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# An Elementary Proof of Nonexistence of Isometries between $\ell_p^k$ and $\ell_q^k$

*For $k = 2$, the two-dimensional coordinate spaces $\ell_1^2$ and $\ell_\infty^2$ are isometric. Consequently, results on computational complexity for one space can be transplanted to the other in a natural way. In this note, an elementary proof is given for the nonisometry between $\ell_p^k$ and $\ell_q^k$ for general k, p, and q.*

## Introduction

Let $Q_i$, $Q_j$ be two points in $R^k$ with coordinates $(x_{i1}, x_{i2}, \cdots, x_{ik})$ and $(x_{j1}, x_{j2}, \cdots, x_{jk})$, respectively. The distance $d_p(Q_i, Q_j)$ between these two points is defined as $d_p(Q_i, Q_j) = (|x_{i1} - x_{j1}|^p + |x_{i2} - x_{j2}|^p + \cdots + |x_{ik} - x_{jk}|^p)^{1/p}$ for $p = 1, 2, \cdots$ and $d_\infty(Q_i, Q_j) = \max(|x_{i1} - x_{j1}|, \cdots, |x_{ik} - x_{jk}|)$. Following the notation used in [1, p. 374], we denote the space with such distance function by $\ell_p^k$.

Recently, some attention has been given to the spaces $\ell_1^2$ and $\ell_\infty^2$ because of their natural applications to computer science. (See, for example [2-4]). The isometry $f((x, y)) = ((y + x)/2, (y - x)/2)$, $(x, y) \in \ell_\infty^2$, between $\ell_\infty^2$ and $\ell_1^2$ makes it sufficient to study only one of these two spaces as far as computational complexity is concerned, since the existence of a polynomial time algorithm for a problem in one space implies the existence of a polynomial time algorithm for the same problem in the other space. More recently, some other applications have been reported that require the study of $\ell_1^k$ and $\ell_\infty^k$ (See, for example [5].) The question of their being isometric came up naturally. After a search of the literature, it seems that people always assume they are nonisometric. However, this fact usually has to be deduced by the reader from some very powerful theorems. (See, for example [6].)

The purpose of this note is to give an elementary proof of a more general fact concerning nonisometries between $\ell_p^k$ and $\ell_q^k$. We give a very simple proof of nonisometries between $\ell_p^k$ and $\ell_\infty^k$. In the same spirit but with a more complicated proof, we then establish the final result.

### Lemma 1

It is impossible to put more than $2k$ points inside the unit sphere in $\ell_p^k$ which are separated by a distance $\geq 2$, for $1 \leq p < \infty$ and $k \geq 1$.

*Proof* We proceed by induction on $k$. Obviously, it is true for $k = 1$. By induction hypothesis, we can put at most $2k - 2$ points on the intersection of the unit sphere and the hyperplane $x_1 = 0$. Suppose we have two points $a = (a_1, \cdots, a_k)$ and $b = (b_1, \cdots, b_k)$ in the positive half of the unit sphere: $a_1 \geq b_1 > 0$. Then define $a' = (-a_1, a_2, \cdots, a_k)$. We have

$$\begin{aligned} d_p(a, b) &< d_p(a', b) \\ &\leq d_p(a', O) + d_p(O, b) \\ &\leq 1 + 1 \\ &= 2, \end{aligned}$$

where $O$ is the origin. Note that the first inequality is strict only for $1 \leq p < \infty$. The second inequality is the triangle inequality.

Consequently, there is at most one point with $x_1 > 0$ and at most one point with $x_1 < 0$. Thus, there are at most $2k$ points in the unit sphere in $\ell_p^k$, separated by a distance $\geq 2$.

**Theorem 1**

1. For $k = 1$, $\ell_p^k$ and $\ell_\infty^k$ are isometric for $1 \le p < \infty$.
2. For $k = 2$, $\ell_p^k$ and $\ell_\infty^k$ are isometric for $p = 1$, but are not isometric for $1 < p < \infty$.
3. For $k \ge 3$, $\ell_p^k$ and $\ell_\infty^k$ are not isometric for $1 \le p < \infty$.

*Proof* It suffices to prove (3) and the second part of (2).

Suppose $f$ were an isometry from $\ell_\infty^k$ to $\ell_p^k$ mapping $O$ to $O$. Then $f$ maps the $\ell_\infty^k$ unit sphere to the $\ell_p^k$ unit sphere. But the $\ell_\infty^k$ unit sphere has $2^k$ points all distance 2 apart, namely, the $2^k$ vertices. Since $f$ is an isometry, the images of these points must satisfy the same property. For $k \ge 3$, this is impossible by Lemma 1. For $k = 2$, again by Lemma 1, the only possible candidates for the four points distance 2 apart in the $\ell_p^2$ unit sphere are $(1, 0)$ $(-1, 0)$ $(0, 1)$ $(0, -1)$ for $p < \infty$. However, they are distance 2 apart only if $p = 1$. Thus, here is a contradiction for $1 < p < \infty$.

We now prove the following final result:

**Theorem 2**

For $k \ge 2$, $1 \le p \le \infty$, $1 \le q \le \infty$, the spaces $\ell_p^k$ and $\ell_q^k$ are nonisometric, except that $\ell_1^2 = \ell_\infty^2$.

*Remark* For $k = 1$, $\ell_p^1 = \ell_q^1$.

**Lemma 2 (unique midpoint)**

If $1 < p < \infty$, $k \ge 2$, then for $x, y \in \ell_p^k$, if $d(x, y) = d(y, -x) = d(x, -x)/2$, then $y = 0$.

*Proof* We have $0 = 2(d(x, -x)/2)^p - d(x, y)^p - d(y, -x)^p = \Sigma(2|x_i|^p - |x_i - y_i|^p - |x_i + y_i|^p)$. Call the summand $S_i$. Then $S_i = 0$ when $y_i = 0$. Its derivative $\partial S_i/\partial y_i$ is $-p\{|x_i + y_i|^{p-1} - |x_i - y_i|^{p-1}\}$ sgn $(x_i)$ for $|y_i| < |x_i|$. Thus, for $0 < y_i < |x_i|$, $\partial S_i/\partial y_i < 0$ so that $S_i < 0$ for $0 < y_i \le |x_i|$. For $0 > y_i > -|x_i|$, $\partial S_i/\partial y_i > 0$ so that $S_i < 0$ for $0 > y_i \ge -|x_i|$. For $|y_i| > |x_i|$ we have $S_i < 0$ by inspection.

So each $S_i \le 0$, and for $\Sigma S_i = 0$ we must have each $S_i = 0$; thus each $y_i = 0$, implying $y = 0$.

*Corollary* If $1 < p < \infty$, $k \ge 2$, $x, y, z \in \ell_p^k$, $d(x, y) = d(y, z) = d(x, z)/2$, then $y = (x + z)/2$.

*Proof* By linearity.

*Remark* For $p = 1$ or $p = \infty$, $k \ge 2$, the lemma is not true. To see this, just consider the points:

$x = (1, 1, 0, \cdots, 0)$, $y = (1, -1, 0, \cdots, 0)$ for $p = 1$

and the points

$x = (1, 0, 0, \cdots, 0)$, $y = (0, 1, 0, \cdots, 0)$ for $p = \infty$.

Thus for $k \ge 2$ this property immediately separates the case ($p = 1$ or $p = \infty$) from the case ($1 < p < \infty$).

Given a space $\ell_p^k$ where we know $k$ but not $p$, we may first test the "unique midpoint" property to see whether $1 < p < \infty$.

If $p = 1$ or $p = \infty$, there are only finitely many directions in which the unique midpoint property holds. That is, fixing an arbitrary origin $O$, there are only finitely many $x \in \ell_1^k$ or $x \in \ell_\infty^k$, such that $d(O, x) = 1$ and such that there is a unique point $y$ with $d(O, y) = d(y, x) = 1/2$. Namely, if $p = 1$, there are $2k$ such values of $x$, namely those with one coordinate equal to $\pm 1$, the rest 0: $x = (\pm 1, 0, \cdots, 0), (0, \pm 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1)$. If $p = \infty$, there are $2^k$ such values of $x$, namely $(\pm k^{-p}, \pm k^{-p}, \cdots, \pm k^{-p})$. For $k \ge 3$, we have $2^k > 2k$, so that this count serves to distinguish $\ell_1^k$ from $\ell_\infty^k$ for $k \ge 3$. As noted above, for $k = 2$, the spaces are actually isometric, as they are for $k = 1$.

Now suppose $1 < p < \infty$. For each point $x$ define $-x$ to be the point $z$ such that $d(x, O) = d(z, O) = d(x, z)/2$. (By Lemma 2, $0 = (x + z)/2$, so that $z = -x$.) Now choose two points $x, y$ on the unit sphere ($d(x, O) = d(y, O) = 1$) with $d(x, y) = d(x, -y) = A(x, y)$. Choose $x$ and $y$ to maximize $A(x, y)$. Let the maximum value of $A(x, y)$ be $A$, *i.e.*,

$$A = \sup \{d(x, y) | x, y \in \ell_p^k,\ d(x, O) = d(y, O) = 1,\ d(x, y) = d(x, -y)\}.$$

$A$ is an invariant of the space and almost serves to distinguish $\ell_p^k$ and $\ell_q^k$.

**Lemma 3**

If $1 < p < 2$, then $A = 2^{1/p}$ and is achieved for precisely those $x$ and $y$ such that for each coordinate $i$, either $x_i = 0$ or $y_i = 0$.

*Proof* Set $S = d(x, y)^p + d(x, -y)^p - 2d(x, O)^p - 2d(y, O)^p$. For $x, y$ such that $A(x, y) = 2^{1/p}$ we have $S = 0$. For $A(x, y) > 2^{1/p}$ we would have $S > 0$. Now $S = \Sigma(|x_i - y_i|^p + |x_i + y_i|^p - 2|x_i|^p - 2|y_i|^p)$. Again call the summand $S_i$.

If $x_i = 0$ or $y_i = 0$, then $S_i = 0$.

Otherwise assume, without loss of generality, that $x_i \ge y_i > 0$. We may set $x_i = 1$, $y_i = r$. If $r = 1$, then $S_i = 0^p + 2^p - 2 \cdot 1 - 2 \cdot 1 = 2^p - 4 < 0$ since $p < 2$.

If $0 < r < 1$, set $t = S_i/r^p = \{(1 - r)^p + (1 + r)^p - 2 - 2r^p\}/r^p = \{2[1 + C(p, 2)r^2 + C(p, 4)r^4 + \cdots] - 2 - 2r^p\}/r^p = 2[C(p, 2)r^{2-p} + C(p, 4)r^{4-p} + \cdots - 1]$. As $r \to 1$, $t \to 2^p - 4 < 0$. Clearly, since $t$ is a sum of positive multiples of positive powers of $r$ [note that $C(p, 2N)$, the binomial coefficient, is positive since $1 < p < 2$], $t$ gets smaller as $r$ gets smaller in the range $0 < r < 1$. Thus in that range, $t < 0$ and $S_i < 0$. So the only way for $S_i = 0$ is for $r = 0$, *i.e.*, $x_i = 0$ or $y_i = 0$. Also it is impossible to have $S_i > 0$.

So for $S = 0$ we must have, for each $i$, $x_i = 0$ or $y_i = 0$, and we always have $S \le 0$.

**Lemma 4**
If $2 < p < \infty$, then $A = 2^{1-1/p}$ and is achieved when $x_i = \pm y_i$ for all $i$.

*Proof* Set $S = 2d(x, y)^p + 2d(x, -y)^p - 2^p d(x, O)^p - 2^p d(y, O)^p$. Again, if $A(x, y) = 2^{1-1/p}$, then $S = 0$, and if $A(x, y) > 2^{1-1/p}$, then $S > 0$. We rewrite $S$ as

$$S = \Sigma\{2|x_i - y_i|^p + 2|x_i + y_i|^p - |2x_i|^p - |2y_i|^p\},$$

with the summand denoted $S_i$.

If $y_i = 0$, then $S_i = (2 + 2 - 2^p)|x_i|^p < 0$ unless $x_i = 0$ also.

If $y_i = \pm x_i$, then $S_i = 2 \cdot 2^p + 0 - 2^p - 2^p = 0$.

Assume $x_i > y_i > 0$, and we may assume $x_i = 1$, $0 < y_i = r < 1$. We want to show

$$2(1 - r)^p + 2(1 + r)^p - 2^p(1 + r^p) < 0.$$

Call the left-hand expression $t$; then as $r \to 1$, $t \to 0$.

For $0 < r < 1$, we have $\partial t/\partial r = -2p(1 - r)^{p-1} + 2p(1 + r)^{p-1} - 2^p p r^{p-1} = 2p\,\{(1 + r)^{p-1} - (1 - r)^{p-1} - (2r)^{p-1}\} > 0$ by the triangle inequality on $\ell^k_{p-1}$. Thus for $0 < r < 1$ we have $t < 0$.

Thus we have $S_i \le 0$, and $S_i = 0$ iff $x_i = \pm y_i$.

Thus $S \le 0$, and $S = 0$ iff for all $i$, $x_i = \pm y_i$.

**Lemma 5**
If $p = 2$, $A = 2^{1/2}$.

*Proof* Continuity in $p$.

Now, given a space $\ell^k_p$ of unknown $p$, with $1 < p < \infty$, first find $A$ as above.

If $A = 2^{1/2}$, then $p = 2$.

If $2 > A > 2^{1/2}$, then either

$p = 1/\log_2 A$ $(1 < p < 2)$, or

$p = 1/(1 - \log_2 A)$ $(2 < p < \infty)$.

If $k \ge 3$, we can distinguish these two cases immediately: Try to find an $x$, with $d(O, x) = 1$, such that there are infinitely many $y$ with $d(O, y) = 1$, $d(x, y) = d(x, -y) = A$. We will succeed iff $1 < p < 2$.

If $1 < p < 2$, then $x = (1, 0, \cdots, 0)$, $y = (0, a, \cdots, b)$ will work as long as $d(O, y) = 1$.

If $2 < p < \infty$, for any $x$ there are at most $2^k$ choices of $y$, namely, the $k$ choices of sign in $y_i = \pm x_i$.

If $k = 2$ we still have problems. For a given value of $A$, with $2^{1/2} < A < 2$, we have two possible values of $p$, say $p$ and $q$, with $1 < p < 2 < q < \infty$. They are related by $(1/p) + (1/q) = 1$.

The solutions of $(x, y)$ which led to our maximum value of $A(x, y)$ are (for $1 < p < 2$) $x = (\pm 1, 0)$, $y = (0, \pm 1)$ or vice versa; or (for $2 < q < \infty$) $x = \pm(2^{-1/q}, 2^{-1/q})$, $y = \pm(2^{-1/q}, -2^{-1/q})$ or vice versa. In either case, by our unique midpoint lemma, we can determine the behavior of $d(x, 2^{-n}y)$ as $n \to \infty$. That is, take the midpoint of $(0, y)$ and call it $2^{-1}y$; then find $2^{-2}y$ = midpoint $(0, 2^{-1}y)$, etc.

For $1 < p < 2$, we have $d(x, 2^{-n}y) = (1^p + (2^{-n})^p)^{1/p} = (1 + 2^{-np})^{1/p} = 1 + (1/p)2^{-np} + O(2^{-2np})$.

For $2 < q < \infty$, we have $d(x, 2^{-n}y) = [(2^{-1/q} + 2^{-n-1/q})^q + (2^{-1/q} - 2^{-n-1/q})^q]^{1/q} = \{(1/2)[(1 + 2^{-n})^q + (1 - 2^{-n})^q]\}^{1/q} = \{1 + C(q, 2)2^{-2n} + O(2^{-4n})\}^{1/q} = 1 + ((q - 1)/2)2^{-2n} + O(2^{-4n})$.

The leading terms differ unless $q = p = 2$. [Compare $(1/p)2^{-np}$ with $((q - 1)/2)2^{-2n}$.] So this test distinguishes $\ell^2_p$ from $\ell^2_q$ for $1 < p < 2 < q < \infty$.

To summarize, given an $\ell^2_p$ space with $k \ge 2$, $1 \le p \le \infty$, with $k$ known and $p$ unknown, these tests can determine $p$ intrinsically unless $k = 2$ and $p = 1$ or $\infty$. First we distinguish among ($p = 1$ or $p = \infty$) and ($1 < p < \infty$). In the first case we count the directions along which the unique midpoint lemma holds. In the latter case we find $A$, which narrows the possible values of $p$ to two values, $p$ and $q$, $1 < p < 2 < q < \infty$, or else identifies $p = 2$. If $k \ge 3$, we can then ask the number of solutions to the $A$ equation for a given $x$, which separates $p < 2$ from $p > 2$. If $k = 2$, the behavior of $d(x, 2^{-n}y)$ as $n \to \infty$ distinguishes large $p$ from small $p$. So the only isometries that exist are between $\ell^1_p$ and $\ell^1_q$ and between $\ell^2_1$ and $\ell^2_\infty$.

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