + V'_i \partial\mathscr{H}'/\partial x_i. \tag{4}$$

In (4) and henceforth, lower-case Latin indices range over the values 1, 2. Greek indices extend over 1, 2, 3. In either case, a repeated index denotes summation. When it is convenient, we shall write z in place of x_3. With these conventions, the continuity equation can be written

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} + \frac{\partial(\rho v_3)}{\partial z} = 0, \tag{5}$$

where ρ is the fluid density and v_κ are the velocity components. Integrating (5) across the film yields

$$\frac{\partial}{\partial t}\int_{\mathscr{H}}^{\mathscr{H}'} \rho dz = \rho(\mathscr{H}')\left\{\frac{\partial\mathscr{H}'}{\partial t} + \left[v_i \frac{\partial\mathscr{H}'}{\partial x_i} - v_3\right]_{z=\mathscr{H}'}\right\}$$

$$- \rho(\mathscr{H})\left\{\frac{\partial\mathscr{H}}{\partial t} + \left[v_i \frac{\partial\mathscr{H}}{\partial x_i} - v_3\right]_{z=\mathscr{H}}\right\} - \frac{\partial}{\partial x_i}\int_{\mathscr{H}}^{\mathscr{H}'} \rho v_i dz. \tag{6}$$

Now $[v_3 - v_i \partial\mathscr{H}'/\partial x_i]_{z=\mathscr{H}'}$ is the velocity of the fluid normal to the upper boundary. Hence, it is equal to the component of that boundary's motion normal to itself, viz., $V'_3 - V'_i \partial\mathscr{H}'/\partial x_i$. With the second of constraints (4), the first term in braces on the right side of (6) vanishes. By a similar observation about the lower boundary, the other term in braces also vanishes. Thus,

$$\frac{\partial}{\partial t}\int_{\mathscr{H}}^{\mathscr{H}'} \rho dz = -\nabla\cdot\int_{\mathscr{H}}^{\mathscr{H}'} \rho \mathbf{v} dz, \tag{7}$$

where ∇ denotes the two-dimensional del operator $(\partial/\partial x_1, \partial/\partial x_2)$ and $\mathbf{v}$ denotes (v_1, v_2). So far this is exact. The next step is to use the equations of motion to express the right side in terms of available quantities. This will involve the lubrication approximation, viz., that the film thickness is small compared with its lateral dimensions.

When fluid inertia is neglected, the velocity components and pressure within the film are governed by the creeping

flow equations

$$\frac{\partial p}{\partial x_\kappa} = \frac{\partial}{\partial x_\alpha}\left(\mu\frac{\partial v_\kappa}{\partial x_\alpha}\right) + (\lambda + \mu)\frac{\partial \Delta}{\partial x_\kappa} + \frac{\partial v_\alpha}{\partial x_\kappa}\frac{\partial \mu}{\partial x_\alpha} + \Delta\frac{\partial \lambda}{\partial x_\kappa} + \rho F_\kappa. \tag{8}$$

In (8), μ and λ denote the first and second viscosity coefficients, $\Delta = \partial v_\alpha/\partial x_\alpha$ is the dilation and F_κ are the components of body force per unit mass. In order to determine the magnitude of various terms, we normalize with respect to the following quantities:

B, a typical lateral length of the configuration;
h_0, a typical value of the film thickness;
V, a typical value of the lateral surface motion;
ω, a typical frequency of the squeeze motion;
μ_0, a typical value of the viscosity.

The lubrication approximation is that the ratio $\varepsilon = h_0/B$ is small compared with unity. Since variations across the film are steep compared with those along it, the normalized variables are chosen in a way that "stretches" the coordinate across the film. Following the usual procedure of lubrication theory, we take

$$\begin{aligned} &x_i = BX_i, \qquad z = x_3 = h_0\zeta = \varepsilon B\zeta, \qquad t = T/\omega, \\ &v_i = (\omega B + V)u_i, \qquad v_3 = h_0\omega w, \qquad \Delta = (\omega + V/B)D, \\ &\mu = \mu_0\mu^*, \qquad \lambda = \mu_0\lambda^*, \\ &p = \mu_0(\omega + V/B)\varepsilon^{-2}\pi. \end{aligned} \tag{9}$$

The choice of a pressure scale proportional to ε^{-2} is standard in lubrication theory. If, instead, an arbitrary exponent n is used *a priori*, a straightforward analysis [3] leads to the result that $n = -2$.

Introducing the normalized variables into the $\kappa = 3$ component of (8) yields

$$\frac{\partial \pi}{\partial \zeta} = \varepsilon^2\left\{\frac{\omega B}{\omega B + V}\left[\frac{\partial}{\partial \zeta}\left(\mu^*\frac{\partial w}{\partial \zeta}\right) + \frac{\partial w}{\partial \zeta}\frac{\partial \mu^*}{\partial \zeta}\right] + (\lambda^* + \mu^*)\frac{\partial D}{\partial \zeta} + \frac{\partial v_i}{\partial \zeta}\frac{\partial \mu^*}{\partial X_i} + D\frac{\partial \lambda^*}{\partial \zeta}\right\} + O(\varepsilon^3). \tag{10}$$

Thus, to second order in ε, the pressure is constant across the film.

The $\kappa = 1, 2$ components of (8) become

$$\frac{\partial \pi}{\partial X_i} = \frac{\partial}{\partial \zeta}\left(\mu^*\frac{\partial u_i}{\partial \zeta}\right) + \varepsilon^2\left[\frac{\partial}{\partial X_j}\left(\mu^*\frac{\partial u_i}{\partial X_j}\right) + (\lambda^* + \mu^*)\frac{\partial D}{\partial X_i} + \frac{\partial u_j}{\partial X_i}\frac{\partial \mu^*}{\partial X_j} + \frac{\omega B}{\omega B + V}\frac{\partial w}{\partial X_i}\frac{\partial \mu^*}{\partial \zeta} + \frac{B^2\rho F_i}{\mu_0(\omega B + V)}\right]. \tag{11}$$

If terms of order ε^2 are neglected, this simply equates $(\partial/\partial\zeta)(\mu^*\partial u_i/\partial\zeta)$ to $\partial\pi/\partial X_i$. Returning to the physical variables, we then have

$$\frac{\partial}{\partial z}\left(\mu\frac{\partial \boldsymbol{v}}{\partial z}\right) = \nabla p. \tag{12}$$

Since the pressure variation across the film is negligibly small, the right side of (12) is independent of z. Hence, (12) is easily integrated if we assume that the viscosity is independent of z. For fluids with variable viscosity, this is tantamount to assuming that the temperature is constant across the film. The argument goes as follows.

In general, viscosity varies with density and temperature. But there is an equation of state relating pressure, density, and temperature. Since we have already established that the pressure is effectively constant across the film, uniform temperature guarantees uniform density—a fact which we shall explicitly use in what follows. Thus, unless the viscosity depends on variables other than density and temperature, its variation across the film can be neglected.

For the extremely thin films of interest here, the cross-film temperature variations are indeed negligible. Hence we do assume that μ (and ρ) are independent of z. Integration of (12) then yields

$$\boldsymbol{v} = \left(\frac{1}{2\mu}\nabla p\right)z^2 + \mathbf{c}_1 z + \mathbf{c}_2, \tag{13}$$

where $\mathbf{c}_1$ and $\mathbf{c}_2$ are constants of integration. In the liquid film, these can be determined by equating the velocities at $z = \mathscr{A}$ and $z = \mathscr{A}'$ to the boundary velocities. In the gas, matters are more complicated, because in some situations of practical importance the mean free path of the gas molecules may be a significant fraction of the film thickness. One must then deal with partial slip between gas and boundary. In general, the degree of slip depends in a complicated fashion upon the geometry and thermodynamics of the situation. An approximation sometimes used [2] relates $\boldsymbol{v}$ and its z-derivative to the boundary velocities $\mathbf{V} = (V_1, V_2)$ and $\mathbf{V}' = (V_1', V_2')$ according to

$$\begin{aligned} \boldsymbol{v}(\mathscr{A}) &= \mathbf{V} + \gamma\lambda\left.\frac{\partial \boldsymbol{v}}{\partial z}\right|_{z=\mathscr{A}}, \\ \boldsymbol{v}(\mathscr{A}') &= \mathbf{V}' - \gamma'\lambda\left.\frac{\partial \boldsymbol{v}}{\partial z}\right|_{z=\mathscr{A}'}, \end{aligned} \tag{14}$$

where λ is the mean free path and γ, γ' are dimensionless constants whose values depend on the nature of the interaction between the gas molecules and the bounding surface. Since we are interested in a gas film that is bounded at the top by a solid and at the bottom by a liquid, we cannot assume *a priori* that γ and γ' are equal. However, it seems possible that, on the molecular scale, both surfaces will often be coarse-grained enough that the gas molecules undergo diffuse reflection from each of them. In this case, γ

and γ' are the same, and are, in fact, nearly equal to unity. So that our results will appear in a form for which this important special case can easily be discerned, we introduce the notation

$$\alpha = (\gamma + \gamma')/2,$$
$$\beta = (\gamma - \gamma')/2. \tag{15}$$

Since the mean free path is inversely proportional to the density, we may write

$$\lambda = \lambda_a \rho_a/\rho, \tag{16}$$

where λ_a is the mean free path at the ambient density ρ_a. With (15) and (16), the boundary conditions (14) become

$$\boldsymbol{v}(\mathscr{H}) = \mathbf{V} + \frac{(\alpha + \beta)K_n h_0 \rho_a}{\rho} \left.\frac{\partial \boldsymbol{v}}{\partial z}\right|_{z=\mathscr{H}},$$
$$\boldsymbol{v}(\mathscr{H}') = \mathbf{V}' - \frac{(\alpha - \beta)K_n h_0 \rho_a}{\rho} \left.\frac{\partial \boldsymbol{v}}{\partial z}\right|_{z=\mathscr{H}'}, \tag{17}$$

where the dimensionless parameter K_n is the Knudsen number λ_a/h_0.

Since we are assuming that the density is constant across the film, the same value of ρ is used in both of the boundary conditions (17). No-slip is recovered when the Knudsen number approaches zero. Hence, conditions (17) apply to the liquid as well as the gas. Using them to evaluate the constants of integration in (13) yields

$$\begin{aligned}\boldsymbol{v} = &-\left(\frac{1}{2\mu}\nabla\rho\right)(z - \mathscr{H})(\mathscr{H}' - z) \\ &+ (\mathbf{V}' - \mathbf{V})z/h + \mathbf{V}\mathscr{H}'/h - \mathbf{V}'\mathscr{H}/h - \frac{\alpha K_n h_0 \rho_a}{\rho} \\ &\times \left\{\frac{h}{2\mu}\nabla p + \frac{\mathbf{V}' - \mathbf{V}}{h}\frac{(z - \mathscr{H}) - (\mathscr{H}' - z)}{h + 2\alpha K_n h_0 \rho_a/\rho}\right\} \\ &+ \frac{\beta K_n h_0 \rho_a/\rho}{h + 2\alpha K_n h_0 \rho_a/\rho} \\ &\times \left\{\left[(z - \mathscr{H}) - (\mathscr{H}' - z) + \frac{2\beta K_n h_0 \rho_a}{\rho}\right]\frac{h}{2\mu}\nabla p \right. \\ &\left. + \mathbf{V}' - \mathbf{V}\right\}.\end{aligned} \tag{18}$$

Thus,

$$\begin{aligned}\int_{\mathscr{H}}^{\mathscr{H}'} \boldsymbol{v}\,dz &= -\frac{h^3}{12\mu}\left[1 + \frac{6\alpha K_n h_0 \rho_a}{\rho h} - \frac{12(\beta K_n h_0 \rho_a/\rho)^2}{h(h + 2\alpha K_n h_0 \rho_a/\rho)}\right]\nabla p \\ &+ \frac{h}{2}(\mathbf{V} + \mathbf{V}') + \frac{\beta K_n h_0 (\rho_a/\rho)h}{h + 2\alpha K_n h_0 \rho_a/\rho}(\mathbf{V}' - \mathbf{V}).\end{aligned} \tag{19}$$

Since we have taken the density to be independent of z, the factor ρ on the right side of (7) can be taken outside the integral sign. Substituting (19) into (7) then yields a generalized Reynolds equation:

$$\begin{aligned}&\nabla\cdot\left\{\rho h^3\left[1 + \frac{6\alpha K_n h_0 \rho_a}{\rho h} - \frac{12(\beta K_n h_0 \rho_a/\rho)^2}{h(h + 2\alpha K_n h_0 \rho_a/\rho)}\right]\nabla p\right\} \\ &= 6\mu\nabla\cdot[\rho h(\mathbf{V} + \mathbf{V}')] + 12\mu\frac{\partial(\rho h)}{\partial t} \\ &\quad + 12\mu\beta K_n h_0 \rho_a \nabla\cdot\left[\frac{h(\mathbf{V}' - \mathbf{V})}{h + 2\alpha K_n h_0 \rho_a/\rho}\right].\end{aligned} \tag{20}$$

In an isothermal ideal gas, the density is proportional to the pressure. Thus,

$$\rho/\rho_a = p/p_a \tag{21}$$

where p_a is the ambient pressure. If we simplify the notation by defining

$$\sigma = \alpha K_n h_0 p_a = \alpha \lambda_a p_a,$$
$$\tau = \beta K_n h_0 p_a = \beta \lambda_a p_a, \tag{22}$$

the velocity expression (18) becomes

$$\begin{aligned}\boldsymbol{v} = &-\left(\frac{1}{2\mu_g}\nabla p\right)(z - \mathscr{H}_I)(\mathscr{H}_T - z) \\ &+ (\mathbf{V}_T - \mathbf{V}_I)z/G + \mathbf{V}_I\mathscr{H}_T/G - \mathbf{V}_T\mathscr{H}_I/G \\ &- \sigma\left\{\frac{G}{2\mu_g}\frac{\nabla p}{p} + \frac{\mathbf{V}_T - \mathbf{V}_I}{G}\frac{(z - \mathscr{H}_I) - (\mathscr{H}_T - z)}{pG + 2\sigma}\right\} \\ &+ \frac{\tau}{pG + 2\sigma}\left\{\left[(z - \mathscr{H}_I) - (\mathscr{H}_T - z) + \frac{2\tau}{p}\right]\right. \\ &\left.\times\frac{G}{2\mu_g}\nabla p + \mathbf{V}_T - \mathbf{V}_I\right\},\end{aligned} \tag{23}$$

where the subscripts I and T signify the interface and the top boundary of the gas film, respectively. The generalized Reynolds equation (20) becomes

$$\begin{aligned}&\nabla\cdot\left\{G^3 p\left[1 + \frac{6\sigma}{Gp} - \frac{12\tau^2}{Gp(Gp + 2\sigma)}\right]\nabla p\right\} \\ &= 6\mu_g\nabla\cdot[Gp(\mathbf{V}_T + \mathbf{V}_I)] + 12\mu_g\frac{\partial(Gp)}{\partial t} \\ &\quad + 12\mu\tau\nabla\cdot\left[\frac{Gp(\mathbf{V}_T - \mathbf{V}_I)}{Gp + 2\sigma}\right].\end{aligned} \tag{24}$$

For the important special case in which the two boundary surfaces have identical slip characteristics, i.e., for $\tau = 0$, this reduces to a time-dependent extension of Burgdorfer's well-known result [2] for gas lubrication at spacings nearly as small as the mean free path.

The case $\tau \neq 0$ is qualitatively different, since the Reynolds equation then includes a term involving the difference of the surface velocities.

4. The liquid phase

Since the mean free path in a liquid is virtually zero, the Knudsen number of the liquid film is negligibly small. Hence, the general expression (18) for the velocity reduces to

$$\mathbf{v} = -\left(\frac{1}{2\mu_\ell}\nabla p\right)(z - \mathscr{H}_B)(\mathscr{H}_I - z) + (\mathbf{V}_I - \mathbf{V}_B)z/L + \mathbf{V}_B\mathscr{H}_I/L - \mathbf{V}_I\mathscr{H}_B/L, \tag{25}$$

where the subscript B signifies the bottom of the liquid film. Also, liquids are almost incompressible, so that we may take $\rho = \rho_a$. Hence, (20) reduces to the familiar Reynolds equation for a liquid lubricant:

$$\nabla\cdot(L^3\nabla p) = 6\mu_\ell\nabla\cdot[L(\mathbf{V}_B + \mathbf{V}_I)] + 12\mu_\ell\frac{\partial L}{\partial t}. \tag{26}$$

This is linear in p but not in L—a distinction which is important in the present context, because the thickness of the liquid film depends on the dynamics of the coupled gas-liquid system. Specifically, once the interface velocity $\mathbf{V}_I$ has been eliminated from the problem (next section), the total gap $\mathscr{H}_T - \mathscr{H}_B$ must be apportioned between L and G in such a way that the two Reynolds equations (24) and (26), each subject to the boundary condition of ambient pressure at the configuration periphery, are compatible with a common solution for the pressure field.

The last point is, technically, not precise. Because of interfacial tension, the pressure in the liquid will differ slightly from that in the gas. However, in the notation of Section 2, the interface curvature is of order ε/B, whereas the pressure is proportional to ε^{-2}. Hence the correction is negligible in the context of the present analysis. If the coefficient of interfacial tension is extremely large, the numerical factor could possibly override this asymptotic argument, since ε never really goes to zero in practice. Since the value of the interfacial tension doesn't enter the derivation of the two Reynolds equations, the results obtained so far would still apply, except that p would stand for different pressures in Sections 3 and 4. The apportionment of gap between L and G would then take place in such a way that the difference between these pressures would be that demanded by interfacial tension.

5. Elimination of the interface velocity

The coupled Reynolds equations (24) and (26), with appropriate subsidiary conditions, are not yet sufficient to determine the pressure and film thicknesses, because both contain the unknown velocity $\mathbf{V}_I$. This velocity is determined by requiring that the tangential tractions that the two fluids exert on each other be equal and opposite. To the degree of approximation used in Section 2, this is equivalent to requiring that $\mu\partial v/\partial z$ be continuous at the interface $z = \mathscr{H}_I$. Now, with (25) and (23),

$$\mu_\ell\left.\frac{\partial \mathbf{v}}{\partial z}\right|_{z=\mathscr{H}_I^-} = \frac{L}{2}\nabla p + \frac{\mu_\ell}{L}(\mathbf{V}_I - \mathbf{V}_B), \tag{27}$$

and

$$\mu_g\left.\frac{\partial \mathbf{v}}{\partial z}\right|_{z=\mathscr{H}_I^+} = -\frac{G}{2}\left(1 - \frac{2\tau/pG}{1 + 2\sigma/pG}\right)\nabla p + \frac{\mu_g}{G}\frac{\mathbf{V}_T - \mathbf{V}_I}{1 + 2\sigma/pG}. \tag{28}$$

Equating these expressions and solving for $\mathbf{V}_I$ leads to

$$(G\mu_\ell + L\mu_g)\mathbf{V}_I = G\mu_\ell\mathbf{V}_B + L\mu_g\mathbf{V}_T - \frac{1}{2}(L + G)LG\nabla p + \frac{2\sigma M}{p}\left[\mu_\ell(\mathbf{V}_B - \mathbf{V}_T) - \frac{1}{2}(L + G)L\nabla p\right] + \frac{\tau LG}{p + 2\sigma/G}\left(1 + \frac{2\sigma M}{Gp}\right)\nabla p, \tag{29}$$

where the dimensionless quantity M is defined by

$$M = \frac{L\mu_g}{G\mu_\ell + L\mu_g + 2\mu_\ell\sigma/p}. \tag{30}$$

If the interfacial tension is great enough to give rise to the possibility that the gas and liquid pressures differ significantly, then p in Eqs. (29) and (30) should be interpreted as the gas pressure, and the right side of (29) should be augmented by

$$\left(\frac{1}{2}G + \frac{M\sigma}{p}\right)L^2\nabla(p - p_L), \tag{31}$$

where p_L denotes the pressure in the liquid.

6. Asymptotic results for a very viscous liquid phase

An important special case arises when the shear viscosity of the liquid greatly exceeds that of the gas. For this case, we may write

$$\mu_\ell = \mu_g/\delta, \tag{32}$$

with $\delta \ll 1$. The complicated expression (29) for the interface velocity then reduces to

$$\mathbf{V}_I = \mathbf{V}_B + \delta\left[\frac{L}{G}(\mathbf{V}_T - \mathbf{V}_B) - \frac{L(L + G)}{2\mu_g}\nabla p + \frac{2\sigma(L/G)(\mathbf{V}_B - \mathbf{V}_T)}{Gp + 2\sigma} + \frac{\tau LG\nabla p}{\mu_g(Gp + 2\sigma)}\right] + O(\delta^2). \tag{33}$$

Thus, as δ approaches zero, shearing motions in the liquid layer become vanishingly small. With (33), the Reynolds equations (24) and (26) become

$$\nabla\cdot\left\{G^3p\left[1+\frac{6\sigma}{Gp}-\frac{12\tau^2}{Gp(Gp+2\sigma)}\right]\nabla p\right\}$$
$$=6\mu_g\nabla\cdot[Gp(\mathbf{V}_T+\mathbf{V}_B)]+12\mu_g\frac{\partial(Gp)}{\partial t}$$
$$+12\mu_g\tau\nabla\cdot\left[\frac{Gp(\mathbf{V}_T-\mathbf{V}_B)}{Gp+2\sigma}\right]+O(\delta), \quad (34)$$

and

$$\nabla\cdot(L^3\nabla p)=12\mu\left[\nabla\cdot(L\mathbf{V}_B)+\frac{\partial L}{\partial t}\right]+O(\delta). \quad (35)$$

7. A configuration which isolates the effect of different slip coefficients

Suppose now that the ratio of viscosities is high enough that the terms of order δ can be neglected in (34). The only remaining effect of the liquid is through the terms in τ, which result from the difference in slip coefficients between the solid and liquid boundaries. Such terms also arise in a single-phase gas bearing if one of the bounding surfaces consists of a single crystal or other solid surface which might be expected to have a significant component of specular reflection.

For the configuration shown in **Figure 1,** *only* those terms arising from the difference in slip coefficients contribute to the lubricating pressure. Two identical circular cylinders rotate with the same angular velocity Ω and in the same sense, so that the lateral velocities are equal but opposite, and there is no squeeze motion. Hence, the lubricating pressure in the nip between the cylinders vanishes unless there is asymmetrical slip. In principle, this offers the experimental possibility of isolating the effect [4]. In practice, of course, a slight misalignment of the cylinders or a tiny amount of vibration in their bearings would mask the small effect one is trying to observe. The purpose of the present section is to present a relatively simple analysis that offers some insight, rather than to support an experimental technique.

In the notation of Fig. 1,

$$G=h_0+2r(1-\sqrt{1-x^2/r^2}), \quad (36)$$

$$\mathbf{V}_T=-\mathbf{V}_B=\Omega r\sqrt{1-x^2/r^2}\,\mathbf{i}_x. \quad (37)$$

The first two terms on the right side of (34) then vanish. Hence, there is no lubricating pressure unless τ differs from zero. If we assume that the cylinders are long enough that side leakage may be neglected, (34) reduces to the one-dimensional equation

$$\frac{d}{dx}\left\{G^3p\left[1+\frac{6\sigma}{Gp}-\frac{12\tau^2}{Gp(Gp+2\sigma)}\right]\frac{dp}{dx}\right\}$$
$$=24\,\mu_g\tau\Omega r\frac{d}{dx}\left[\frac{Gp\sqrt{1-x^2/r^2}}{Gp+2\sigma}\right]. \quad (38)$$

We now normalize (38), choosing h_0 as the length scale and

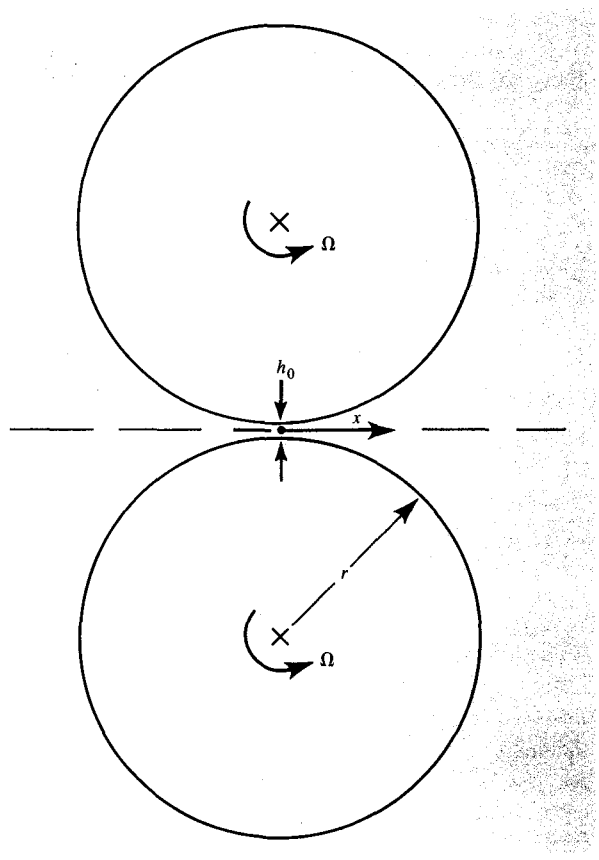

Figure 1

A configuration for which the lubricating pressure vanishes unless there is differential slip.

p_a as the pressure scale. The ambient pressure should be taken as that obtaining in the region where the lubrication approximation ceases to be valid. This may be indistinguishable from the actual atmospheric pressure. In some parameter ranges, however, there may be enough inlet and exit flow that p_a is somewhat different. Using h_0 and the appropriate p_a, we define dimensionless variables X, P, dimensionless parameters R, A, B, N, and dimensionless functions $H(X)$, $F(P, X)$, $\Phi(P, X)$ according to

$$X=x/h_0, \qquad P=p/h_0, \qquad R=r/h_0,$$
$$A=\sigma/h_0p_a, \qquad B=\tau/h_0p_a, \qquad N=24\,\mu_g\Omega R/p_a,$$
$$H(X)=G/h_0=1+2R(1-\sqrt{1-X^2/R^2})$$
$$\approx 1+X^2/R \text{ for } X\ll R,$$
$$F(P,X)=H^3P\left[1+\frac{6A}{HP}-\frac{12B^2}{HP(HP+2A)}\right],$$
$$\Phi(P,X)=HP/(HP+2A). \quad (39)$$

In passing, we note from (22) that the normalized slip coefficients may also be written

$$A = K_n\alpha, \qquad B = K_n\beta.$$

With (39), the differential Eq. (38) becomes

$$\frac{d}{dX}\left[F(P, X)\frac{dP}{dX}\right] = BN\frac{d}{dX}\left[\sqrt{1 - X^2/R^2}\,\Phi(P, X)\right], \tag{40}$$

which has the first integral

$$F(P, X)\frac{dP}{dX} = BN\sqrt{1 - X^2/R^2}\,\Phi(P, X) + C,$$

where C is a constant of integration. This is equivalent to the integral equation

$$P = BN\int_{X_0}^{X}\frac{\sqrt{1 - X^2/R^2}\,\Phi(P, X)dX}{F(P, X)} + C\int_{X_0}^{X}\frac{dX}{F(P, X)}, \tag{41}$$

where X_0 is a second constant of integration.

Far from the nip in either direction, the pressure is ambient. This supplies subsidiary conditions for the elimination of C and X_0, but the phrase "far from the nip" requires some discussion. If we were to regard this purely as a boundary value problem for Eq. (40), it would be natural to require that $P = 1$ at the geometrical limits of the configuration, i.e., at $X = \pm R$. However, the lubrication approximation underlying (40) breaks down for much smaller values of $|X|$. Moreover, one expects that, for physically realizable parameters, the pressure will be indistinguishable from ambient for values of $|X|$ that are smaller still. Hence, we take

$$P(\pm X_m) = 1, \tag{42}$$

where $1 \ll X_m \ll R$, the actual value of X_m being chosen by some preliminary numerical experimentation. Using (42) in (41) leads to

$$P(X) = 1 + BN\int_{-X_m}^{X}\frac{\sqrt{1 - X^2/R^2}\,\Phi(P, X) - Q}{F(P, X)}\,dX, \tag{43}$$

where

$$Q = \left[\int_{-X_m}^{X_m}\frac{\sqrt{1 - X^2/R^2}\,\Phi(P, X)}{F(P, X)}\,dX\right] \div \left[\int_{-X_m}^{X_m}\frac{dX}{F(P, X)}\,dX\right]. \tag{44}$$

The problem is now in a form suitable for a variant of Picard iteration, based on (43). The variation is that Q must be recalculated from (44) at each step of the iteration.

To illustrate, we consider the case of cylinders of radius 2 cm rotating in air at normal conditions ($\mu = 1.8 \times 10^{-5}$ Pa-s, $p_a = 10^5$ N/m^2, $\lambda_a = 64$ nm), with a 200-nm gap between them. The Knudsen number is then 0.32. Two series of computations were carried out. In the first, the slip coefficients were fixed at arbitrarily chosen but reasonable values, and the rotation speed was allowed to vary. In the second series, the effect of asymmetric slip was examined for a fixed value of rotation speed. In both series, α was set to 0.78125. Thus, for both series,

$$R = 10^5, \qquad A = 0.25, \qquad N = 4.524 \times 10^{-5}\omega,$$

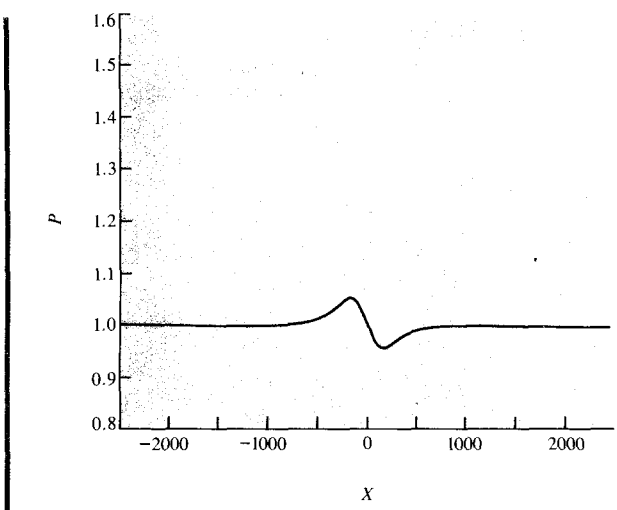

Figure 2

The distribution of dimensionless pressure for $B = 0.05$, $N = 0.5$.

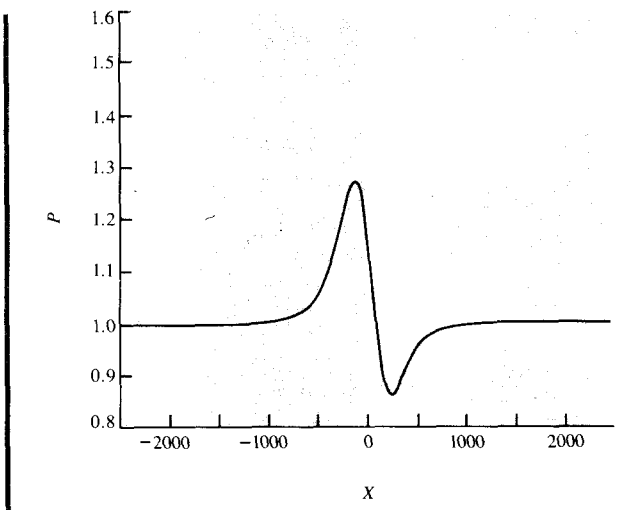

Figure 3

The distribution of dimensionless pressure for $B = 0.05$, $N = 2.5$.

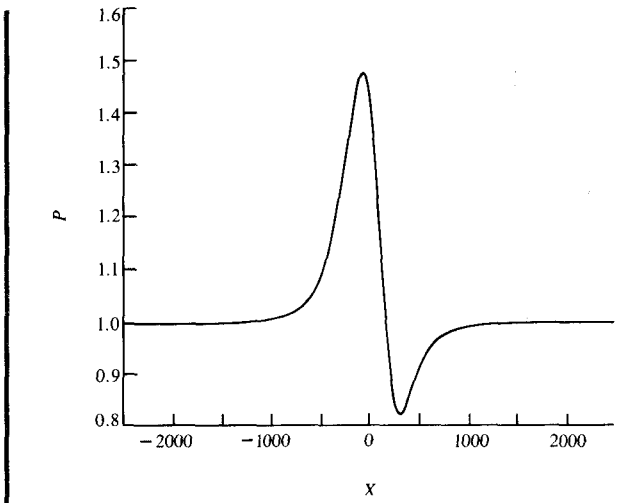

Figure 4

The distribution of dimensionless pressure for $B = 0.05$, $N = 4.9$.

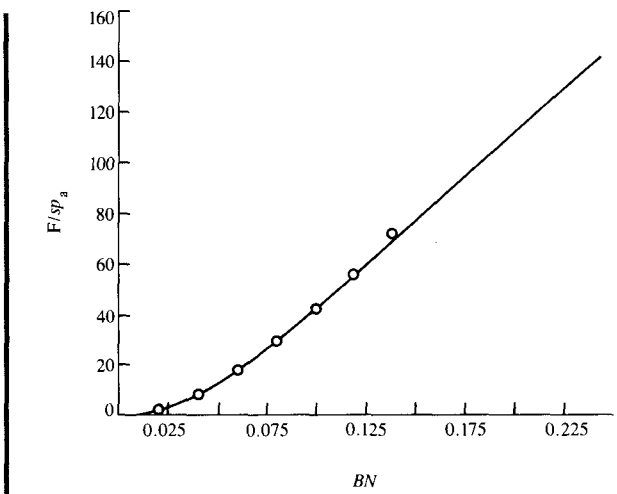

Figure 5

The total force (F) per unit axial length, plotted versus the product BN. Results obtained with B fixed at 0.05 are represented by the solid curve; those obtained with N fixed at 2.0 are represented by the circles.

where ω denotes the rotation rates in revolutions per minute.

The numerical integrations in (43) and (44) were carried out trapezoidally with a uniform ΔX of 0.5, i.e., $\Delta x = 100$ nm, and with $X_m = 2500$.

• *Results for B fixed and N varying*

For this series, β was set to 0.15625, so that $B = 0.05$. The pressure distribution was calculated for twenty-four different values of N, ranging from 0.1 up to 4.9. Representative cases are shown in **Figures 2, 3,** and **4**. For the largest values of N convergence was quite sluggish, presumably because of the steep gradients near the nip. To proceed to higher values of N would require a more sophisticated scheme which concentrates grid points in the region of small $|X|$.

• *Results for N fixed and B varying*

Varying B is almost the same thing as varying N, but not quite. The pressure field is strongly influenced by B through the coefficient BN in Eq. (41), but it is also influenced to some extent by the appearance of B^2 in the dimensionless function F defined in (39). With N fixed at 2.0, the pressure distribution was calculated for seven values of B, ranging from 0.01 to 0.07. The results were qualitatively similar to those obtained for $B = 0.05$, with N adjusted to preserve the same value of BN. For the larger values of B, there were perceptible quantitative differences. One of these is illustrated in **Figure 5,** which plots the net normalized force generated between the cylinders versus BN. The solid curve connects the points computed for $B = 0.05$ with N varying. The circles were computed with N fixed at 2.0 and B varying. The point for $B = 0.07$ lies somewhat above the curve, the actual difference in ordinates being 0.355.

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