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Existence of Good δ-Decodable Codes for the Two-User Multiple-Access Adder Channel

This paper defines a class of δ-decodable codes for the two-user multiple-access adder channel with binary inputs. This class is a generalization of the class of two-user codes investigated by Kasami and Lin (1978). Lower bounds on the achievable rates of codes in this class are derived. We show that, for a wide range of error correcting capability, this class contains good two-user δ-decodable codes with rates lying above the timesharing line.

1. Introduction

Consider the multiple-access communication system shown in Fig. 1 in which two independent sources are attempting to transmit data to two users over a common channel. During a message interval, the two messages emanating from the two sources are encoded independently according to two binary block codes C_1 and C_2 of the same length n. The encoders and the decoder are assumed to maintain bit and word synchronization. The two code vectors emanating from the two encoders are combined by the channel into a single vector $\mathbf{r}$ with symbols from a certain alphabet. The single decoder at the receiver decodes $\mathbf{r}$ into two code words, one in C_1 and the other in C_2, for the two users.

Block coding for the two memoryless multiple-access channels shown in Fig. 2 has been investigated by Kasami and Lin [1, 2], Tilborg [3], and Weldon [4]. The channel shown in Fig. 2(a) is called a *noiseless adder channel*. At any time, the input to the channel is a binary 2-tuple (a_1, a_2) with a_i chosen from the set $\{0, 1\}$; the output b of the channel is the real sum of the two input bits a_1 and a_2, *i.e.*, $b = a_1 + a_2$ where + denotes real number addition. The two-dimensional capacity region of the noiseless two-user adder channel is shown in Fig. 3 [5–7]. The second channel model shown in Fig. 2(b) is also a two-user adder channel except noise is introduced. For this channel we say that a single transmission error has occurred if any of the following transitions occur: (1) from (0, 0) to 1; (2) from (1, 1) to 1; (3) from (0, 1) or (1, 0) to 0 or 2. We say that two transmission errors occur if the transition is either from (0, 0) to 2 or from (1, 1) to 0. For both models of the two-user adder channel, the two code words transmitted from the two encoders are combined into a single vector $\mathbf{r}$ with symbols from the alphabet $\{0, 1, 2\}$.

Let $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ be two n-tuples in $\{0, 1\}^n$. Then $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$ is an n-tuple in $\{0, 1, 2\}^n$. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ be two n-tuples in $\{0, 1, 2\}^n$. Define the L-distance between $\mathbf{x}$ and $\mathbf{y}$, denoted by $d_L(\mathbf{x}, \mathbf{y})$, as follows:

$$d_L(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|,$$

where $-$ denotes real number subtraction and $|x_i - y_i|$ denotes the absolute value of $x_i - y_i$.

Let C_1 and C_2 be two binary block codes of length n used for the noisy two-user adder channel. These two codes are referred to as a *two-user* code, denoted by (C_1, C_2). A two-user code (C_1, C_2) is said to be *δ-decodable* ($\delta > 0$), if and only if, for any two distinct pairs $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}', \mathbf{v}')$ in (C_1, C_2), $d_L(\mathbf{u} + \mathbf{v}, \mathbf{u}' + \mathbf{v}') \geq \delta$. Kasami and Lin [1] showed that a two-user δ-decodable code is capable of correcting $\lfloor(\delta - 1)/2\rfloor$ or fewer transmission errors in the noisy two-user adder channel where $\lfloor(\delta - 1)/2\rfloor$ denotes the greatest integer equal to or less than $(\delta - 1)/2$. Moreover, they proved that, if (C_1, C_2) is δ-decodable, then the minimum Hamming distances of both C_1 and C_2 must be greater than or equal to δ. Let R_1 and R_2 denote the rates of C_1 and C_2, respectively. For a given δ and a given code length n, it is desired to construct a two-user δ-decodable code (C_1, C_2) with maximum achievable rates (R_1, R_2).

Upper and lower bounds on the achievable rates of two-user δ-decodable codes have been derived by Kasami and Lin [2] and Tilborg [3]. Kasami and Lin [2] also introduced a class of two-user δ-decodable codes, and they showed that, for a certain range of δ/n, their class contains good two-user δ-decodable codes (C_1, C_2) with rates (R_1, R_2) lying above the time-sharing line, *i.e.*, $R_1 + R_2 > 1$.

In this paper, we extend Kasami and Lin's [2] results. We define a class of two-user δ-decodable codes which contains the two-user δ-decodable codes studied by Kasami and Lin as a subclass. Lower bounds on the achievable rates of two-user codes in this class are derived. We show that, for a wide range of δ/n, there exist good two-user δ-decodable codes in the class with rates (R_1, R_2) lying above the time-sharing line.

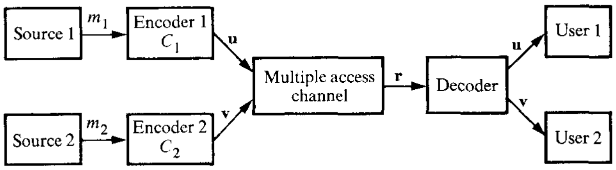

Figure 1 A multiple-access communication system with two users.

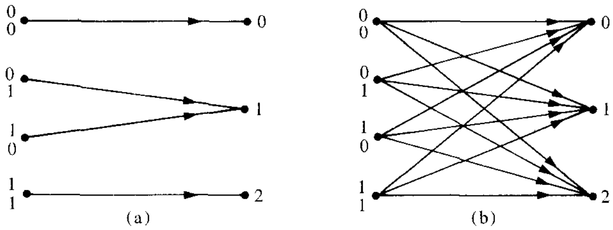

Figure 2 Two-user adder channel models. (a) Noiseless two-user adder channel; (b) noisy two-user adder channel.

2. A class of two-user δ-decodable codes

In this section, we define a class of two-user δ-decodable codes for the noisy adder channel. In the following section, we show that this class contains good two-user δ-decodable codes.

Let n_1 be a positive integer which is divisible by 3. Let C_{10} be an (n_1, k_1) linear code with minimum distance greater than t. For simplicity, we assume that C_{10} is a systematic code with the generator matrix in the following form:

$$\mathbf{G}_{10} = [\mathbf{I}_{k_1}\mathbf{P}_{10}], \tag{1}$$

where $\mathbf{I}_{k_1}$ is a $k_1 \times k_1$ identity matrix and $\mathbf{P}_{10}$ is a $k_1 \times (n_1 - k_1)$ matrix over $GF(2)$, where GF is a Galois field. Now, form a $(2n_1, k_1)$ linear code C_{11} by interleaving C_{10} as follows: a 0-digit in a code word of C_{10} is replaced by two 0-digits, and a 1-digit in a code word of C_{10} is replaced by two 1-digits. That is, C_{11} is obtained by interleaving C_{10} with a degree of 2. Clearly, the code C_{11} has minimum distance greater than or equal to $2t + 2$. The generator matrix of C_{11} is of the following form:

$$\mathbf{G}_{11} = \left[\begin{array}{l|c} 110000 \;\ldots\; 00 & \\ 001100 \;\ldots\; 00 & \\ 00001100 \;\ldots\; 00 & \\ \cdot & \mathbf{P}_{11} \\ \cdot & \\ \cdot & \\ 0000000 \;\ldots\; 11 & \end{array}\right], \tag{2}$$

where $\mathbf{P}_{11}$ is a $k_1 \times (2n_1 - 2k_1)$ matrix which is obtained from $\mathbf{P}_{10}$ by repeating each column of $\mathbf{P}_{10}$ twice, *i.e.*, the $(2i - 1)$th and the $2i$th columns of $\mathbf{P}_{11}$ are identical to the ith column of $\mathbf{P}_{10}$ for $1 \le i \le n_1 - k_1$.

Now, we consider a binary $(2n_1 + n_2, k_1 + k_2)$ linear code C_1 with the generator matrix in the following form:

$$\mathbf{G}_1 = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\mathbf{G}_{11}} & \multicolumn{2}{c}{\mathbf{O}_{k_1\times n_2}} \\ \hline \mathbf{O}_{k_2\times 2k_1} & \mathbf{P}_{21} & \mathbf{I}_{k_2} & \mathbf{P}_{22} \end{array}\right], \tag{3}$$

where $\mathbf{G}_{11}$ is the $k_1 \times 2n_1$ matrix given by (2); $\mathbf{O}_{k_2\times 2k_1}$ and $\mathbf{O}_{k_1\times n_2}$ are two zero matrices; $\mathbf{P}_{21}$ is a $k_2 \times (2n_1 - 2k_1)$ matrix over $GF(2)$ such that the $(2i - 1)$th and the $2i$th columns are identical for $1 \le i \le n_1 - k_1$; and $\mathbf{P}_{22}$ is a $k_2 \times (n_2 - k_2)$ matrix over $GF(2)$.

Let R_{10} and R_1 be the rates of C_{10} and C_1, respectively. Then, we have

$$R_1 = \frac{(1 + k_2/k_1)}{(2 + n_2/n_1)} R_{10}. \tag{4}$$

• *Lemma 1*

If the parameters n_1, n_2, k_1, k_2, and t satisfy the inequality

$$\sum_{0\le 2i_1+i_2\le 2t} \binom{n_1}{i_1}\binom{n_2}{i_2} \le 2^{n_1+n_2-k_1-k_2}, \tag{5}$$

then there exists a $(2n_1 + n_2, k_1 + k_2)$ linear code C_1 with minimum distance greater than $2t$ that has a generator matrix of the form given by (3).

Proof
Since $\mathbf{G}_{11}$ is a fixed matrix and since the $(2i-1)$th column and the $2i$th column of $\mathbf{P}_{21}$ are alike for $1 \le i \le n_1 - k_1$, the total number of matrices of the form $\mathbf{G}_1$ is

$$2^{k_2(n_1+n_2-k_1-k_2)}. \tag{6}$$

A code vector $\mathbf{u}$ generated by $\mathbf{G}_1$ has $2n_1 + n_2$ components. The first $2n_1$ components of $\mathbf{u}$ can be divided into n_1 blocks, with the ith block consisting of the $(2i-1)$th and the $2i$th components of $\mathbf{u}$ and each block being either (0, 0) or (1, 1). Let Γ denote those vectors of length $2n_1 + n_2$ over $GF(2)$ such that

1. The Hamming weight of each vector in Γ is $2t$ or less,
2. The first n_1 blocks of a vector in Γ are chosen from the set $\{(0, 0), (1, 1)\}$.

Clearly, Γ contains the all-zero vector. The number of vectors in Γ is

$$|\Gamma| = \sum_{0 \le 2i_1+i_2 \le 2t} \binom{n_1}{i_1}\binom{n_2}{i_2}. \tag{7}$$

The nonzero vectors in Γ can be classified into two types:

1. A type-I vector, in which the k_2 components from the $(2n_1 + 1)$th position to the $(2n_1 + k_2)$th position are not all zero.
2. A type-II vector, in which the k_2 components from the $(2n_1 + 1)$th position to the $(2n_1 + k_2)$th position are all zero.

Since $\mathbf{G}_{11}$ generates a code with minimum weight greater than $2t$, no nonzero vector in Γ can be a linear combination of the first k_1 rows of $\mathbf{G}_1$. Therefore, a type-I nonzero vector in Γ is either a linear combination of the last k_2 rows of some $\mathbf{G}_1$ or a linear combination of the first k_1 rows *and* the last k_2 rows of some $\mathbf{G}_1$. Since $\mathbf{G}_{11}$ is fixed, a type-I nonzero vector in Γ is in exactly

$$2^{(k_2-1)(n_1+n_2-k_1-k_2)}$$

codes generated by matrices of the form $\mathbf{G}_1$ given by (3) (use an argument similar to Peterson and Weldon ([8, Theorem 4.9, p. 92]). However, a type-II nonzero vector in Γ cannot be in any code generated by a matrix of the form $\mathbf{G}_1$. Therefore, the number of matrices of the form $\mathbf{G}_1$ that generate codes containing nonzero vectors from Γ is upper bounded by

$$2^{(k_2-1)(n_1+n_2-k_1-k_2)}(|\Gamma| - 1).$$

Hence, if

$$2^{(k_2-1)(n_1+n_2-k_1-k_2)}|\Gamma| \le 2^{k_2(n_1+n_2-k_1-k_2)}, \tag{8}$$

then there exists a $(2n_1 + n_2, k_1 + k_2)$ linear code C_1 with minimum distance greater than $2t$ that has a generator matrix $\mathbf{G}_1$ of the form given by (3). From (7) and (8), we obtain the inequality of (5). □

Now, choose C_1 as a $(2n_1 + n_2, k_1 + k_2)$ linear code with minimum distance greater than $2t$ and a generator matrix $\mathbf{G}_1$ of the form given by (3). For the next step, we want to define a code C_2 of length $2n_1 + n_2$ such that C_1 and C_2 form a two-user $(2t + 1)$-decodable code. Let C_{20} be the set of all those binary vectors of length $2n_1 + n_2$ such that the first $2n_1$ components consist of $n_1/3$ (0, 0) blocks and $2n_1/3$ blocks over $\{(0, 1), (1, 0)\}$, and the last n_2 components are arbitrary binary digits. The size of C_{20} is

$$|C_{20}| = \binom{n_1}{n_1/3} \cdot 2^{2n_1/3} \cdot 2^{n_2}. \tag{9}$$

Next, let C_{12} be a $(2n_1 + n_2, k_2)$ linear code whose generator matrix is

$$\mathbf{G}_{12} = [\mathbf{O}_{k_2 \times 2n_1}\mathbf{I}_{k_2}\mathbf{P}_{22}], \tag{10}$$

where $\mathbf{O}_{k_2 \times 2n_1}$ is a $k_2 \times 2n_1$ zero matrix, $\mathbf{I}_{k_2}$ is a $k_2 \times k_2$ identity matrix, and $\mathbf{P}_{22}$ is the $k_2 \times (n_2 - k_2)$ matrix given in (3). For any binary vector $\mathbf{v}$ of length $2n_1 + n_2$, let $w_1(\mathbf{v})$ and $w_2(\mathbf{v})$ denote the Hamming weights of the first $2n_1$ components and the last n_2 components of $\mathbf{v}$, respectively. Now, we define C_2 as follows: Let C_2 be a maximal subset of C_{20}, such that, for any two different vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in C_2,

$$w_1(\mathbf{v}_1 \oplus \mathbf{v}_2) + \min_{\mathbf{w} \in C_{12}} w_2(\mathbf{v}_1 \oplus \mathbf{v}_2 \oplus \mathbf{w}) > 2t, \tag{11}$$

where $\oplus$ denotes modulo-2 addition. It follows from (11) that the minimum Hamming distance of C_2 is greater than $2t$.

So far, we have defined two codes C_1 and C_2 with minimum distances greater than $2t$. Next, we want to show that (C_1, C_2) is a two-user $(2t + 1)$-decodable code. This is given in the following theorem.

• *Theorem 1*
Let C_1 be a $(2n_1 + n_2, k_1 + k_2)$ linear code with minimum distance greater than $2t$ that is generated by a matrix $\mathbf{G}_1$ of the form given by (3). Let C_2 be a code of length $2n_1 + n_2$ defined by (11). Then (C_1, C_2) is a two-user $(2t + 1)$-decodable code.

Proof
Let $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}', \mathbf{v}')$ be two distinct pairs in (C_1, C_2). Suppose that $\mathbf{v} = \mathbf{v}'$. Then $\mathbf{u} \ne \mathbf{u}'$ and $w(\mathbf{u} \oplus \mathbf{u}') > 2t$. It follows from the definition of L-distance that

$$d_L(\mathbf{u} + \mathbf{v}, \mathbf{u}' + \mathbf{v}') = w(u \oplus u') > 2t. \tag{12}$$

Suppose that $\mathbf{v} \ne \mathbf{v}'$. Define

$$d_L^{(1)}(\mathbf{u} + \mathbf{v}, \mathbf{u}' + \mathbf{v}') = \sum_{j=1}^{2n_1} |(u_j + v_j) - (u_j' + v_j')|,$$

$$d_L^{(2)}(\mathbf{u} + \mathbf{v}, \mathbf{u}' + \mathbf{v}') = \sum_{j=2n_1+1}^{2n_1+n_2} |(u_j + v_j) - (u_j' + v_j')|.$$

Let us consider the first $2n_1$ components of $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{v}$, and $\mathbf{v}'$. We pointed out earlier that these $2n_1$ components can be divided into n_1 blocks, and the ℓth block consists of the $(2\ell - 1)$th and the 2ℓth components with $1 \le \ell \le n_1$. Due to the structure of C_1, the ℓth block of a vector in C_1 is either (0, 0) or (1, 1). And due to the structure of C_2, the ℓth block of a vector in C_2 is one of the three combinations (0, 0), (0, 1), and (1, 0). Let $\mathbf{u}(\ell)$, $\mathbf{u}'(\ell)$, $\mathbf{v}(\ell)$, and $\mathbf{v}'(\ell)$ denote the ℓth blocks of $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{v}$, and $\mathbf{v}'$, respectively, for $1 \le \ell \le n_1$. If we compute the L-distance between $\mathbf{u}(\ell) + \mathbf{v}(\ell)$ and $\mathbf{u}'(\ell) + \mathbf{v}'(\ell)$ for all the possible combinations of $\mathbf{u}(\ell)$, $\mathbf{u}'(\ell)$, $\mathbf{v}(\ell)$, and $\mathbf{v}'(\ell)$, we can show that, for $1 \le \ell \le n_1$,

$$d_L[\mathbf{u}(\ell) + \mathbf{v}(\ell), \mathbf{u}'(\ell) + \mathbf{v}'(\ell)] \ge w[\mathbf{v}(\ell) \oplus \mathbf{v}'(\ell)],$$

where $w[\mathbf{v}(\ell) \oplus \mathbf{v}'(\ell)]$ is the Hamming weight of $\mathbf{v}(\ell) \oplus \mathbf{v}'(\ell)$. Therefore, we have

$$\begin{aligned} d_L^{(1)}(\mathbf{u} + \mathbf{v}, \mathbf{u}' + \mathbf{v}') &= \sum_{\ell=1}^{n_1} d_L[\mathbf{u}(\ell) + \mathbf{v}(\ell), \mathbf{u}'(\ell) + \mathbf{v}'(\ell)] \\ &\ge \sum_{\ell=1}^{n_1} w[\mathbf{v}(\ell) \oplus \mathbf{v}'(\ell)] \\ &= w_1(\mathbf{v} \oplus \mathbf{v}'). \end{aligned} \tag{13}$$

By the definition of L-distance, we have

$$\begin{aligned} d_L^{(2)}(\mathbf{u} + \mathbf{v}, \mathbf{u}' + \mathbf{v}') &\ge w_2(\mathbf{v} \oplus \mathbf{v}' \oplus \mathbf{u} \oplus \mathbf{u}') \\ &\ge \min_{\mathbf{w} \in C_{12}} w_2(\mathbf{v} \oplus \mathbf{v}' \oplus \mathbf{w}). \end{aligned} \tag{14}$$

(Note that the last n_2 components of $\mathbf{u} \oplus \mathbf{u}'$ are identical to the last n_2 components of a certain vector $\mathbf{w}$ in C_{12}.) It follows from (11), (13), and (14) that, for $\mathbf{v} \ne \mathbf{v}'$, we have

$$\begin{aligned} d_L(\mathbf{u} + \mathbf{v}, \mathbf{u}' + \mathbf{v}') &= d_L^{(1)}(\mathbf{u} + \mathbf{v}, \mathbf{u}' + \mathbf{v}') \\ &\quad + d_L^{(2)}(\mathbf{u} + \mathbf{v}, \mathbf{u}' + \mathbf{v}') \\ &\ge w_1(\mathbf{v} \oplus \mathbf{v}') + \min_{\mathbf{w} \in C_{12}} w_2(\mathbf{v} \oplus \mathbf{v}' \oplus \mathbf{w}) \\ &> 2t. \end{aligned} \tag{15}$$

From (12) and (15), we conclude that (C_1, C_2) is $(2t + 1)$-decodable. □

Next, we need to determine the size of C_2 defined by (11). For this purpose, we define the following set: For a vector $\mathbf{v}$ in C_{20}, let

$$\begin{aligned} C_{20}(\mathbf{v}) = \{\mathbf{x} \mid \mathbf{x} \in C_{20} &\text{ and } w_1(\mathbf{x} \oplus \mathbf{v}) \\ &+ \min_{\mathbf{w} \in C_{12}} w_2(\mathbf{x} \oplus \mathbf{v} \oplus \mathbf{w}) \le 2t\}. \end{aligned} \tag{16}$$

Define

$$|C_{20}(\mathbf{v})|_{\max} = \max_{\mathbf{v} \in C_{20}} |C_{20}(\mathbf{v})|.$$

Then, the number of vectors in C_2 is

$$|C_2| \ge \frac{|C_{20}|}{|C_{20}(\mathbf{v})|_{\max}}. \tag{17}$$

It can be shown that

$$\begin{aligned} &|C_{20}(\mathbf{v})|_{\max} \\ &\le \sum_{0 \le 2t_1 + t_2 \le 2t} \left\{ \sum_{s=0}^{t_1} \binom{2n_1/3}{s} \sum_{j_1=0}^{s} \binom{n_1/3}{j_1} \binom{s}{j_1} \right\} \cdot \left\{ \sum_{i=0}^{t_2} \binom{n_2}{i} 2^{k_2} \right\}. \end{aligned} \tag{18}$$

The derivation of (18) is given in Appendix A. From (9), (17), and (18), we obtain a lower bound on the number of vectors in C_2.

In this section, we have defined a class of two-user δ-decodable codes. In the next section, we show that this class contains efficient two-user δ-decodable codes. For $n_2 = 0$ and $k_2 = 0$, the two-user δ-decodable code presented here reduces to a two-user δ-decodable code introduced by Kasami and Lin [2].

The requirement that n_1 be divisible by 3 is not necessary. If n_1 is not divisible by 3, we use $\lceil n_1/3 \rceil$ (which denotes the least integer greater than or equal to $n_1/3$) and $\lfloor 2n_1/3 \rfloor$ to replace $n_1/3$ and $2n_1/3$ in (9) and (18). The results will be the same.

3. Lower bounds on the achievable rates

In Section 2, we defined a class of two-user δ-decodable codes. For each two-user code (C_1, C_2) in this class, C_1 is a linear code with a generator matrix of the form given by (3) and C_2 is defined by (11). In this section, we derive lower bounds on the achievable rate of C_2 for various ranges of δ. We show that, for arbitrarily large code length, there exist good two-user δ-decodable codes with rates lying above the time-sharing line.

It was proved by Gilbert [9] and Varsharmov [10] that for arbitrarily large n_1, there exists a binary (n_1, k_1) linear code with minimum distance greater than t for which the following inequality holds:

$$\frac{k_1}{n_1} \ge 1 - H(t/n_1),$$

where $H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$. This bound on code rate is referred to as the *Gilbert-Varsharmov bound.* In our construction of code C_1, we start with choosing an (n_1, k_1) linear code C_{10} with minimum distance greater than t; next we interleave C_{10} by degree 2 to obtain a $(2n_1, k_1)$ code C_{11}; and then we form C_1 with minimum distance greater than $2t$ and a generator matrix of the form given by (3). The existence of C_1 is guaranteed if its parameters satisfy the inequality given in Lemma 1.

Table 1 Upper bound on $R_i(\phi)$ [11].

$\phi = 2t/n$	$B(\phi)$
0.00	1.000
0.01	0.954
0.02	0.918
0.03	0.885
0.04	0.854
0.05	0.825
0.06	0.797
0.08	0.744
0.10	0.693
0.12	0.644

Now, we choose C_{10} as an (n_1, k_1) linear code with arbitrarily large n_1 and minimum distance greater than t which meets the Gilbert-Varsharmov bound, *i.e.*,

$$R_{10} = \frac{k_1}{n_1} \geq 1 - H(t/n_1). \tag{19}$$

With this choice, the rate of C_{11} is

$$R_{11} \geq \frac{1}{2}[1 - H(t/n_1)].$$

Based on the chosen code C_{10}, we form a $(2n_1 + n_2, k_1 + k_2)$ linear code C_1 with minimum distance greater than $2t$ and a generator matrix $\mathbf{G}_1$ of the form given by (3). Let

$n = 2n_1 + n_2$, $a = 2n_1/n$, $b = n_2/n$, $\phi = 2t/n$.

It follows from Lemma 1 that such a code C_1 exists if

$$R_1 = \frac{k_1 + k_2}{n} \leq 1 - \frac{a}{2}\left[1 + H\left(\frac{1}{z_1^2 + 1}\right) + \frac{2b}{a} H\left(\frac{1}{z_1 + 1}\right)\right] - o(1), \tag{20}$$

where (a) $z_1 \geq 1$ and is a root of

$$g(z) = z^3 + \left(1 - \frac{b}{\phi}\right) z^2 + \left(1 - \frac{a}{\phi}\right) z + 1 - \frac{1}{\phi},$$

and (b) $o(1)$ approaches zero as n becomes large.

The derivation of (20) is given in Appendix B. From (4) and (19), we have

$$R_1 \geq \frac{a}{2}(1 + k_2/k_1)[1 - H(\phi/a)], \tag{21}$$

where equality holds for $\phi = 0$.

Based on C_1 with rates satisfying (20) and (21), we define C_2 as given by (11). It follows from Theorem 1 that (C_1, C_2) is a two-user $(2t + 1)$-decodable code which is capable of correcting t or fewer transmission errors over the noisy adder channel. It follows from (17) that the rate R_2 of C_2 satisfies the following bound:

$$R_2 = \frac{1}{n}\log_2|C_2| \geq \frac{1}{n}\{\log_2|C_{20}| - \log_2|C_{20}(\mathbf{v})|_{\max}\}. \tag{22}$$

From (9), (18), and (22), we obtain the following lower bounds on R_2 for large n and various ranges of $\phi = 2t/n$ (derivations are given in Appendix C):

(a) For $0 \leq \phi \leq \dfrac{a}{3} + \dfrac{b}{1 + \sqrt{2}}$,

$$R_2 \geq \frac{a}{2}\left[\log_2 6 - \bar{H}\left(\frac{\phi}{a}\right) - \frac{2}{3}H\left(\frac{\rho}{2}\right) - \frac{1}{3}(1 + \rho)H\left(\frac{1}{1 + \rho}\right)\right] + b[1 - H(\sigma)] - R_1 + o(1); \tag{23}$$

(b) For $\dfrac{a}{3} + \dfrac{b}{1 + \sqrt{2}} < \phi \leq \dfrac{a}{3} + \dfrac{b}{2}$,

$$R_2 \geq \frac{a}{2}\left[\log_2 3 - \frac{1}{3} - \bar{H}\left(\frac{\phi}{a}\right)\right] + b\left[1 - H\left(\frac{3\phi - a}{3b}\right)\right] - R_1 + o(1); \tag{24}$$

(c) For $\phi > \dfrac{a}{3} + \dfrac{b}{2}$,

$$R_2 \geq \frac{a}{2}\left[\log_2 3 - \frac{1}{3} - \bar{H}\left(\frac{\phi}{a}\right)\right] - R_1, \tag{25}$$

where

1. $\bar{H}(x) = H(x)$ for $0 \leq x \leq \frac{1}{2}$ and

   $\bar{H}(x) = 1$ for $\frac{1}{2} < x \leq 1$;

2. $\rho = \dfrac{4}{\sqrt{8z_2^2 + 9} - 1}$ and $\sigma = \dfrac{1}{z_2 + 1}$ with $z_2 > 1$

   and as a root of

$$\frac{4a}{3(\sqrt{8z^2 + 9} - 1)} + \frac{b}{z + 1} = \phi. \tag{26}$$

For various values of ϕ and $0 \leq a, b \leq 1$ with $a + b = 1$, the above lower bounds on the achievable rates (R_1, R_2) of two-user δ-decodable codes defined in Section 2 are plotted in Fig. 3 [R_1 must satisfy the constraint of (20)]. Each line corresponds to a specific value of $\phi = 2t/n$. The high-end point of each line corresponds to $a = 1$ and $b = 0$, and the low-end point corresponds to $a = 0$ and $b = 1$.

For $0 \le \phi \le 0.035$ and for various values of a and b, the class of two-user δ-decodable codes defined in Section 2 contains codes (C_1, C_2) with rate pairs (R_1, R_2) lying above the time-sharing line, *i.e.*, $R_1 + R_2 > 1$.

It has been proved by Kasami and Lin [2] that, for (C_1, C_2) to be a two-user $(2t + 1)$-decodable code, the minimum Hamming distances of the component codes C_1 and C_2 must be greater than or equal to $2t + 1$. Therefore, for any given $\phi = 2t/n$, the rates $R_1(\phi)$ and $R_2(\phi)$ of C_1 and C_2 must satisfy the upper bound $B(\phi)$ derived by McEliece *et al.* [11], *i.e.*,

$$R_i(\phi) \le B(\phi)$$

for $i = 1$ or 2. For various ϕ, the values of $B(\phi)$ are given in Table 1. If the code words from the encoders are transmitted by using the time-sharing scheme, the total transmission rate of the system is no greater than $B(\phi)$. For various ϕ, time-sharing lines based on the McEliece *et al.* upper bound $B(\phi)$ are plotted in Fig. 4. We see that, for any $\phi < 0.095$, there exist two-user $(2t + 1)$-decodable codes (C_1, C_2) with rates $[R_1(\phi), R_2(\phi)]$ lying above the time-sharing line corresponding to the same ϕ, *i.e.*, $R_1(\phi) + R_2(\phi) > B(\phi)$. This indicates that the class of two-user $(2t + 1)$-decodable codes defined in Section 2 contains good two-user $(2t + 1)$-decodable codes.

Consider the special case where $a = 1$, $b = 0$, and $\phi \le 1/3$. For this case, the code C_1 is simply obtained by interleaving C_{10} with a degree of two, *i.e.*, $C_1 = C_{11}$. It follows from (20) and (21) that we have

$$[1 - H(\phi)]/2 \le R_1 \le \frac{1}{2}.$$

From (26), we have $\rho = 3\phi$. Thus, the bound on R_2 given by (23) reduces to

$$R_2 \ge \frac{1}{2}\left[\log_2 6 - H(\phi) - \frac{2}{3} H\left(\frac{3}{2}\phi\right) - \frac{1}{3}(1 + 3\phi) \cdot H\left(\frac{1}{1 + 3\phi}\right)\right] - R_1 + o(1). \quad (27)$$

From (27), we obtain

$$R_1 + R_2 \ge 1.2925 - \frac{1}{2} H(\phi) - \frac{1}{3} H\left(\frac{3\phi}{2}\right) - \frac{1}{6}(1 + 3\phi) H\left(\frac{1}{1 + 3\phi}\right) + o(1).$$

This special case was first investigated by Kasami and Lin [2].

Appendix A: Derivation of (18)

For convenience, we repeat the definition of $C_{20}(\mathbf{v})$ here: For a vector $\mathbf{v}$ in C_{20}, let

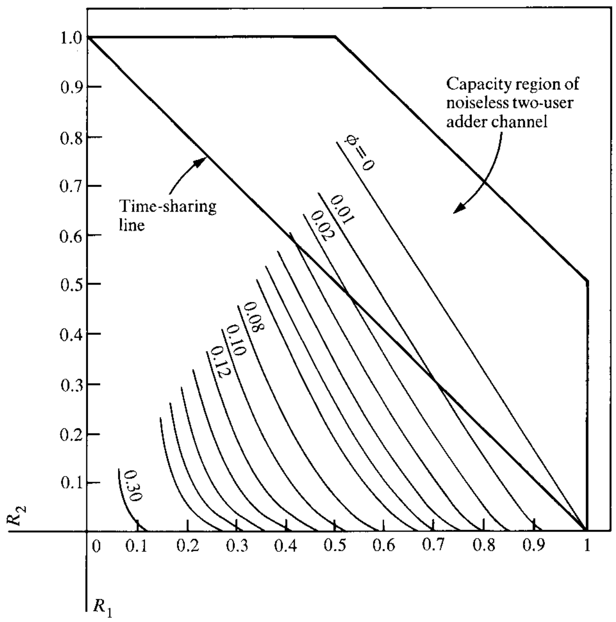

Figure 3 Lower bounds on the achievable rates of $(2t + 1)$-decodable code pairs for various values of $\phi = (2t + 1)/n$. The high-end point for each line corresponds to $a = 1$ and $b = 0$, and the low-end point of each line corresponds to $a = 0$ and $b = 1$.

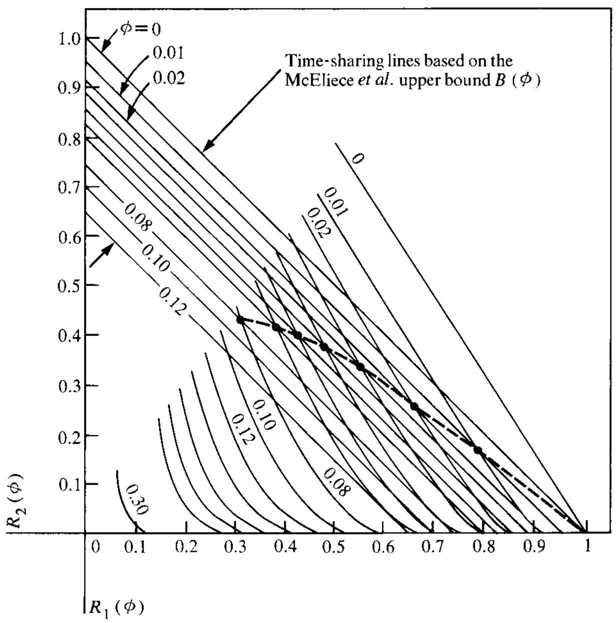

Figure 4 Comparison between the lower bounds on the achievable rates of $(2t + 1)$-decodable code pairs and the time-sharing lines obtained by using the upper bound of McEliece *et al.* [11] on the rates of the component codes for various ϕ. The dots are the intersections of the lower bounds on $[R_1(\phi), R_2(\phi)]$ and the time-sharing lines of the same ϕ.

$$C_{20}(\mathbf{v}) = \{\mathbf{x} \mid \mathbf{x} \in C_{20} \text{ and } w_1(\mathbf{x} \oplus \mathbf{v}) + \min_{\mathbf{w} \in C_{12}} w_2(\mathbf{x} \oplus \mathbf{v} \oplus \mathbf{w}) \le 2t\}. \tag{A1}$$

To determine the size of $C_{20}(\mathbf{v})$, we take two steps. Let $\mathbf{x}$ be a vector in C_{20}. We compare $\mathbf{x}$ and $\mathbf{v}$ in the first $2n_1$ components. For $1 \le i \le n_1$, define the following numbers:

1. Let j_0 denote the number of blocks i such that the ith block of $\mathbf{v}$ is $(0, 0)$ and the ith block of $\mathbf{x}$ is either $(0, 1)$ or $(1, 0)$.
2. Let j_1 denote the number of blocks i such that the ith block of $\mathbf{v}$ is either $(0, 1)$ or $(1, 0)$ and the ith block of $\mathbf{x}$ is $(0, 0)$.
3. Let j_2 denote the number of blocks i such that the ith block of $\mathbf{v}$ is $(0, 1)$ [or $(1, 0)$] and the ith block of $\mathbf{x}$ is $(1, 0)$ [or $(0, 1)$].

Since both $\mathbf{x}$ and $\mathbf{v}$ are in C_{20} and since they have the same weight in the first $2n_1$ components, $j_0 = j_1$ and the weight of the first $2n_1$ components of $\mathbf{x} \oplus \mathbf{v}$ is

$$w_1(\mathbf{x} \oplus \mathbf{v}) = j_0 + j_1 + 2j_2 = 2(j_1 + j_2). \tag{A2}$$

Clearly $w_1(\mathbf{x} \oplus \mathbf{v})$ is even.

Let $0 \le t_1 \le t$. Let $C^1_{20}(\mathbf{v}, t_1)$ be a subset of C_{20} such that (1) all the vectors in $C^1_{20}(\mathbf{v}, t_1)$ have the same last n_2 components; (2) for each vector $\mathbf{x}$ in $C^1_{20}(\mathbf{v}, t_1)$, $w_1(\mathbf{x} \oplus \mathbf{v}) \le 2t_1$. Then, we have

$$\begin{aligned}
|C^1_{20}(\mathbf{v}, t_1)| &= \sum_{0 \le j_0 + j_1 + 2j_2 \le 2t_1} \binom{n_1/3}{j_0} \binom{2n_1/3}{j_1} \binom{2n_1/3 - j_1}{j_2} \\
&= \sum_{0 \le j_1 + j_2 \le t_1} \binom{n_1/3}{j_1} \binom{2n_1/3}{j_1} \binom{2n_1/3 - j_1}{j_2} \\
&= \sum_{s=0}^{t_1} \sum_{j_1=0}^{s} \binom{n_1/3}{j_1} \binom{2n_1/3}{j_1} \binom{2n_1/3 - j_1}{s - j_1} \\
&= \sum_{s=0}^{t_1} \binom{2n_1/3}{s} \sum_{j_1=0}^{s} \binom{n_1/3}{j_1} \binom{s}{j_1}.
\end{aligned} \tag{A3}$$

Let $0 \le t_2 \le 2t$. Let $C^2_{20}(\mathbf{v}, t_2)$ be a subset of C_{20} such that (1) all the vectors in $C^2_{20}(\mathbf{v}, t_2)$ have the same first $2n_1$ components; and (2) for each vector $\mathbf{x}$ in $C^2_{20}(\mathbf{u}, t_2)$,

$$\min_{\mathbf{w} \in C_{12}} w_2(\mathbf{x} \oplus \mathbf{v} \oplus \mathbf{w}) \le t_2.$$

Then, we have

$$|C^2_{20}(\mathbf{v}, t_2)| \le 2^{k_2} \sum_{i=0}^{t_2} \binom{n_2}{i}. \tag{A4}$$

It follows from the definitions of $C_{20}(\mathbf{v})$, $C^1_{20}(\mathbf{v}, t_1)$, and $C^2_{20}(\mathbf{v}, t_2)$ that

$$|C_{20}(\mathbf{v})| \le \sum_{0 \le 2t_1 + t_2 \le 2t} |C^1_{20}(\mathbf{v}, t_1)| \cdot |C^2_{20}(\mathbf{v}, t_2)|. \tag{A5}$$

Combining (A3), (A4), and (A5), we obtain

$$|C_{20}(\mathbf{v})| \le \sum_{0 \le 2t_1 + t_2 \le 2t} \left\{ \sum_{s=0}^{t_1} \binom{2n_1/3}{s} \sum_{j_1=0}^{s} \binom{n_1/3}{j_1} \binom{s}{j_1} \right\} \times \left\{ \sum_{i=0}^{t_2} \binom{n_2}{i} 2^{k_2} \right\}. \tag{A6}$$

Since the bound on $|C_{20}(\mathbf{v})|$ given by (A6) holds for any $\mathbf{v}$ in C_{20}, we therefore obtain (18).

Appendix B: Derivation of (20)

Let $n = 2n_1 + n_2$, $a = 2n_1/n$, and $b = n_2/n$. Taking the logarithm of both sides of (5), dividing it by n, and rearranging it, we obtain

$$R_1 \le 1 - \frac{a}{2} - \frac{1}{n} \log_2 \left\{ \sum_{0 \le 2i_1 + i_2 \le 2t} \binom{an/2}{i_1} \binom{bn}{i_2} \right\}. \tag{B1}$$

The binomial coefficient

$$\binom{m}{\ell}$$

can be bounded as follows [12]:

$$\sqrt{\frac{m}{8\ell(m - \ell)}}\, 2^{mH(\ell/m)} \le \binom{m}{\ell} \le \sqrt{\frac{m}{2\pi\ell(m - \ell)}}\, 2^{mH(\ell/m)}. \tag{B2}$$

Let $x = 2i_1/n$, $y = i_2/n$, and $\phi = 2t/n$. It is known in coding theory that, for an (n, k) code with minimum distance greater than $2t$, we have $\phi = 2t/n \le 1/2$ except for the trivial case with two code words. It follows from (B1) and (B2) that

$$R_1 \le 1 - \frac{a}{2} - \max_{0 \le x + y \le \phi} \left\{ \frac{a}{2} H\left(\frac{x}{a}\right) + bH\left(\frac{y}{b}\right) \right\} - o(1), \tag{B3}$$

where $o(1)$ approaches zero as n becomes large.

Let

$$f_1(x, y) = \frac{a}{2} H\left(\frac{x}{a}\right) + bH\left(\frac{y}{b}\right), \qquad x + y = \phi', \tag{B4}$$

with $0 < \phi' \le \phi$. To find $\max_{0 \le x + y \le \phi} f_1(x, y)$, we use Lagrange's method of indeterminate multipliers. Consider

$$g_1(x, y) = f_1(x, y) - \lambda_1(x + y). \tag{B5}$$

Taking the first and second derivatives of $g_1(x, y)$, we have

$$\begin{aligned}
&\frac{\partial g_1}{\partial x} = \frac{1}{2} \log_2\left(\frac{a - x}{x}\right) - \lambda_1, \qquad \frac{\partial g_1}{\partial y} = \log_2\left(\frac{b - y}{y}\right) - \lambda_1, \\
&\frac{\partial^2 g_1}{\partial x^2} = -\frac{a/\log_e 2}{2x(a - x)}, \qquad \frac{\partial^2 g_1}{\partial y^2} = -\frac{b/\log_e 2}{y(b - y)}, \\
&\frac{\partial^2 g_1}{\partial x \partial y} = \frac{\partial^2 g_1}{\partial y \partial x} = 0.
\end{aligned} \tag{B6}$$

Note that the second derivatives of $g_1(x, y)$ are nonpositive for $0 < x < a$ and $0 < y < b$. Setting $\partial g_1/\partial x = 0$ and $\partial g_1/\partial y = 0$, we have

$$x = \frac{a}{2^{2\lambda_1} + 1}, \qquad y = \frac{b}{2^{\lambda_1} + 1} \tag{B7}$$

with $x + y = \phi'$. Since $a + b = 1$ and since $\phi' \le \phi \le 1/2$, we have

$$\min\left\{\frac{1}{2^{2\lambda_1} + 1}, \frac{1}{2^{\lambda_1} + 1}\right\} \le \phi' \le \frac{1}{2}. \tag{B8}$$

From (B8), we conclude that $\lambda_1 \ge 0$. It follows from (B7) that $x/a \le 1/2$ and $y/b \le 1/2$. This implies that $f_1(x, y)$ takes its maximum value at

$$x + y = \phi. \tag{B9}$$

Let $z = 2^{\lambda_1}$. From (B7) and (B9), we have

$$g(z) = z^3 + \left(1 - \frac{b}{\phi}\right) z^2 + \left(1 - \frac{a}{\phi}\right) z + 1 - \frac{1}{\phi} = 0. \tag{B10}$$

Since $0 \le \phi \le 1/2$, $g(1) \le 0$. Also, we see that

$$\lim_{z \to +\infty} g(z) = +\infty.$$

Therefore, $g(z)$ has at least one real root in the range $z \ge 1$. Since $g(z)$ has at most one extremal point for $z \ge 1$, then $g(z)$ has exactly one real root in the range of $z \ge 1$. Let z_1 be the real root of $g(z)$ in the range $z \ge 1$. Then, we obtain

$$\max_{0 \le x+y \le \phi} f_1(x, y) = f_1\left(\frac{a}{z_1^2 + 1}, \frac{b}{z_1 + 1}\right). \tag{B11}$$

Combining (B3), (B4), and (B11), we obtain (20).

Appendix C: Derivation of lower bounds on R_2

The rate of C_2 is given by (22). For convenience, we repeat (22) here:

$$R_2 = \frac{1}{n} \log_2 |C_2| \ge \frac{1}{n} \{\log_2 |C_{20}| - \log_2 |C_{20}(\mathbf{v})|_{\max}\}. \tag{C1}$$

To bound R_2, we need to determine $\log_2 |C_{20}|$ and $\log_2 |C_{20}(\mathbf{v})|$. It follows from (9) that

$$\frac{1}{n} \log_2 |C_{20}| = \frac{a}{3} + b + \frac{1}{n} \log \binom{n_1}{n_1/3}, \tag{C2}$$

where $a = 2n_1/n$ and $b = n_2/n$. Bounding the binomial coefficient

$$\binom{n_1}{n_1/3}$$

as shown in (B2), (C2) becomes

$$\frac{1}{n} \log_2 |C_{20}| = \frac{a}{2} \log_2 3 + b + o(1), \tag{C3}$$

where $o(1)$ approaches zero as n becomes large.

It follows from (18) that

$$|C_{20}(\mathbf{v})|_{\max} \le \sum_{0 \le 2t_1 + t_2 \le 2t} \left\{\sum_{s=0}^{t_1} \binom{an/3}{s} \sum_{j_1=0}^{s} \binom{an/6}{j_1} \binom{s}{j_1}\right\} \cdot \left\{\sum_{i=0}^{t_2} \binom{bn}{i} 2^{k_2}\right\}. \tag{C4}$$

Let

$$S_1 = \frac{1}{n} \log \left\{\sum_{j_1=0}^{s} \binom{an/6}{j_1} \binom{s}{j_1}\right\}. \tag{C5}$$

Upper bounding the binomial coefficients

$$\binom{an/6}{j_1} \text{ and } \binom{s}{j_1},$$

we have, for large n,

$$S_1 \le \max_{0 \le j_1 \le s} \left\{\frac{a}{6} H\left(\frac{6j_1}{an}\right) + \frac{s}{n} H\left(\frac{j_1}{s}\right)\right\} + o(1). \tag{C6}$$

Set $z = 2j_1/n$ and $x = 2s/n$. Since $s \le t_1$ and $t_1 \le t$, we have $x \le \phi = 2t/n$. Based on the structure of C_{20} and $C_{20}(\mathbf{v})$, we have

$$2s \le 2t_1 \le \frac{2n_1}{3}.$$

This implies that $x \le a/3$. Now, we can put (C6) into the following form:

$$S_1 \le \max_{0 \le z \le x} \left\{\frac{a}{6} H\left(\frac{3z}{a}\right) + \frac{x}{2} H\left(\frac{z}{x}\right)\right\} + o(1). \tag{C7}$$

The function $(a/6)H(3z/a) + (x/2)H(z/x)$ is convex over $0 \le z \le x$, and it takes its maximum value at

$$z = \frac{ax}{3x + a}. \tag{C8}$$

Combining (C7) and (C8) and using the fact $H(\rho) = H(1 - \rho)$, we obtain the following bound on S_1:

$$S_1 \le \frac{a}{6}\left(1 + \frac{3x}{a}\right) H\left(\frac{a}{3x + a}\right) + o(1). \tag{C9}$$

Let $y = t_2/n$. Since $t_2 \le n_2$, we have $y \le b$. Using (C5), (C9), and upper bounding the binomial coefficients

$$\binom{an/3}{s} \text{ and } \binom{bn}{i}$$

as shown in (B2), we can manipulate (C4) into the following form:

$$\frac{1}{n} \log_2 |C_{20}(\mathbf{v})|_{\max} \le \max_{\substack{0 \le x+y \le \phi \\ 0 \le x \le a/3 \\ 0 \le y \le b}} \left\{\frac{a}{3} H\left(\frac{3x}{2a}\right) + \frac{a}{6}\left(1 + \frac{3x}{a}\right) H\left(\frac{a}{3x + a}\right) + b\bar{H}\left(\frac{y}{b}\right)\right\} + \frac{k_2}{n} + o(1), \tag{C10}$$

where

$$\bar{H}(Y) = \begin{cases} H(Y) & \text{for } 0 \le Y \le 1/2, \\ 1 & \text{for } 1/2 < Y \le 1. \end{cases} \tag{C11}$$

Define

$$h(X) = H\left(\frac{X}{2}\right) + \frac{1}{2}(1 + X)\, H\left(\frac{1}{1 + X}\right). \tag{C12}$$

The first and second derivatives of $h(X)$ are

$$h'(X) = \frac{1}{2} \log_2 \frac{(2 - X)(1 + X)}{X^2}, \tag{C13}$$

$$h''(X) = -\left(\frac{X + 4}{2X(2 - X)(1 + X)}\right) \cdot \left(\frac{1}{\log_e 2}\right). \tag{C14}$$

From (C13) and (C14), we can see that $h(X)$ increases monotonically as X increases from 0 to $(1 + \sqrt{17})/4$. Let

$$f_2(x, y) = \frac{a}{3}\, h\left(\frac{3x}{a}\right) + b\bar{H}\left(\frac{y}{b}\right). \tag{C15}$$

Since $x \le a/3$, we have $3x/a \le 1 < (1 + \sqrt{17})/4$. Therefore $h(3x/a)$ increases monotonically as x goes from 0 to $a/3$. Also, we note that $\bar{H}(y/b)$ increases monotonically for $0 \le y \le b/2$, and it is equal to 1 for $b/2 < y \le 1$. Let

$$F_2(a, b, \phi) = \max_{\substack{0 \le x+y \le \phi \\ 0 \le x \le a/3 \\ 0 \le y \le b}} f_2(x, y). \tag{C16}$$

Combining (C10), (C12), (C15), and (C16), we obtain

$$\frac{1}{n} \log_2 |C_{20}(\mathbf{v})| \le F_2(a, b, \phi) + k_2/n + o(1). \tag{C17}$$

In the following, we determine $F_2(a, b, \phi)$ for various ranges of ϕ. For $(a/3) + (b/2) < \phi$, we have

$$F_2(a, b, \phi) = \frac{2a}{3} + b. \tag{C18}$$

For $\phi \le (a/3) + (b/2)$, we have

$$F_2(a, b, \phi) = \max_{\substack{x+y=\phi \\ 0 \le x \le a/3 \\ 0 \le y \le b/2}} f_2(x, y). \tag{C19}$$

For this case, we use Lagrange's method of indeterminate multipliers to determine the maximum value of $f_2(x, y)$. Consider

$$g_2(x, y) = f_2(x, y) - \lambda_2(x + y) \tag{C20}$$

with constraint $x + y = \phi$. Setting $\partial g_2/\partial x$ and $\partial g_2/\partial y$ to zero, we obtain

$$\log_2 \frac{(2a - 3x)(a + 3x)}{9x^2} = 2\lambda_2, \qquad \log_2 \frac{b - y}{y} = \lambda_2,$$

with $x + y = \phi$. Since $0 \le x \le a/3$ and $0 \le y \le b/2$, then $\lambda_2 \ge 0$. We can also show that the second derivatives of $g_2(x, y)$ are nonpositive for $0 \le x \le a/3$ and $0 \le y \le b/2$. Let $z = 2^{\lambda_2}$. From (C20) and the constraint $x + y = \phi$, we find that $f_2(x, y)$ takes its maximum value at

$$x = \frac{4a}{3(\sqrt{8z_2^2 + 9} - 1)}, \qquad y = \frac{b}{z_2 + 1}, \tag{C21}$$

where z_2 is a root of

$$\frac{4a}{3(\sqrt{8z^2 + 9} - 1)} + \frac{b}{z + 1} = \phi. \tag{C22}$$

Since the lefthand side of (C22) decreases monotonically for $z \ge 0$ and is equal to

$$\frac{a(1 + \sqrt{17})}{12} + \frac{b}{2} > \phi \tag{C23}$$

at $z = 1$, there exists exactly one root z_2 of (C22) such that $z_2 > 1$. If $z_2 \ge \sqrt{2}$, then $x \le a/3$, $y < b/2$, and

$$\frac{a}{3} + \frac{b}{1 + \sqrt{2}} \ge \phi. \tag{C24}$$

It follows from (C19), (C21), and (C24) that, for $\phi \le (a/3) + b/(1 + \sqrt{2})$, we have

$$F_2(a, b, \phi) = f_2\left(\frac{4a}{3(\sqrt{8z_2^2 + 9} - 1)}, \frac{b}{z_2 + 1}\right). \tag{C25}$$

However, for $(a/3) + b/(1 + \sqrt{2}) < \phi \le (a/3) + (b/2)$, we find that

$$F_2(a, b, \phi) = f_2\left(\frac{a}{3}, \phi - \frac{a}{3}\right) = \frac{2a}{3} + bH\left(\frac{3\phi - a}{3b}\right). \tag{C26}$$

It follows from (C1), (C3), and (C17) that we obtain the following lower bound on R_2:

$$R_2 \ge \frac{a}{2} \log_2 3 + b - F_2(a, b, \phi) - \frac{k_2}{n} + o(1), \tag{C27}$$

where $F_2(a, b, \phi)$ is given by (C18), (C25), and (C26) for different ranges of ϕ. Since $R_1 = (k_1 + k_2)/n$, (C27) becomes

$$R_2 \ge \frac{a}{2} \log_2 3 + b - F_2(a, b, \phi) - R_1 + \frac{k_1}{n} + o(1). \tag{C28}$$

The term k_1/n can be expressed in the following form:

$$k_1/n = (k_1/n_1) \cdot (n_1/n) = (a/2)R_{10}. \tag{C29}$$

From (19) and (C29), we obtain

$$\frac{k_1}{n} \ge \frac{a}{2}\left[1 - \bar{H}\left(\frac{\phi}{a}\right)\right], \tag{C30}$$

where $\bar{H}(\phi/a) = H(\phi/a)$ for $0 \le \phi/a \le 1/2$ and $\bar{H}(\phi/a) = 1$ for $1/2 < \phi/a \le 1$. Combining (C28) and (30), we obtain

$$R_2 \ge \frac{a}{2}\left[\log_2 3 + 1 - \bar{H}\left(\frac{\phi}{a}\right)\right] + b - F_2(a, b, \phi) - R_1 + o(1). \quad \text{(C31)}$$

It follows from (C31), (C18), (C25), and (C26) that we obtain the lower bounds on R_2 given by (23), (24), and (25).

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