

Exploiting Symmetry in Electrical Packaging Analysis

Many properties of physical systems can be expressed by symmetric matrices of order n , where n is the number of components in the system. The computer storage requirement for inverting the most general symmetric matrix is $n(n+1)/2$ storage locations. For large values of n , the number of multiplications required is proportional to n^3 . If the physical system possesses certain geometrical symmetries, both the amount of storage and the number of multiplications can be reduced substantially. It will be shown that if the physical system possesses p orthogonal planes of symmetry, where $p = 1, 2$, or 3 , and if n is sufficiently larger, then the storage requirement can be reduced approximately by $1/2^p$ and the number of multiplications by $1/4^p$.

Introduction

Real or complex matrices of rather high orders are commonly encountered in electrical packaging analysis. Inductance, capacitance, impedance, and admittance matrices are familiar examples. Quite often it is necessary to construct the inverse of such matrices, or to solve systems of linear equations in which these matrices appear as coefficient matrices. The purpose of this paper is to show how the physical symmetry of a given package design can be exploited to reduce the amount of computer storage and computer time required for these operations.

With few exceptions, the real matrices encountered in electrical packaging analysis are usually symmetric and positive definite. The complex matrices are usually symmetric and have positive definite imaginary parts and positive semidefinite real parts. These properties are quite general in that they are independent of any particular geometric symmetry that a package might exhibit. For real symmetric positive definite matrices, the square root method of Cholesky provides a convenient and efficient technique for matrix inversion or solution of linear equations. By considering complex square roots, one can extend the Cholesky factorization to include complex matrices of the type described above. When applying the Cholesky method to a symmetric matrix of order n , it is necessary to store only the $n(n+1)/2$ elements of the upper triangle rather than the full n^2 elements.

If a physical package possesses sufficient geometrical symmetry, it is often possible to reduce the storage requirements to well below the generally required $n(n+1)/2$ elements. The following discussion will introduce an abstraction, called an interaction matrix, of which inductance, capacitance, impedance, and admittance matrices are special cases. The element a_{ij} of an interaction matrix, $\mathbf{A}$, represents, in some sense, the physical interaction between components i and j of a physical system. The distinguishing feature of an interaction matrix is that, in general, the element a_{ij} depends only on the relative positions of components i and j in the array of components which make up the physical package.

As an example of the effect of symmetry, consider an interaction matrix for the four-component system shown in Fig. 1. This configuration possesses a vertical plane of symmetry, in that the left half of the system is the mirror image of the right half. The interaction matrix will have the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{13} \\ a_{13} & a_{23} & a_{22} & a_{12} \\ a_{14} & a_{13} & a_{12} & a_{11} \end{bmatrix}$$

In addition to the usual symmetry with respect to reflection

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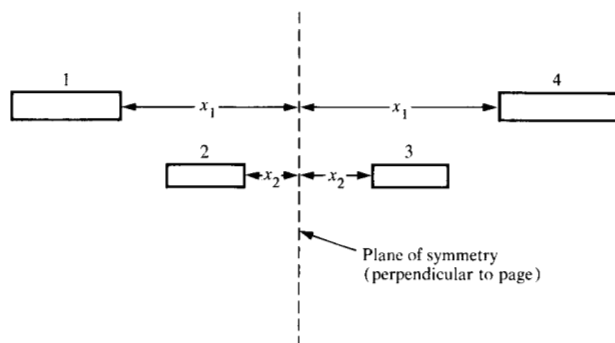


Figure 1 Single plane of symmetry.

tion through the diagonal, this matrix is also symmetric with respect to reflection through the quasi-diagonal extending from the lower left-hand corner to the upper right-hand corner. The latter symmetry is a direct consequence of the right-left geometrical symmetry of the configuration of Fig. 1. Not all the ten elements of the upper triangle are distinct. There are only six unique elements, and one might suspect, quite correctly, that the inverse matrix can be constructed using only six storage locations. Consider now the generalization to matrices of arbitrary order.

One plane of symmetry

An $n \times n$ matrix with elements a_{ij} will be called bisymmetric if $a_{ij} = a_{ji}$ and $a_{ij} = a_{n+1-j, n+1-i}$ for $i, j = 1, 2, \dots, n$. The first condition states that the matrix is symmetric with respect to reflection through the main diagonal, which extends from the element a_{11} in the upper left-hand corner to the element a_{nn} in the lower right-hand corner. The second condition states that the matrix is unchanged when it is reflected through the quasi-diagonal, which extends from the element a_{nn} in the lower left-hand corner to the element a_{1n} in the upper right-hand corner. Bisymmetric matrices are a subset of a class of matrices, called centrosymmetric, which are distinguished by the property that $a_{ij} = a_{n+1-i, n+1-j}$. The reduction of bisymmetric matrices that will be discussed below is closely related to the reduction of centrosymmetric matrices given by Good [1] and Andrew [2].

Bisymmetric matrices are encountered in physical problems which have a plane of symmetry that divides the configuration into two halves which are mirror images of each other. The bisymmetric form will be obtained if the individual components of the physical system are numbered so that if component i' is the mirror image of component i , then $i + i' = n + 1$, where n is the total number of components.

The general form for a bisymmetric matrix of even order, $n = 2m$, is

$$A_{2m} = \begin{bmatrix} B_m & C_m J_m \\ J_m C_m & J_m B_m J_m \end{bmatrix}, \quad (1)$$

where B_m and C_m are $m \times m$ symmetric matrices and J_m is the $m \times m$ reversal matrix,

$$J_m = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad (2)$$

having ones on the quasi-diagonal and zeros everywhere else. Postmultiplication of C_m by J_m reverses the order of the columns of C_m , whereas premultiplication of C_m reverses the order of the rows. The pre- and postmultiplications of B_m by J_m reverse the orders of both the rows and columns of B_m .

Only the upper triangles of B_m and C_m need be stored. The remaining matrix elements can be found by reflection through the diagonal and quasi-diagonal. The number of storage locations required is

$$\frac{m(m+1)}{2} + \frac{m(m+1)}{2} = m(m+1) = \frac{n(n+2)}{4}.$$

The general form for a bisymmetric matrix of odd degree, $n = 2m + 1$, is

$$A_{2m+1} = \begin{bmatrix} B_m & D_m & C_m J_m \\ D_m' & E & D_m' J_m \\ J_m C_m & J_m D_m & J_m B_m J_m \end{bmatrix}, \quad (3)$$

where B_m and C_m are $m \times m$ symmetric matrices, J_m is the $m \times m$ reversal matrix, D_m is an $m \times 1$ column vector, D_m' is the transpose of D_m , and E is a scalar. It is necessary only to store the upper triangles of the matrices B_m and C_m , and the vector D_m and scalar E . The number of storage locations required is

$$\begin{aligned} \frac{m(m+1)}{2} + \frac{m(m+1)}{2} + m + 1 &= (m+1)^2 \\ &= \frac{(n+1)^2}{4}. \end{aligned}$$

For comparison, it should be recalled that a general symmetric matrix of order n requires $n(n+1)/2$ storage locations.

Thus, approximately half the storage for a symmetric matrix is needed for a bisymmetric matrix.

The inversion method is based on the identity

$$TA^{-1}T^{-1} = [TAT^{-1}]^{-1}, \quad (4)$$

where T is a non-singular matrix.

For the matrix A_{2m+1} of Eq. (3) the appropriate T matrix is

$$T_{2m} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_m & J_m \\ -J_m & I_m \end{bmatrix}. \quad (5)$$

Observing that the square of J_m is equal to I_m , it is easy to show that the inverse of T_{2m} is

$$T_{2m}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_m & -J_m \\ J_m & I_m \end{bmatrix}. \quad (6)$$

From Eqs. (1), (5), and (6) one finds

$$T_{2m} A_{2m} T_{2m}^{-1} = \begin{bmatrix} B_m + C_m & 0 \\ 0 & J_m(B_m - C_m)J_m \end{bmatrix}. \quad (7)$$

For the matrix A_{2m+1} of Eq. (3) the appropriate T matrix is

$$T_{2m+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_m & 0 & J_m \\ 0 & \sqrt{2} & 0 \\ -J_m & 0 & I_m \end{bmatrix}. \quad (8)$$

The inverse is

$$T_{2m+1}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_m & 0 & -J_m \\ 0 & \sqrt{2} & 0 \\ J_m & 0 & I_m \end{bmatrix}. \quad (9)$$

from Eqs. (3), (8), and (9),

$$T_{2m+1} A_{2m+1} T_{2m+1}^{-1} = \begin{bmatrix} B_m + C_m & \sqrt{2}D_m & 0 \\ \sqrt{2}D_m' & E & 0 \\ 0 & 0 & J_m(B_m - C_m)J_m \end{bmatrix}. \quad (10)$$

Thus, the similarity transformation TAT^{-1} reduces a bisymmetric matrix A to a block diagonal matrix with two blocks. Both blocks are symmetric matrices. If A is of order $2m$, both blocks are of order m . If A is of order $2m+1$, one block is of order $m+1$, the other is of order m . Note also that if A is positive definite, then both blocks of TAT^{-1} are positive definite. To invert TAT^{-1} one inverts the two blocks separately. Since the inverse of a symmetric matrix is symmetric, one obtains

$$[T_{2m} A_{2m} T_{2m}^{-1}]^{-1} = [T_{2m} A_{2m}^{-1} T_{2m}^{-1}] = \begin{bmatrix} F_m & 0 \\ 0 & J_m G_m J_m \end{bmatrix}, \quad (11)$$

where

$$F_m = [B_m + C_m]^{-1}$$

and

$$G_m = [B_m - C_m]^{-1}$$

are $m \times m$ symmetric matrices, or

$$[T_{2m+1} A_{2m+1} T_{2m+1}^{-1}]^{-1} = [T_{2m+1} A_{2m+1}^{-1} T_{2m+1}^{-1}] = \begin{bmatrix} F_m & H_m & 0 \\ H_m' & K & 0 \\ 0 & 0 & J_m G_m J_m \end{bmatrix}, \quad (12)$$

where

$$\begin{bmatrix} F_m & H_m \\ H_m' & K \end{bmatrix} = \begin{bmatrix} B_m + C_m & \sqrt{2}D_m \\ \sqrt{2}D_m' & E \end{bmatrix}^{-1}$$

and

$$G_m = [B_m - C_m]^{-1}.$$

In Eq. (12), F_m and G_m are $m \times m$ symmetric matrices, H_m is an $m \times 1$ column vector, and K is a scalar.

Inversion of an $n \times n$ symmetric matrix by conventional techniques requires, for large n , approximately cn^3 multiplications, where c is a constant independent of n . Inversion of two symmetric $m \times m$ matrices requires

$$cm^3 + cm^3 = 2cm^3 \approx 2c \left(\frac{n}{2}\right)^3 = \frac{1}{4} cn^3$$

operations. Hence, for large bisymmetric matrices the two block diagonal submatrices in Eqs. (7) and (10) can be inverted with about one-quarter the number of multiplications required to invert the original matrix by conventional methods.

The inverse of A can be obtained from Eqs. (11) or (12) by noting that

$$A^{-1} = T^{-1}[TA^{-1}T^{-1}]T.$$

Using Eqs. (5), (6), and (11) one finds

$$A_{2m}^{-1} = \frac{1}{2} \begin{bmatrix} F_m + G_m & (F_m - G_m)J_m \\ J_m(F_m - G_m) & J_m(F_m + G_m)J_m \end{bmatrix}. \quad (13)$$

Using Eqs. (8), (9), and (12) one finds

$$A_{2m+1}^{-1} = \frac{1}{2} \begin{bmatrix} F_m + G_m & \sqrt{2}H_m & (F_m - G_m)J_m \\ \sqrt{2}H_m' & 2K & \sqrt{2}H_m'J_m \\ J_m(F_m - G_m) & \sqrt{2}J_m H_m & J_m(F_m + G_m)J_m \end{bmatrix}. \quad (14)$$

Comparison of Eqs. (1) with (13) and (3) with (14) shows that A^{-1} is bisymmetric. That is, the inverse of a bisymmetric matrix is bisymmetric.

No multiplications are required to evaluate $T_{2m} A_{2m} T_{2m}^{-1}$, whereas m multiplications by $\sqrt{2}$ are required to evaluate $T_{2m+1} A_{2m+1} T_{2m+1}^{-1}$. Also, $m(m+1)$ multiplications by $1/2$ are required to construct A_{2m}^{-1} from $T_{2m} A_{2m} T_{2m}^{-1}$, according to Eq. (13), whereas $(m+1)^2$ multiplications are required to construct A_{2m+1}^{-1} from Eq. (14). The inversion of the diagonal blocks in Eqs. (7) and (10) requires of the order of

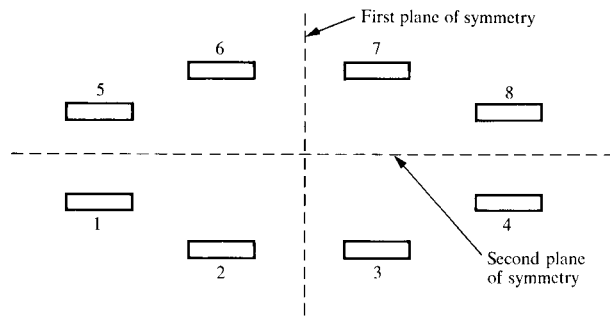


Figure 2 Two planes of symmetry, case 1.

m^3 multiplications each and hence dominates the computation. Thus, in theory, the method described here can be up to four times faster than conventional methods for inverting symmetric matrices. Also, the entire computation can be done in $n(n+2)/4$ storage locations if n is even, or in $(n+1)^2/4$ locations if n is odd, compared with $n(n+1)/2$ for conventional methods.

The above technique is easily modified to obtain the solution $\mathbf{x}$ of the linear system

$$\mathbf{Ax} = \mathbf{R}, \quad (15)$$

where $\mathbf{A}$ is a known $n \times n$ bisymmetric matrix, $\mathbf{R}$ is a known $n \times 1$ column vector, and $\mathbf{x}$ is an unknown $n \times 1$ column vector. Applying the transformation matrix $\mathbf{T}$ to Eq. (15), one obtains

$$[\mathbf{TAT}^{-1}]\mathbf{T}\mathbf{x} = \mathbf{TR}. \quad (16)$$

It is convenient to introduce the auxiliary solution vector $\mathbf{Y}$ defined by

$$\mathbf{Y} = \sqrt{2}\mathbf{T}\mathbf{x}. \quad (17)$$

In the case $n = 2m$, let

$$\mathbf{Y} = \begin{bmatrix} \mathbf{U}_m \\ \mathbf{J}_m \mathbf{V}_m \end{bmatrix} \text{ and } \mathbf{R} = \begin{bmatrix} \mathbf{P}_m \\ \mathbf{J}_m \mathbf{Q}_m \end{bmatrix}, \quad (18)$$

where $\mathbf{U}_m$, $\mathbf{V}_m$, $\mathbf{P}_m$, and $\mathbf{Q}_m$ are $m \times 1$ column vectors. Then Eq. (16) partitions into the two independent sets of linear equations

$$(\mathbf{B}_m + \mathbf{C}_m)\mathbf{U}_m = \mathbf{P}_m + \mathbf{Q}_m \quad (19)$$

and

$$(\mathbf{B}_m - \mathbf{C}_m)\mathbf{V}_m = -\mathbf{P}_m + \mathbf{Q}_m, \quad (20)$$

which can be solved for the vectors $\mathbf{U}_m$ and $\mathbf{V}_m$ by any method suitable for symmetric matrices. The solution

vector $\mathbf{x}$ is obtained by multiplying Eq. (17) on the left by $\mathbf{T}^{-1}$. In particular,

$$\mathbf{x} = \frac{1}{2} \begin{bmatrix} \mathbf{U}_m - \mathbf{V}_m \\ \mathbf{J}_m(\mathbf{U}_m + \mathbf{V}_m) \end{bmatrix}. \quad (21)$$

Since the case $n = 2m + 1$ is analogous, details will be omitted.

Two planes of symmetry

Consider next the arrangement of eight components as shown in Fig. 2. In this configuration the right half is the mirror image of the left half, and the top half is the mirror image of the bottom. Again assuming that the electromagnetic interaction between components is a function only of relative position in the array, the interaction matrix would have the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{12} & a_{22} & a_{23} & a_{13} & a_{16} & a_{26} & a_{27} & a_{17} \\ a_{13} & a_{23} & a_{22} & a_{12} & a_{17} & a_{27} & a_{26} & a_{16} \\ a_{14} & a_{13} & a_{12} & a_{11} & a_{18} & a_{17} & a_{16} & a_{15} \\ \hline a_{15} & a_{16} & a_{17} & a_{18} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{16} & a_{26} & a_{27} & a_{17} & a_{12} & a_{22} & a_{23} & a_{13} \\ a_{17} & a_{27} & a_{26} & a_{16} & a_{13} & a_{23} & a_{22} & a_{12} \\ a_{18} & a_{17} & a_{16} & a_{15} & a_{14} & a_{13} & a_{12} & a_{11} \end{bmatrix}$$

Note that $\mathbf{A}$ consists of four submatrices, each of which is bisymmetric; the two diagonal submatrices are identical, and the two off-diagonal submatrices are identical.

Generally, if an array consists of $n = 4m$ components that can be generated by reflecting m components through a plane and then reflecting the resulting $2m$ components through a second orthogonal plane, the interaction matrix has the form

$$\mathbf{A}_{4m} = \begin{bmatrix} \mathbf{A}_{2m}^{(1)} & \mathbf{A}_{2m}^{(2)} \\ \mathbf{A}_{2m}^{(2)} & \mathbf{A}_{2m}^{(1)} \end{bmatrix}, \quad (22)$$

where $\mathbf{A}_{2m}^{(1)}$ and $\mathbf{A}_{2m}^{(2)}$ are $2m \times 2m$ bisymmetric matrices. The matrix $\mathbf{A}_{4m}$ is then a bisymmetric matrix of order $n = 4m$.

In order to store the matrix $\mathbf{A}_{4m}$, it is necessary only to store the $m(m+1)$ distinct elements of $\mathbf{A}_{2m}^{(1)}$ and the $m(m+1)$ distinct elements of $\mathbf{A}_{2m}^{(2)}$. Thus, the total amount of storage required is

$$2m(m+1) = \frac{n(n+4)}{8},$$

as opposed to the $n(n+1)/2$ storage locations required for a general symmetric matrix. Thus, for large n , only one-fourth the storage required for a general symmetric matrix is needed.

Since A_{4m} is bisymmetric, it can be reduced to two symmetric diagonal blocks by means of a similarity transformation with the matrix:

$$T_{4m}^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{2m} & J_{2m} \\ -J_{2m} & I_{2m} \end{bmatrix} \quad (23)$$

Since $A_{2m}^{(2)}$ is bisymmetric, it follows that $J_{2m}A_{2m}^{(2)} = A_{2m}^{(2)}J_{2m}$; therefore

$$T_{4m}^{(1)}A_{4m}T_{4m}^{(1)-1} = \begin{bmatrix} A_{2m}^{(1)} + J_{2m}A_{2m}^{(2)} & 0 \\ 0 & A_{2m}^{(1)} - J_{2m}A_{2m}^{(2)} \end{bmatrix} \quad (24)$$

Since $J_{2m}A_{2m}^{(2)}$ is bisymmetric and the sum and the difference of two bisymmetric matrices are bisymmetric, the two diagonal blocks in Eq. (24) are bisymmetric and can be reduced further. Let

$$T_{4m}^{(2)} = \begin{bmatrix} T_{2m} & 0 \\ 0 & T_{2m} \end{bmatrix}, \quad (25)$$

where T_{2m} is given by Eq. (5), and define

$$T_{4m} = T_{4m}^{(2)}T_{4m}^{(1)}. \quad (26)$$

Then

$$T_{4m}A_{4m}T_{4m}^{-1} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix}, \quad (27)$$

where A_1, A_2, A_3 , and A_4 are symmetric $m \times m$ matrices. Thus, the problem of inverting A_{4m} has been reduced to that of inverting four symmetric $m \times m$ matrices and forming the inverse transformation

$$A_{4m}^{-1} = T_{4m}^{-1}(T_{4m}A_{4m}^{-1}T_{4m}^{-1})T_{4m}. \quad (28)$$

The inverse transformation requires no multiplications, only additions and subtractions. For large m , the inversion of the four $m \times m$ matrices requires approximately 1/16 the number of multiplications that would be required to invert the original $4m \times 4m$ matrix.

Another interesting case is obtained by starting with an array of $2m+1$ components which are symmetrically placed with respect to a plane of symmetry and reflecting them through a second orthogonal plane as shown in Fig. 3. The interaction matrix for the resulting system of $4m+2$ components has the form

$$A_{4m+2} = \begin{bmatrix} A_{2m+1}^{(1)} & A_{2m+1}^{(2)} \\ A_{2m+1}^{(2)} & A_{2m+1}^{(1)} \end{bmatrix}, \quad (29)$$

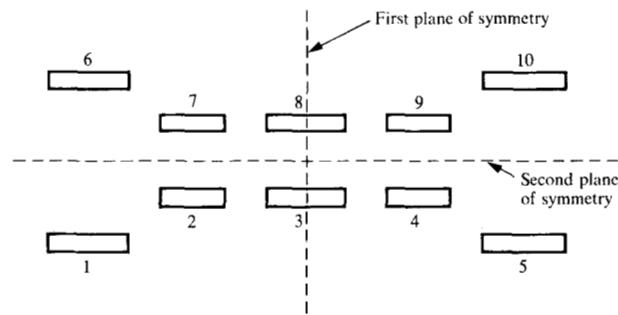


Figure 3 Two planes of symmetry, case 2.

where $A_{2m+1}^{(1)}$ and $A_{2m+1}^{(2)}$ are bisymmetric. This matrix can be reduced to block diagonal form by means of a similarity transformation with the matrix

$$T_{4m+2} = T_{4m+2}^{(2)}T_{4m+2}^{(1)}, \quad (30)$$

where

$$T_{4m+2}^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{2m+1} & J_{2m+1} \\ -J_{2m+1} & I_{2m+1} \end{bmatrix} \quad (31)$$

and

$$T_{4m+2}^{(2)} = \begin{bmatrix} T_{2m+1} & 0 \\ 0 & T_{2m+1} \end{bmatrix}, \quad (32)$$

where T_{2m+1} is given by Eq. (8). Upon forming the similarity transformation, one finds

$$T_{4m+2}A_{4m+2}T_{4m+2}^{-1} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix}, \quad (33)$$

where A_1 and A_3 are symmetric matrices of order $m+1$, and A_2 and A_4 are symmetric matrices of order m . Thus, A_{4m+2} can be inverted by inverting two matrices of order $m+1$ and two of order m .

The two cases described above clearly do not exhaust the possibilities with two planes of symmetry. They are, however, suggestive of the approach that can be used.

Three planes of symmetry

A single class of configurations with three orthogonal planes of symmetry will be examined. This class of configurations is obtained by starting with m components. The m components are reflected through a plane to obtain

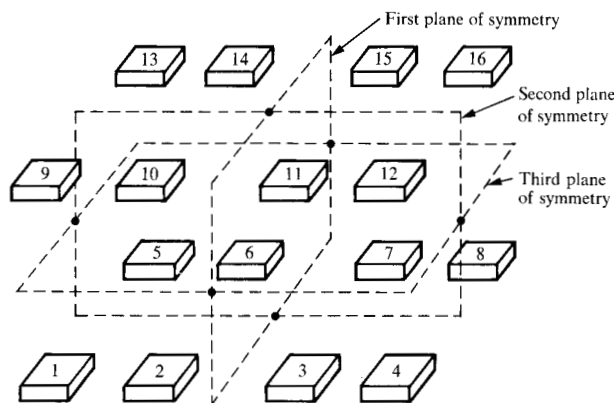


Figure 4 Three planes of symmetry.

a total of $2m$ components. The $2m$ components are reflected through a second orthogonal plane to obtain $4m$ components and the $4m$ components are reflected through a third orthogonal plane to obtain an array of $n = 8m$ components. Figure 4 illustrates the case of $m = 2$. The interaction matrix for such a configuration has the form

$$\mathbf{A}_{8m} = \begin{bmatrix} \mathbf{A}_{2m}^{(1)} & \mathbf{A}_{2m}^{(2)} & \mathbf{A}_{2m}^{(3)} & \mathbf{A}_{2m}^{(4)} \\ \mathbf{A}_{2m}^{(2)} & \mathbf{A}_{2m}^{(1)} & \mathbf{A}_{2m}^{(4)} & \mathbf{A}_{2m}^{(3)} \\ \mathbf{A}_{2m}^{(3)} & \mathbf{A}_{2m}^{(4)} & \mathbf{A}_{2m}^{(1)} & \mathbf{A}_{2m}^{(2)} \\ \mathbf{A}_{2m}^{(4)} & \mathbf{A}_{2m}^{(3)} & \mathbf{A}_{2m}^{(2)} & \mathbf{A}_{2m}^{(1)} \end{bmatrix}, \quad (34)$$

where $\mathbf{A}_{2m}^{(i)}$ is bisymmetric for $i = 1, 2, 3, 4$. Note that $4m(m+1) = n(n+8)/16$ storage locations are required to store $\mathbf{A}_{8m}$, as opposed to $n(n+1)/2$ locations for a general symmetric matrix of the same order.

In this case, the appropriate transformation matrix is

$$\mathbf{T}_{8m} = \mathbf{T}_{8m}^{(3)} \mathbf{T}_{8m}^{(2)} \mathbf{T}_{8m}^{(1)}, \quad (35)$$

where

$$\mathbf{T}_{8m}^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_{4m} & \mathbf{J}_{4m} \\ -\mathbf{J}_{4m} & \mathbf{I}_{4m} \end{bmatrix}, \quad (36)$$

$$\mathbf{T}_{8m}^{(2)} = \begin{bmatrix} \mathbf{T}_{4m} & 0 \\ 0 & \mathbf{T}_{4m} \end{bmatrix}, \quad (37)$$

and

$$\mathbf{T}_{8m}^{(3)} = \begin{bmatrix} \mathbf{T}_{2m} & 0 & 0 & 0 \\ 0 & \mathbf{T}_{2m} & 0 & 0 \\ 0 & 0 & \mathbf{T}_{2m} & 0 \\ 0 & 0 & 0 & \mathbf{T}_{2m} \end{bmatrix}, \quad (38)$$

where $\mathbf{T}_{4m}$ is given by Eq. (26), and $\mathbf{T}_{2m}$ is given by Eq. (5). A straightforward calculation shows that

$$\mathbf{T}_{8m} \mathbf{A}_{8m} \mathbf{T}_{8m}^{-1} = \begin{bmatrix} \mathbf{A}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{A}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{A}_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{A}_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{A}_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{A}_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{A}_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{A}_8 \end{bmatrix}, \quad (39)$$

where the matrices $\mathbf{A}_i$ for $i = 1, 2, \dots, 8$ are $m \times m$ symmetric matrices. Thus, the problem of inverting $\mathbf{A}_{8m}$ is reduced to that of inverting eight $m \times m$ symmetric matrices. Consequently, for large value of m , $\mathbf{A}_{8m}$ can be inverted using approximately $1/64$ the number of multiplications required to invert a general symmetric matrix of order $n = 8m$.

Conclusion

It has been shown that substantial reductions in storage and computing time can be achieved for the inversion of matrices describing the electromagnetic interaction between components in an electrical package by taking geometrical symmetry into account. In particular, it has been shown that the storage requirements for large matrices can be reduced by approximately $1/2^p$, and the number of multiplications by $1/4^p$, where p is the number of orthogonal planes of symmetry.

The storage saving has the greater importance since it is highly desirable to include as many components as possible when analyzing a section from a very large electrical package.

The transformations required to take advantage of the symmetry are relatively easy to code, since the results of the transformation are easily recognized patterns of sums and differences of the elements of the original matrix.

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