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Bending and Stretching an Elastic Strip Around a Narrow Cylindrical Drum

An analysis is made of the deflected forms assumed by an elastic strip when it is bent and stretched around a rigid drum. A summary is first given of the method of analysis used and the results obtained in a previous study where the drum width is equal to or greater than that of the strip. This work is then extended to the case where the strip width exceeds that of the drum. Deflected forms and contact regions are delineated for a strip-width/drum-width ratio of two and for various values of two parameters: anticlastic deformation and tension.

Introduction

Elastic strips or tapes are commonly used in industrial applications. When these strips are bent by end couples, the longitudinal strains which are induced are accompanied by lateral strains in the width direction of the strip. As a result, the strip bends to a surface in which two principal curvatures are initially opposite in sign—a so-called anticlastic surface.

If the longitudinal radius of curvature R into which the strip is bent is large, the cross section is found to deform to an arc of a circle of radius R/ν, where ν is Poisson's ratio. However, as the radius R is progressively decreased by further bending, the cross-sectional deformation becomes increasingly confined to the edges of the strip [1]. The maximum deformation is then found to be practically independent of the radius R and approaches a value equal to about one-tenth of the strip thickness.

Although the above anticlastic deformation is not large, it can cause practical difficulties. For example, the edges of the magnetic tapes used in computer applications are found to wear because of it. Similar difficulties are encountered in the bending of the long metallic plates used to form the adjustable working sections of wind tunnels. The consequent anticlastic deformation of the plates is found to interfere with the air flow. However, it has been found that this deformation can be drastically reduced by tapering the edges on the concave sides of the tapes or plates. The amount of taper necessary to produce optimal reduction of the anticlastic deformation has been investigated theoretically by Pao and Conway [2] and experimentally by Nickola, Conway, and Farnham [3].

In most practical cases of a tape or strip wrapped around a cylindrical drum, the tape or strip is not only bent but also stretched. As a result, the analysis is much more complicated than when the strip is free to deform at will, since the surface of the drum (assumed to be rigid) is in contact with the strip and consequently inhibits the deformation. The problem of wrapping with tension has been the subject of a theoretical investigation by Meier, Lee, Raider, and Conway [4].

In the study of Meier et al. [4], the drum is assumed to have a width greater than that of the strip. Consequently the strip cannot make contact with the edges of the drum, and the drum takes no part in the mode of deformation. However, if drum width is less than strip width, it is clear that, in certain circumstances, the strip can make contact with the edges of the drum and consequently the mode of deformation can be greatly altered. We assume that the tape is symmetrical with respect to the drum.

It is the objective of the present study to investigate the patterns induced when drum width is less than strip width and thus to extend the work of Meier et al. [4]. Before we do so, for purposes of completeness, we summarize the method used and the results obtained in this earlier work. Details of the analysis, however, are not repeated here.

Drum width greater than strip width

Consider first a long strip bent by equal and opposite end couples, M, and imagine that couples νM of the same sign are simultaneously applied to the longitudinal edges. Consequently, there are no strains in the direction of the width, and the strip deforms with no cross-sectional deformation to a circular cylinder of radius R.

If we now apply equal but opposite couples, νM per unit length, to the circular cylinder in order to free these edges, radial displacement occurs as a result (Fig. 1). This radial displacement forms the anticlastic surface with which we are concerned and which was found in [4] to be governed by the differential equation

$$\frac{d^4w}{dy^4} + 4\lambda^4 w = \frac{P(y)}{D}. \tag{1}$$

Here w is the radial deflection at the coordinate y, which is measured in the direction of the width of the strip, P is the distributed radial force on the strip, and D and λ are, respectively, the plate rigidity and anticlastic deformation parameters defined in the nomenclature in Table 1. Equation (1) has the general solution

$$w(y) = C_1 \cos\lambda y \cosh\lambda y + C_2 \cos\lambda y \sinh\lambda y + C_3 \sin\lambda y \cosh\lambda y + C_4 \sin\lambda y \sinh\lambda y + w_p, \tag{2}$$

where C_1 through C_4 are constants to be determined by the appropriate boundary conditions and where w_p is a particular solution. If symmetry should exist with respect to the coordinate origin, it follows that $C_2 = C_3 = 0$. In the absence of external tension, C_1 and C_4 are then found from the edge conditions, which are

$$\left(\frac{d^2w}{dy^2}\right)_{y=\pm b} = \nu/R; \left(\frac{d^3w}{dy^3}\right)_{y=\pm b} = 0, \tag{3}$$

and the problem is solved.

By examining the resulting curvature $d^2w(0)/dy^2$ at the center of the strip, the latter is found to decrease with increases of $\bar{\lambda}$ and finally to become zero at a critical value of the parameter $\bar{\lambda}_c$, which is the first root of the transcendental equation

$$\tan\bar{\lambda}_c + \tanh\bar{\lambda}_c = 0. \tag{4}$$

The first root is found to be $\bar{\lambda}_c = 2.365$. For $\bar{\lambda} > \bar{\lambda}_c$, the central curvature changes sign as indicated in the typical deformation curves shown in Fig. 2. Thus, if the strip is bent without tension around a drum, contact takes place along a single central line if $\bar{\lambda} \le \bar{\lambda}_c$ and along two symmetrically displaced lines if $\bar{\lambda} > \bar{\lambda}_c$.

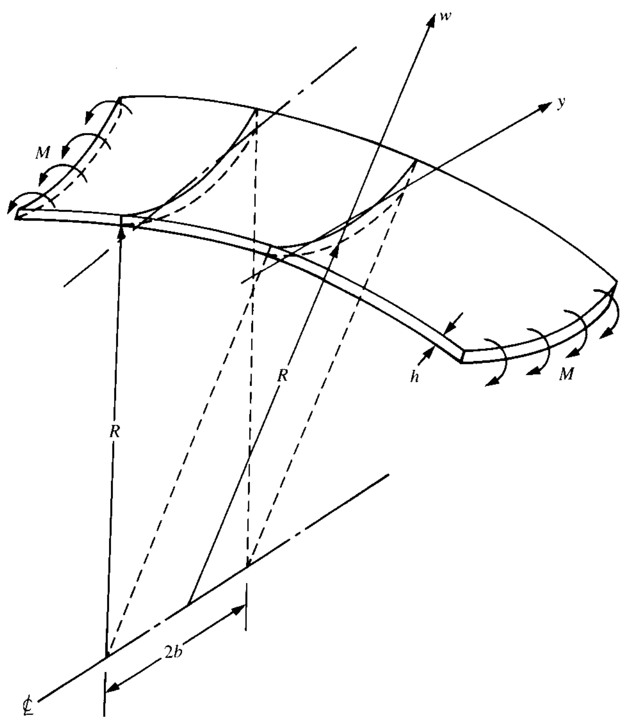

Figure 1 Coordinates for anticlastically deformed strip (without tension).

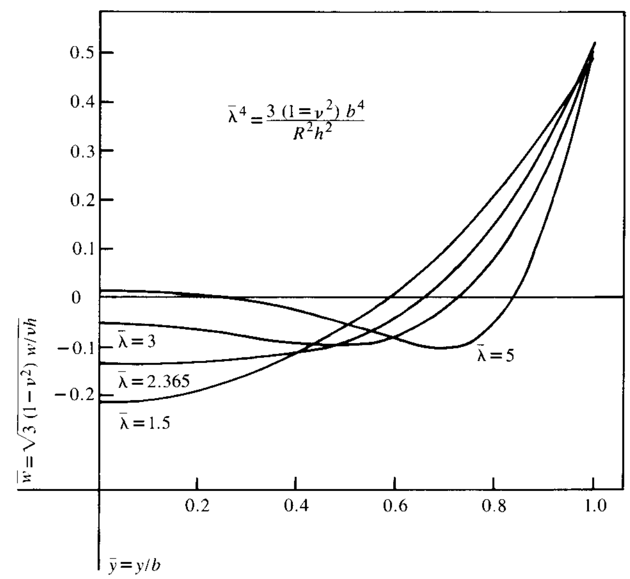

Figure 2 Anticlastic deformation curves without external tension. Reprinted from Ref. [4] with permission of the American Society of Mechanical Engineers.

We now consider the case when the radius R of the drum within the bent strip is assumed to be very slightly increased,

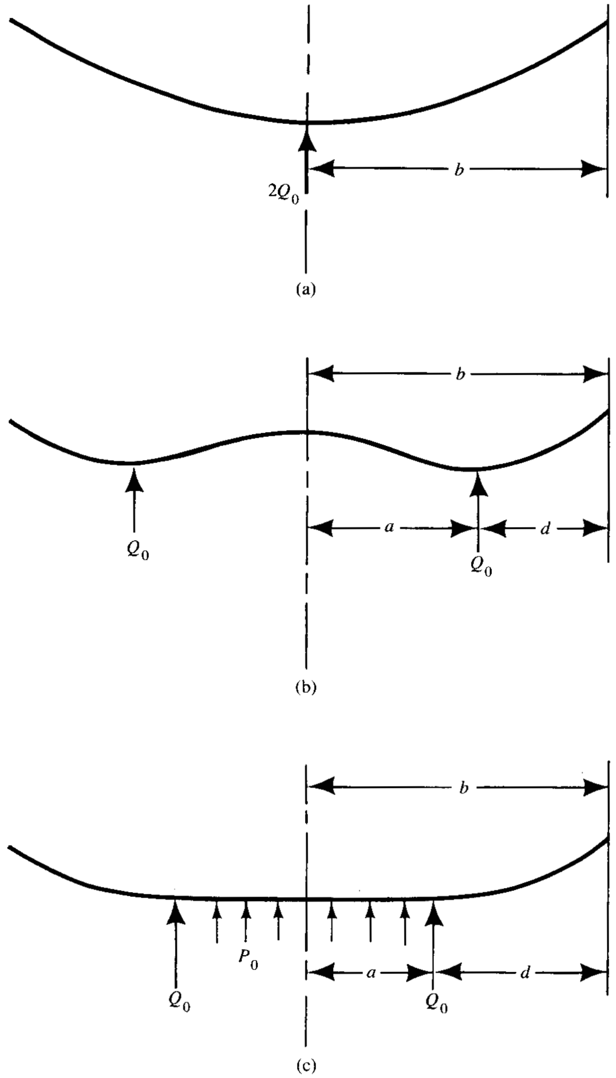

Figure 3 Forms of anticlastic deformation curves under external tension, $b/c < 1$. Reprinted from Ref. [4] with permission of the American Society of Mechanical Engineers.

thus subjecting the strip to tension as well as bending. This is equivalent to stretching the strip around a rigid drum of fixed radius. It is found [4] that, depending on the values of the parameter $\bar{\lambda} = [3(1 - \nu^2)b^4/R^2h^2]^{1/4}$ and a tension parameter $\bar{T} = TR\sqrt{3(1 - \nu^2)}/2bh^2\nu E$, the cross section of the strip can take one of the three forms shown in Fig. 3.

In Fig. 3(a), line contact takes place at the strip center, with a line force there of $2Q_0$ per unit length. In Fig. 3(b), two line contacts take place, each at distance a from the strip center. Finally, for Fig. 3(c), uniform contact pressure P_0 occurs over a central region at the ends of which are line forces Q_0 per unit length. Each of these cases can be analyzed by using a deflected form of Eq. (2) for each range; the

Table 1 Nomenclature.

a	= semidistance between line forces
b	= semiwidth of strip
c	= semiwidth of drum
D	= flexural rigidity = $Eh^3/12\,(1 - \nu^2)$
E	= modulus of elasticity of strip
h	= thickness of strip
M	= moment per unit length
P	= radial pressure in contact area
$\bar{P}$	= $Pb^4\sqrt{3(1 - \nu^2)}/\nu Dh$
Q	= shear force per unit length
$\bar{Q}$	= $Qb^3\sqrt{3(1 - \nu^2)}/\nu hD$
Q_0	= line force (per unit length) within drum width
Q_1	= line force (per unit length) along drum edge
R	= radius of neutral ($w = 0$) surface—approximately equal to drum radius
T	= tensile force in strip
$\bar{T}$	= $TR\sqrt{3(1 - \nu^2)}/2bh^2\nu E$
w	= radial deflections of strip (measured from neutral surface of anticlastic deformation without tension)
$\bar{w}$	= $w\sqrt{3(1 - \nu^2)}/\nu h$
$\bar{w}_0$	= $\bar{w}$ at drum surface
y	= axial distance from center of strip
$\bar{y}$	= y/b
α	= a/b
λ	= $[3(1 - \nu^2)/R^2h^2]^{1/4}$
$\bar{\lambda}$	= λb
ν	= Poisson's ratio

constants are obtained by the use of (a) symmetry or concentrated load conditions for the central range, (b) moment and shear edge conditions, Eq. (3), for the outer ranges, and (c) continuity conditions at the junctions of the two ranges. Again, details of the calculations are omitted but are given in [4].

The tracing of the changing deformation patterns is best visualized by the use of the plot of $\sqrt{2\bar{\lambda}\bar{T}}$ versus $\bar{\lambda}$ shown in Fig. 4. This plot is preferable to one showing $\bar{T}$ versus $\bar{\lambda}$ because it gives better scales for delineating the various patterns. As already pointed out, in the case of zero tension, initial drum contact (with zero force) takes place either on a single central line or on two lines, depending on whether $\bar{\lambda}$ is less than or greater than $\bar{\lambda}_c = 2.365$. This is indicated by the zero ordinate line of Fig. 4.

Consider now the case of initial ($\bar{T} = 0$) single line contact with a specific value of $\bar{\lambda}$. On increasing the tension parameter from zero, it is found that the central curvature of the strip decreases and becomes zero at $\bar{T} = \bar{T}_a$, where

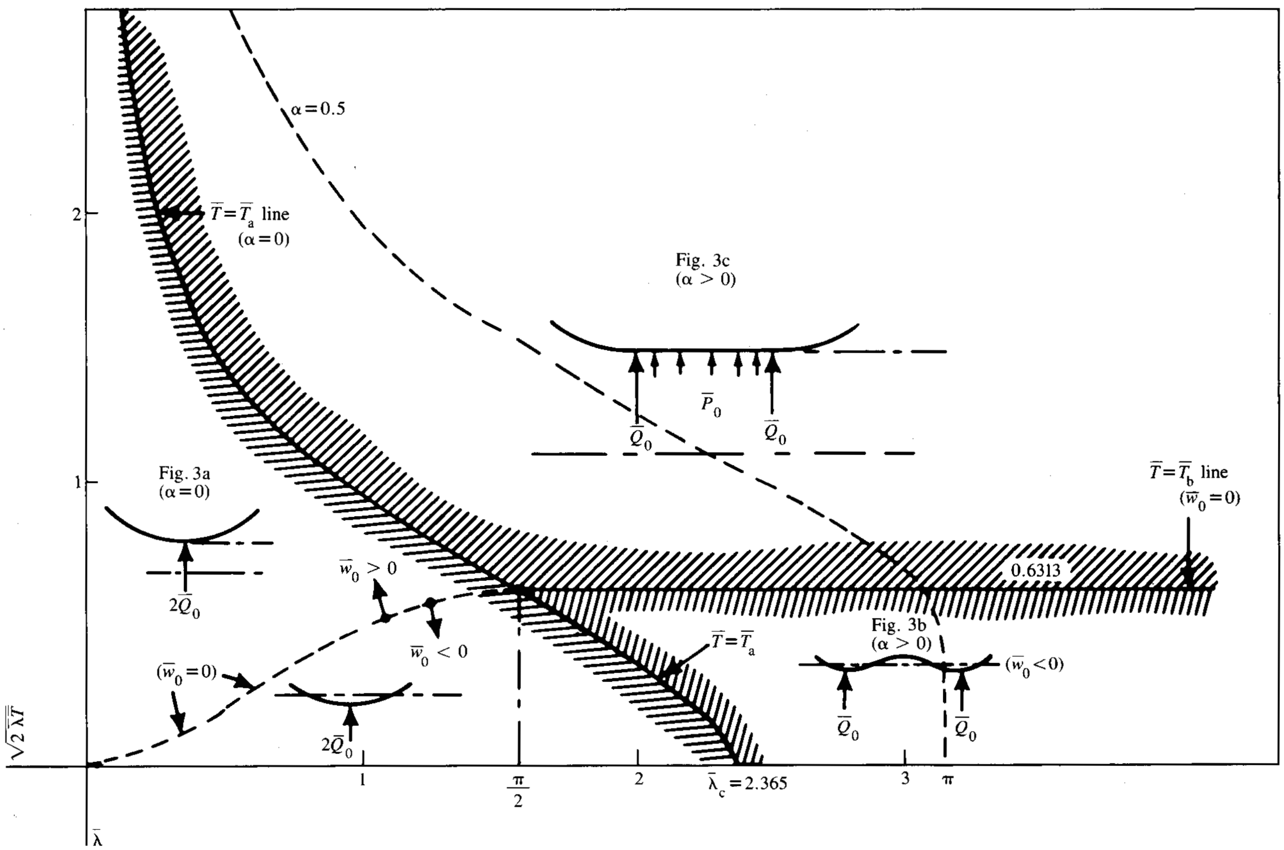

Figure 4 Graphs of $\sqrt{2\overline{\lambda}\overline{T}}$ versus $\overline{T}$, $b/c < 1$. Reprinted from Ref. [4] with permission of the American Society of Mechanical Engineers.

$$\overline{T}_a = \frac{\sin\overline{\lambda}\cosh\overline{\lambda} + \cos\overline{\lambda}\sinh\overline{\lambda}}{2\overline{\lambda}\,(\cosh^2\overline{\lambda} - \cos^2\overline{\lambda})}. \tag{5}$$

If $\overline{\lambda} < \pi/2$, it is found that increasing the tension beyond $\overline{T} = \overline{T}_a$ results in a change from single line contact [Fig. 3(a)] to uniform pressure in a central region [Fig. 3(c)]. Of course, the uniform pressure bandwidth increases with increases of $\overline{T}$.

For the case where $2.365 > \overline{\lambda} > \pi/2$ and there is initial single central line contact [Fig. 3(a)], increasing the tension parameter to $\overline{T} = \overline{T}_a$ results first in the double line contact of Fig. 3(b). Further increasing the value of $\overline{T}$ beyond $\overline{T}_a$ results in progressive reduction of the central curvature. When $\overline{T} > \overline{T}_b$, where

$$\overline{T}_b = \frac{1}{2\overline{\lambda}\cosh(\pi/2)}, \tag{6}$$

progressive central contact takes place with the configuration assuming the form shown in Fig. 3(c).

Finally, if $\overline{\lambda} > 2.365$ and we have initial double line contact [Fig. 3(b)], an increase in the value of $\overline{T}$ causes

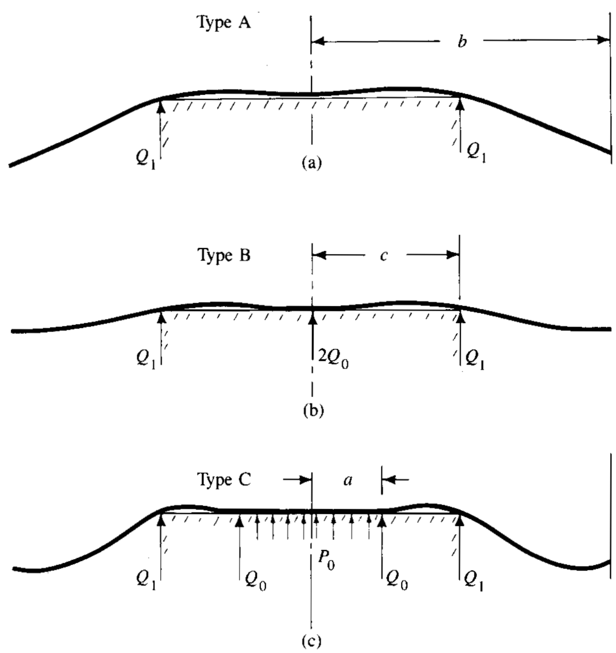

Figure 5 Forms of anticlastic deformation curves under external tension, $b/c > 1$.

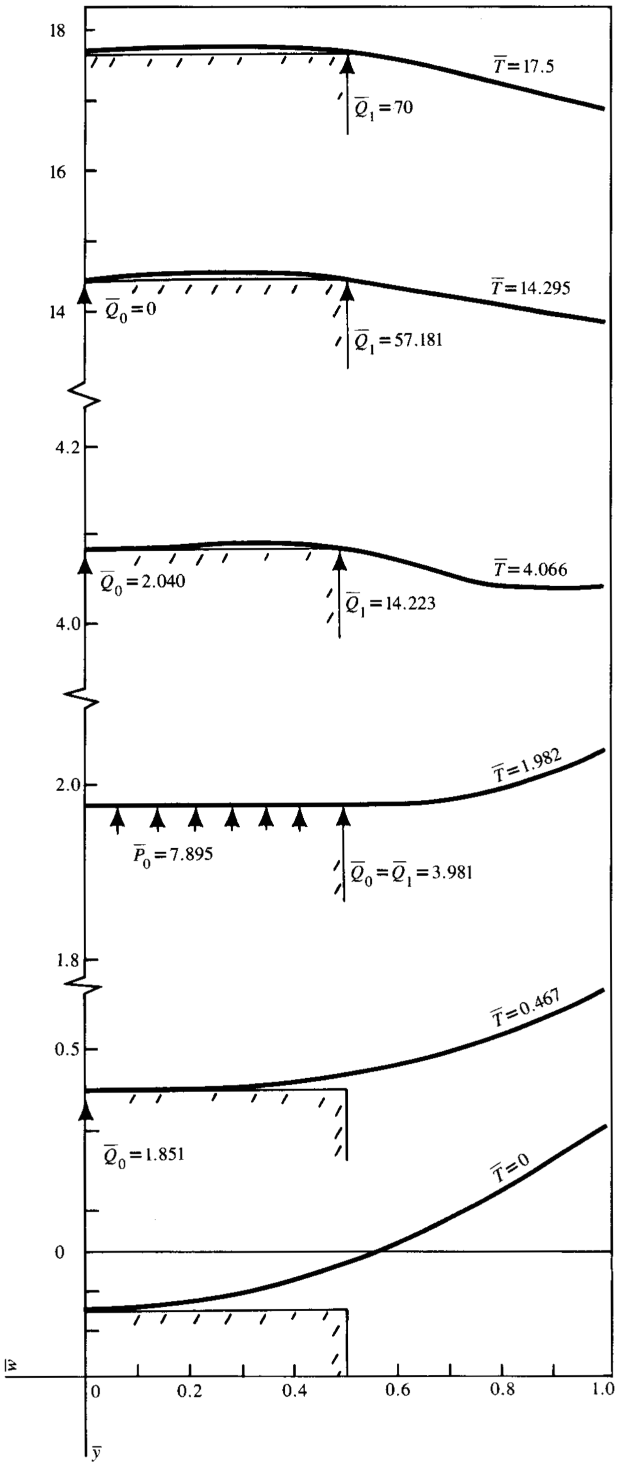

Figure 6 Deflection forms for various values of $\overline{T}$ when $b/c = 2$ and $\overline{\lambda} = 1$.

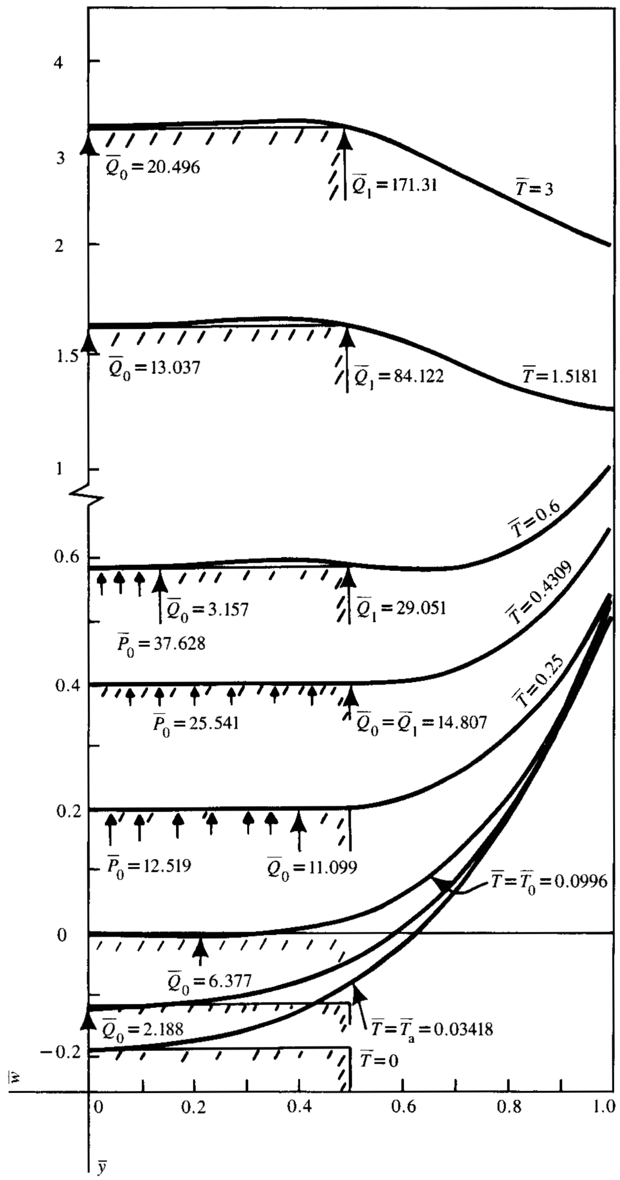

Figure 7 Deflection forms for various values of $\overline{T}$ when $b/c = 2$ and $\overline{\lambda} = 2$.

progressive reduction in the central curvature until finally, at $\overline{T} = \overline{T}_b$, the curvature is reduced to zero. Further increases of tension beyond $\overline{T} = \overline{T}_b$ result in a Fig. 3(c)-type configuration with a progressive increase of the uniform pressure bandwidth.

Drum width less than strip width

We now consider the case where the drum width is less than the strip width, so that the outer portions of the latter can never be in contact with the drum. The method is similar to that of the previous cases, that is, Eq. (2) is applied to each subregion using appropriate boundary and continuity conditions to determine the arbitrary constants. In addition to the patterns (Fig. 3) already discussed in the previous section, there are three other possible forms which the strip can take. These are (see Fig. 5):

A. Contact only along the drum edges, Fig. 5(a), referred to as pattern type A.
B. Contact along a central line and the drum edges, Fig. 5(b), type B.
C. Contact along the drum edges and a uniform pressure contact between two symmetrical line forces, Fig. 5(c), type C.

To illustrate the development of the deformation patterns (types A, B, and C), we consider, as an example, the cases when the drum width is half that of the strip (i.e., $b/c = 2$). There is no special significance in selecting this particular value except that the results obtained exhibit the general characteristics representative of the cases when the drum is narrower than the strip. Assuming first a value of the parameter $\overline{\lambda} = [3(1 - \nu^2)b^4/R^2h^2]^{1/4} = 1$, Fig. 6 shows the shape of the strip for various values of the tension parameter $\overline{T} = TR\sqrt{3(1 - \nu^2)}/2bh^2\nu E$. For $\overline{T} = 0$ (no tension), the strip is just in central line contact with the drum. For $\overline{T} = 0.4672$, contact still takes place only along the drum center with a line force there. As $\overline{T}$ is increased, the outer edges of the strip are pulled down so that, at $\overline{T} = 1.982$, contact takes place over the entire drum width with normalized line forces $\overline{Q}_0 = \overline{Q}_1$ and with normalized pressure $\overline{P}_0$ over the entire drum width. In these cases, the drum width does not enter the calculations. The drum width enters the calculations only after the tension exceeds $\overline{T} = 1.982$ when $\overline{\lambda} = 1$.

As $\overline{T}$ is further increased, line contact still remains at the drum edges (pattern type C), but the uniform pressure contact region is finally reduced to zero at $\overline{T} = 4.0658$, in which case there is still a central line force, $\overline{Q}_0$, as well as edge forces, $\overline{Q}_1$ (type B). Increasing the tension still further progressively decreases the relative value of $\overline{Q}_0$ and increases the relative value of $\overline{Q}_1$. Finally, at $\overline{T} = 14.295$, $\overline{Q}_0$ becomes zero and contact takes place only at the drum edges for $\overline{T} > 14.295$. The final graph for $\overline{T} = 17.5$ is an example of the latter (type A).

Figure 7 shows corresponding deflection patterns for $\overline{\lambda} = 2$, that is, a case of smaller radius, a wider strip, or a thinner strip. In this case (see Fig. 4), we have initial ($\overline{T} = 0$) single line contact. However, the single line loading splits into two line forces for $\overline{T} > 0.03418$, and then proceeds to make contact over a central portion of the drum width for $\overline{T} > 0.0996$. Contact takes place over the entire drum width at $\overline{T} = 0.4309$, when contact with the drum edges first takes place. Thus, the drum width enters the calculations at $\overline{T} = 0.4309$, and the patterns then develop as before, from type C into type B. However, in contrast to the foregoing case, an increase in $\overline{T}$ induces increases in both $\overline{Q}_0$ and $\overline{Q}_1$. Subsequently, pattern type A is never developed when $\overline{T}$ is further increased, and the final pattern is that of three contact lines, at the center and at the edges, an example of which is illustrated in the graph in Fig. 7 for $\overline{T} = 3$.

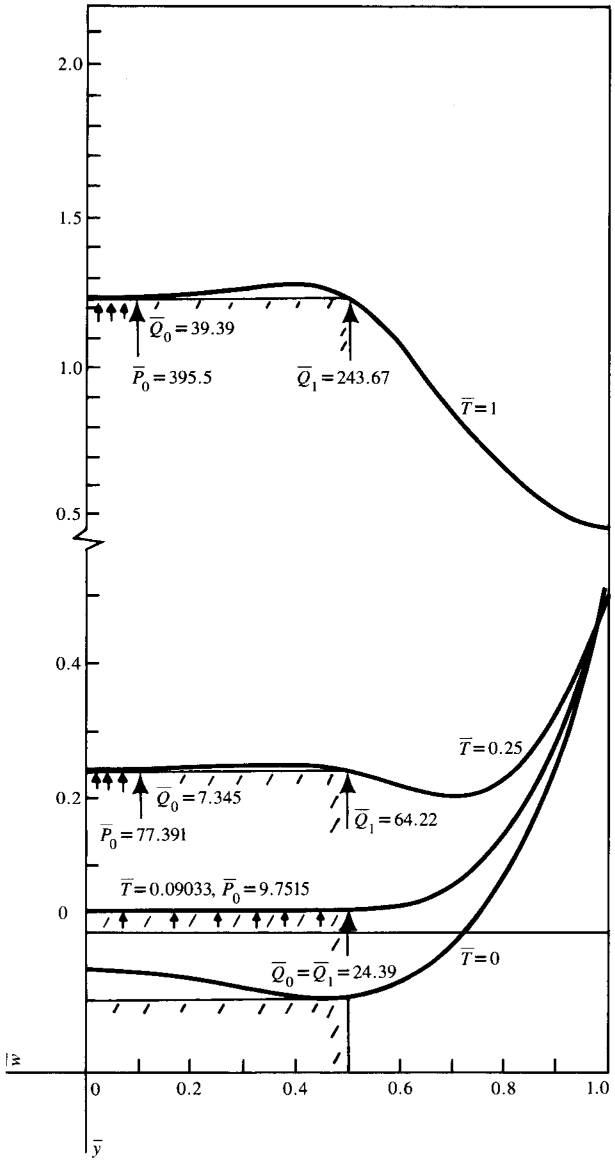

Figure 8 Deflection forms for various values of $\overline{T}$ when $b/c = 2$ and $\overline{\lambda} = 3$.

For $\overline{\lambda} > 2.365$ (see Fig. 3), initial contact ($\overline{T} = 0$) is along two lines. Typical plots for $\overline{\lambda} = 3$ and with various values of $\overline{T}$ are given in Fig. 8. After the full-width pressure contact is developed at $\overline{T} = 0.09033$, as $\overline{T}$ increases, the central pressure region reduces asymptotically to its minimum width ratio of 0.0993. Thus, pattern types A and B are never developed. For $\overline{\lambda} > \pi$, initial contact ($\overline{T} = 0$) occurs at

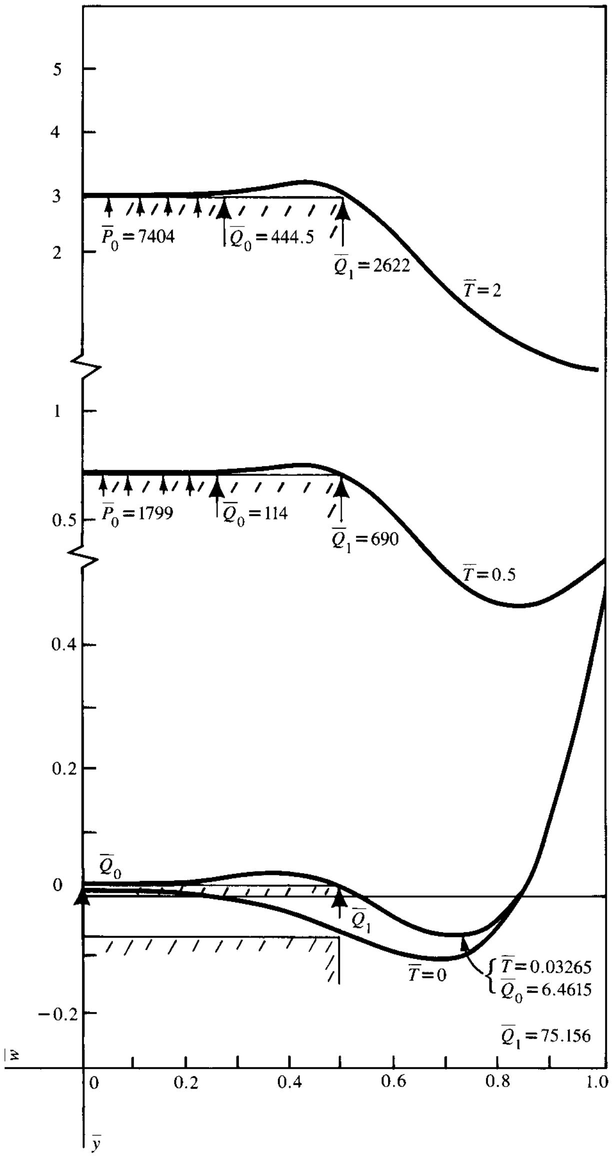

Figure 9 Deflection forms for various values of $\overline{T}$ when $b/c = 2$ and $\overline{\lambda} = 5$.

the edges of the drum. Typical plots for $\overline{\lambda} = 5$ are given in Fig. 9. The pattern starts as type A, which develops into type B as tension increases. At $\overline{T} = 0.03265$, pattern type C sets in with zero width ratio, which grows as $\overline{T}$ increases. The central pressure region grows asymptotically to its maximum width ratio of 0.2655. The graph for $\overline{T} = 2$ in Fig. 9 shows an example of the final patterns when $\overline{\lambda} = 5$.

The above results are, of course, confined to the case when $b/c = 2$. Deflection pattern development plots similar to Fig. 4 may be obtained, but each of these would be confined to the particular b/c value selected. The drum width does not enter the calculations for $b/c < 1$ since the strip cannot touch the drum edges. Finally, as the width of the drum is progressively reduced relative to that of the strip, the various deformation patterns tend to coalesce. Finally, for a very narrow drum, only the Fig. 3(a) pattern occurs, but with the outer edges being pulled down progressively as $\overline{T}$ is increased.

Summary and conclusions

The present paper was concerned with the deflected forms assumed by a strip when it is bent and stretched around a drum. The numerical results for the case when the drum width exceeds that of the strip ($b/c < 1$) were first summarized, and the method of analysis outlined. The case of drum width less than strip width ($b/c > 1$) was then considered.

It was shown that the particular form depends upon the strip-width/drum-width ratio, b/c, and upon the values of two parameters $\overline{\lambda}$ and $\overline{T}$ (see Table 1). Numerical results were then given in Figs. 6–9 for $\overline{\lambda} = 1$, 2, 3, and 5, respectively, for various values of $\overline{T}$ and for $b/c = 2$.

Depending upon the values of b/c, $\overline{\lambda}$, and $\overline{T}$ it was found that the shape can assume one of six possible forms. Three forms are similar to those developed when the drum width is greater than the strip width, as previously investigated [4]. These were shown in Fig. 3. Three additional forms shown in Fig. 5 were

A. Contact along the drum edges (two line forces).
B. Contact along a central line and the drum edges (three line forces).
C. Contact along the drum edges and central region between two symmetrical line forces (one uniformly distributed load plus four line forces).

The development of these forms was discussed, starting with initial contact ($\overline{T} = 0$) cases which, for $\overline{\lambda} < \pi$, are independent of the drum width when $b/c = 2$. The values of $\overline{T}$ for which the drum width enters the calculation were given for $b/c = 2$ and for $\overline{\lambda} = 1$, 2, 3, and 5. These values fall on the dotted line $\alpha = 0.5$ (or $b/a = 2$) in Fig. 4, as it is along this line that the outer contact lines of $\overline{Q}_0$ (see Fig. 3) coincide with the drum edges, that is, $a = c$ for $b/c = 2$. Thus, for parameters ($\overline{\lambda}$, $\overline{T}$; $b/c = 2$) in the region to the left of this curve, the deformation is independent of the drum width, and the deflected forms were investigated in detail in the previous study [4]. For parameter values belonging to the region in the right of the $\alpha = 0.5$ curve, the drum width enters into deformation computations, and the deformed strip takes one of the three new forms considered in the present study.

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