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# Break-Up of a Liquid Jet: Second Perturbation Solution for One-Dimensional Cosserat Theory

*The second perturbation solution is derived within the nonlinear one-dimensional Cosserat theory for a liquid jet emanating from a nozzle with harmonic excitation. Numerical results are presented for parameters relevant to ink-jet printing technology. Satellite drops are predicted but always in the backward merging condition. The results are compared with the corresponding solution obtained by Pimbley and Lee, who used a different one-dimensional set of equations with a different formulation of the problem and obtained forward merging satellite drops under some conditions.*

## Introduction

This work continues the analytical study of the stability and break-up of a liquid jet begun in Bogy [1, 2]. In this study a one-dimensional Cosserat theory of the jet, as published by Green [3], is utilized. The ultimate goal is to predict analytically the experimentally observed satellite drop behavior as reported in Pimbley and Lee [4]. They observed that the satellite drops can be made to merge forward or backward with the main drops, depending on the magnitude of the time harmonic disturbance. For larger disturbance magnitudes, so that break-up occurs closer to the nozzle, the satellite drops merge forward. For smaller magnitudes they merge backward.

In [1] several predictions of this one-dimensional theory were compared with previous stability analyses of Rayleigh [5], Keller et al. [6], Lee [7], and Pimbley [8]. In [2] the stability problem was studied from the point of view of wave propagation in the jet. The primary purpose there was to obtain full understanding of the frequency spectra to aid in the proper formulation of the boundary value problem of a jet emanating from a nozzle. Green [3] proved a uniqueness theorem, which indicates that for the one-dimensional theory under consideration two boundary conditions should be prescribed at two end points of a jet segment. Thus only two conditions can be prescribed at the nozzle and the other two must be set downstream. This is also in agreement with what would be required by the three-dimensional ideal fluid theory for this problem.

Here, as in many fluid flow problems, the downstream boundary conditions are very difficult to set since the flow is unknown. In [2], in the context of the linearized theory, the jet was considered to be semi-infinite for those frequencies at which a jet is stable. In this manner radiation type conditions could be imposed as $z \to \infty$, thereby eliminating two of the four wave eigenfunctions of the equations. This led to a properly formulated problem in which only two boundary conditions need to be specified at the nozzle, in conformity with Green's [3] uniqueness theorem. This was contrasted with the analytical work in [8], and that of Pimbley and Lee [4], wherein four boundary conditions were prescribed at the nozzle while using Lee's [7] one-dimensional formulation.

In this paper the second perturbation solution is derived for the one-dimensional Cosserat jet emanating from a nozzle. The problem is formulated as described in [2]. Numerical results are obtained and compared with the second perturbation results of Pimbley and Lee [4].

## Derivation of the inviscid Cosserat jet perturbation equations

The inviscid form of the straight circular jet equations was given in [1] in terms of jet radius $\phi(z, t)$ and axial velocity $v(z, t)$ as

$$(\phi^2)_t + (v\phi^2)_z = 0, \tag{1}$$

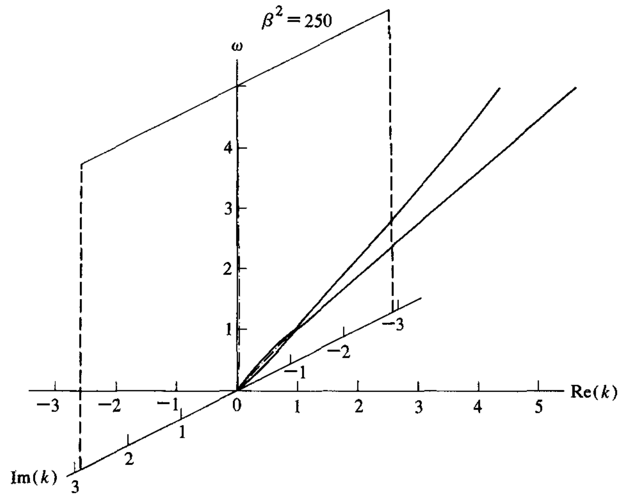


**Figure 1** Frequency spectrum. [Solutions of Eq. (10).] Dashed branches are excluded by radiation conditions—solid branches are retained and designated $k_1$, $k_2$.

and

$$v_t + vv_z - \frac{1}{2}\phi\phi_z\left(v_{zt} + vv_{zz} - \frac{1}{2}v_z^2\right) - \frac{1}{8}\phi^2(v_{zzt} + vv_{zzz})$$

$$= \frac{1}{\beta^2}\left[\frac{\phi_z}{\phi^2(1+\phi_z^2)^{1/2}} + \frac{\phi_z\phi_{zz}}{\phi(1+\phi_z^2)^{3/2}} + \frac{\phi_{zzz}}{(1+\phi_z^2)^{3/2}} - \frac{3\phi_z\phi_{zz}^2}{(1+\phi_z^2)^{5/2}}\right], \tag{2}$$

in which $z$ is axial distance, $t$ is time, subscripts denote partial differentiation, and length and time variables have been nondimensionalized by nominal radius $a$ and velocity $v_0$. The Weber number, $\beta^2$, is related to surface tension $T$ and density $\rho$ by

$$\beta^2 = \rho a v_0^2/T. \tag{3}$$

We wish to obtain a steady time harmonic solution of these equations for a semi-infinite jet, $z > 0$, satisfying the boundary conditions at the nozzle given by

$$\phi(0, t) = 1, \qquad v(0, t) = 1 + \epsilon\cos(\omega t), \tag{4}$$

where $\epsilon << 1$ is the magnitude of the applied disturbance of the axial velocity and will be our perturbation parameter.

Two other boundary conditions must be set for this fourth-order system. As explained in [2] in the context of the linearized theory, we satisfy a radiation condition that energy must be outgoing at $z = \infty$ at disturbance frequencies $\omega$ for which the jet is mathematically stable and does not break up. This allows us to exclude two branches of the frequency spectrum (the dashed branches in Fig. 1) and correspondingly two of the four wave eigenfunctions are deleted. The remaining two branches (solid lines in Fig. 1) are retained and the corresponding eigenfunctions are used to satisfy (4).

A straightforward perturbation expansion is assumed, of the form

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \cdots,$$
$$v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 + \cdots. \tag{5}$$

This substituted into (1) and (2) leads to the following problems:

$$O(1):\ \phi_0 = 1, \quad v_0 = 1, \quad \phi_0(0, t) = 1, \quad v_0(0, t) = 1. \tag{6}$$

$$O(\epsilon):\ 2\phi_{1_t} + 2\phi_{1_z} + v_{1_z} = 0,$$
$$v_{1_t} + v_{1_z} - \frac{1}{8}(v_{1_{zzt}} + v_{1_{zzz}}) - \frac{1}{\beta^2}(\phi_{1_z} + \phi_{1_{zzz}}) = 0,$$
$$\phi_1(0, t) = 0, \quad v_1(0, t) = \cos(\omega t). \tag{7}$$

$$O(\epsilon^2):\ 2\phi_{2_t} + 2\phi_{2_z} + v_{2_z} = -(\phi_1 v_{1_z} + 2v_1\phi_{1_z}),$$
$$v_{2_t} + v_{2_z} - \frac{1}{8}(v_{2_{zzt}} + v_{2_{zzz}}) - \frac{1}{\beta^2}(\phi_{2_z} + \phi_{2_{zzz}})$$
$$= -v_1 v_{1_z} + \frac{1}{2}\phi_{1_z}(v_{1_{zt}} + v_{1_{zz}}) + \frac{1}{8}v_1 v_{1_{zzz}}$$
$$+ \frac{1}{4}\phi_1(v_{1_{zzt}} + v_{1_{zzz}}) + \frac{1}{\beta^2}\phi_{1_z}(\phi_{1_{zz}} - 2\phi_1),$$
$$\phi_2(0, t) = 0, \qquad v_2(0, t) = 0. \tag{8}$$

The O(1) solution (6) satisfies (1), (2), and (4) when $\epsilon = 0$, and it represents the constant radius, constant velocity solution of the unperturbed jet. The $O(\epsilon)$ problem (7) is that of the linearized theory studied in [2], where $\delta$, $w$, and $w_0$ were used in place of $\phi_1$, $v_1$, and $\epsilon$. The solution obtained there for $\epsilon\phi_1$, the radius perturbation, is

$$\epsilon\phi_1(z, t) = \frac{\epsilon}{2\omega}\mathrm{Re}\left(\frac{k_1k_2}{k_1 - k_2}\{\exp[i(\omega t - k_2 z)] - \exp[i(\omega t - k_1 z)]\}\right), \tag{9}$$

in which $k_1$, $k_2$ represent the roots of the dispersion relation

$$\beta^2(\omega - k)^2 = 4k^2(k^2 - 1)/(k^2 + 8), \tag{10}$$

which belong to branches of the frequency spectrum that pass through the point $(\omega, k) = (0, 0)$. When $\omega < 1$, $k_1$ and $k_2$ are complex conjugates and $k_1 = k_1^R + ik_1^I$ is chosen as the root with $k_1^I > 0$. In this case (9) can be written in the real form

$$\epsilon\phi_1(z, t) = -\frac{\epsilon}{2\omega}\frac{|k_1|^2}{k_1^I}\sinh(k_1^I z)\sin(\omega t - k_1^R z). \qquad (11)$$

Thus the disturbance is unstable for $\omega < 1$ and grows exponentially with distance $z$. For high velocity jets (large $\beta^2$) $k_1^R \simeq \omega$ and $k_1^I << 1$. In this case (11) gives

$$|\epsilon\phi_1(z, t)| = 1 \quad \text{at} \quad z = z_B \simeq \frac{1}{k_1^I}\ln\left(\frac{4k_1^I}{\epsilon k_1^R}\right), \qquad (12)$$

where $z_B$ is the distance to breakoff as predicted by the linear theory. Although the linear theory is not expected to give valid results when $\phi_1 = O(\epsilon^{-1})$ this value of $z_B$ has been shown to be in agreement with experimental observation. When $\omega > 1$, $k_1$ and $k_2$ are real. In this case (9) takes the real form

$$\epsilon\phi_1(z, t) = \frac{\epsilon}{2\omega}\frac{k_1 k_2}{k_1 - k_2}[\cos(\omega t - k_2 z) - \cos(\omega t - k_1 z)], \qquad (13)$$

which is a stable solution.

A somewhat different form of the linear solution (9) is given by

$$\phi_1(z, t) = \sum_{j=1}^{2} C_j k_j \cos(\omega t - k_j z), \qquad (14)$$

where

$$C_1 = -\frac{1}{2\omega}\frac{k_2}{k_1 - k_2}, \qquad C_2 = \frac{1}{2\omega}\frac{k_1}{k_1 - k_2}. \qquad (15)$$

This form also reduces to (11) for complex conjugate $k_1$, $k_2$ and it is the same as (13) for real $k_1$, $k_2$. Corresponding to (14) is the expression for $v_1$,

$$v_1(z, t) = \sum_{j=1}^{2} 2C_j(\omega - k_j)\cos(\omega t - k_j z). \qquad (16)$$

The linear theory solution (14), (16) appears to be of the same form as that given by Pimbley [8]; however, there are some important differences. Pimbley's solution, based on different one-dimensional equations than (1) and (2), is a four-term sum rather than two. He prescribes four boundary conditions at $z = 0$ rather than the two in (7) and retains all four eigenfunctions corresponding to the four roots of his equation corresponding to (10). Equation (10) has, for large $\beta^2$, four complex conjugate roots for $\omega < 1$ (see Fig. 1), whereas Pimbley's [8] equation has the two roots corresponding to the solid branches; but the complex dashed branches in Fig. 1, which also occur in the three-dimensional investigation in Keller et al. [6], are replaced in [8] by two real branches. Since these real roots give stable contributions, their inclusion into the solution in [8] causes no obvious discrepancy in the results of interest, i.e., near $z = z_B$. However, as is shown in [2], the well-posed boundary value problem permits only two conditions at $z = 0$.

### Derivation of the second perturbation solution

Next we consider the linear problem represented by (8). Observe that the differential operators in (8) for $\phi_2$ and $v_2$ are the same as in (7) for $\phi_1$ and $v_1$, the right-hand sides are determined by $\phi_1$ and $v_1$, and the boundary conditions are homogeneous. It follows that

$$\phi_2 = \phi_2^H + \phi_2^P, \qquad v_2 = v_2^H + v_2^P, \qquad (17)$$

where $\phi_2^H$, $v_2^H$ satisfy the first two of (8) with the right-hand sides set equal to zero, while $\phi_2^P$, $v_2^P$ are particular solutions. Furthermore,

$$\phi_2^H(0, t) = -\phi_2^P(0, t), \qquad v_2^H(0, t) = -v_2^P(0, t). \qquad (18)$$

Thus we take

$$\phi_2^H(z, t) = \sum_{j=1}^{2}\tilde{C}_j\tilde{k}_j\cos(\tilde{\omega}t - \tilde{k}_j z) + C,$$

$$v_2^H(z, t) = \sum_{j=1}^{2} 2\tilde{C}_j(\tilde{\omega} - \tilde{k}_j)\cos(\tilde{\omega}t - \tilde{k}_j z) + D, \qquad (19)$$

where $\tilde{k}_1$, $\tilde{k}_2$ are the appropriate two roots of (10) corresponding to $\tilde{\omega}$, which must be determined from $\phi_2^P$, $v_2^P$. The need for the constants $C$ and $D$ in (19) will be apparent later. In order to determine $\phi_2^P$, $v_2^P$ we must first calculate the right-hand sides in (8). By use of (14) and (16) it is found that

$$-(\phi_1 v_{1_z} + 2v_1\phi_{1_z}) = \sum_{j=1}^{2} f_{jj}\sin(2\theta_j) + f_{12}^E\sin(\theta_1 + \theta_2) + f_{12}^0\sin(\theta_1 - \theta_2), \qquad (20)$$

in which

$$\theta_j = \omega t - k_j z, \qquad (21)$$

and

$$f_{jj} = 3C_j^2k_j^2(k_j - \omega), \qquad j = 1, 2,$$
$$f_{12}^E = C_1C_2[3k_1k_2(k_1 + k_2) - 2\omega(k_1^2 + k_1k_2 + k_2^2)],$$
$$f_{12}^0 = C_1C_2(k_1 - k_2)[3k_1k_2 - 2\omega(k_1 + k_2)]. \qquad (22)$$

Likewise the right-hand side of the second of (8) is

$$\sum_{j=1}^{2} g_{jj}\sin(2\theta_j) + g_{12}^E\sin(\theta_1 + \theta_2) + g_{12}^0\sin(\theta_1 - \theta_2), \qquad (23)$$

where

$$g_{jj} = C_j^2\left[\frac{1}{2}(\omega - k_j)^2(k_j^3 - 4k_j) - \frac{1}{2\beta^2}k_j^3(k_j^2 + 2)\right],$$
$$j = 1, 2,$$

$$g_{12}^{E} = C_1C_2\Big\{-\frac{1}{4}[8(k_1 + k_2) + (k_1^3 + k_2^3)](\omega - k_1)(\omega - k_2)$$
$$- \frac{1}{2\beta^2}(k_1 + k_2)k_1k_2(2 + k_1k_2)$$
$$+ \frac{1}{4}k_1k_2[(k_1 + 2k_2)(\omega - k_1)^2 + (k_2 + 2k_1)(\omega - k_2)^2]\Big\},$$

$$g_{12}^{0} = C_1C_2\Big\{-\frac{1}{4}[8(k_1 - k_2) + (k_1^3 - k_2^3)](\omega - k_1)(\omega - k_2)$$
$$- \frac{1}{2\beta^2}(k_1 - k_2)k_1k_2(2 - k_1k_2)$$
$$+ \frac{1}{4}k_1k_2[(k_1 - 2k_2)(\omega - k_1)^2$$
$$- (k_2 - 2k_1)(\omega - k_2)^2]\Big\}. \qquad (24)$$

In view of (20), (23) we assume for the particular solution

$$\phi_2^{P} = \sum_{j=1}^{2} A_{jj}\cos(2\theta_j) + A_{12}^{E}\cos(\theta_1 + \theta_2)$$
$$+ A_{12}^{0}\cos(\theta_1 - \theta_2),$$

$$v_2^{P} = \sum_{j=1}^{2} B_{jj}\cos(2\theta_j) + B_{12}^{E}\cos(\theta_1 + \theta_2)$$
$$+ B_{12}^{0}\cos(\theta_1 - \theta_2), \qquad (25)$$

and we determine the $A$ and $B$ coefficients by substitution into (8) with right-hand sides given by (20) and (23). In this manner we obtain the four linear systems

$$2k_jB_{jj} - 4(\omega - k_j)A_{jj} = f_{jj}, \qquad j = 1, 2,$$

$$-(\omega - k_j)(2 + k_j^2)B_{jj} - \frac{2}{\beta^2}(k_j - 4k_j^3)A_{jj} = g_{jj},$$
$$j = 1, 2,$$

$$(k_1 + k_2)B_{12}^{E} + 2(k_1 + k_2 - 2\omega)A_{12}^{E} = f_{12}^{E},$$

$$(k_1 + k_2 - 2\omega)\left[1 + \frac{1}{8}(k_1 + k_2)^2\right]B_{12}^{E}$$
$$- \frac{1}{\beta^2}[k_1 + k_2 - (k_1 + k_2)^3]A_{12}^{E} = g_{12}^{E},$$

$$(k_1 - k_2)B_{12}^{0} + 2(k_1 - k_2)A_{12}^{0} = f_{12}^{0},$$

$$\left[k_1 - k_2 + \frac{1}{8}(k_1 - k_2)^3\right]B_{12}^{0}$$
$$- \frac{1}{\beta^2}[k_1 - k_2 - (k_1 - k_2)^3]A_{12}^{0} = g_{12}^{0}, \qquad (26)$$

which determine $B_{jj}$, $A_{jj}$, $B_{12}^{E}$, $A_{12}^{E}$, $B_{12}^{0}$, $A_{12}^{0}$ and hence $\phi_2^{P}$, $v_2^{P}$ (except at $\omega = 1/2$ or $1$ for which the coefficient determinant for $B_{12}^{E}$, $A_{12}^{E}$ vanishes).

Next we determine the $\tilde{C}_j$ in (20) so that $\phi_2^{H}$, $v_2^{H}$ satisfy the boundary conditions (18) with (25). From (25), (21),

$$\phi_2^{P}(0, t) = \left(\sum_{j=1}^{2} A_{jj} + A_{12}^{E}\right)\cos(2\omega t) + A_{12}^{0},$$

$$v_2^{P}(0, t) = \left(\sum_{j=1}^{2} B_{jj} + B_{12}^{E}\right)\cos(2\omega t) + B_{12}^{0}, \qquad (27)$$

so that (18), (19) require

$$C = -A_{12}^{0}, \qquad D = -B_{12}^{0}, \qquad \tilde{\omega} = 2\omega,$$

$$\tilde{C}_1 = [-(\tilde{\omega} - \tilde{k}_2)(A_{11} + A_{22} + A_{12}^{E})$$
$$+ \tilde{k}_2(B_{11} + B_{22} + B_{12}^{E})/2]/\Delta,$$

$$\tilde{C}_2 = [(\tilde{\omega} - \tilde{k}_1)(A_{11} + A_{22} + A_{12}^{E})$$
$$- \tilde{k}_1(B_{11} + B_{22} + B_{12}^{E})/2]/\Delta,$$

$$\Delta = \tilde{k}_1(\tilde{\omega} - \tilde{k}_2) - \tilde{k}_2(\tilde{\omega} - \tilde{k}_1). \qquad (28)$$

This completes the second perturbation solution.

## Second perturbation numerical results

Summarizing the above derivation we have that the second perturbation jet radius is given by

$$\phi(z, t) = 1 + \epsilon\phi_1(z, t) + \epsilon^2[\phi_2^{H}(z, t) + \phi_2^{P}(z, t)], \qquad (29)$$

where $\epsilon$ is the specified amplitude of the applied disturbance in the axial velocity at the nozzle as given by (4), and $\phi_1(z, t)$ is given by (14) and (15) in which $k_1$, $k_2$ are the roots of (10) corresponding to $\omega$ and $\beta$ that belong to the branches that pass through the origin of the frequency spectra. The second perturbation part $\phi_2^{P}(z, t)$ is determined by (22), (24), (25), and (26), while the part $\phi_2^{H}(z, t)$ is determined by (19), (22), (24), (26), and (28). The wave numbers $\tilde{k}_1$, $\tilde{k}_2$ are determined from (10) for $\tilde{\omega}$. Since $\tilde{\omega} = 2\omega$, it follows from our previous discussion that if $\omega > 1/2$, then $\tilde{\omega} > 1$ and $\tilde{k}_1$, $\tilde{k}_2$ will be real. This means that $\phi_2^{H}$ in (19) will be stable harmonic traveling waves. However, $\phi_2^{P}$ in (25) has wave numbers $2k_1$, $2k_2$, which are complex for $\omega < 1$. These terms have exponential growth in $z$ at twice the rate of $\phi_1$ but are multiplied in (29) by $\epsilon^2$ rather than $\epsilon$. This second perturbation solution is not uniformly valid but is singular at $\omega = 1/2$ and $\omega = 1$ for the reason mentioned above.

The second perturbation radius given by (29) was programmed and numerical results were obtained for $\beta^2 = 250$ and $\omega = 0.525$ (corresponding to the parameter values $\varepsilon^2 = 0.002$ and $\lambda/d = 6$ in Pimbley and Lee [4]). Values of $\epsilon$ were chosen which gave break-off lengths $z_B$

(i.e., the minimum value of $z$ for which $\phi(z, t)$ can be zero) corresponding to those values presented in Fig. 7 of [4]. The results are shown in Fig. 2. Here we see that satellite drops are apt to form but due to the fact that downstream ligament separation occurs first they would be backward merging for all values of the break-off length $z_B$. This is somewhat surprising since Pimbley and Lee [4] obtained in their Fig. 7, for the same set of parameters, upstream ligament separation and hence forward merging satellites for $z_B < 90$. (Recall that their experimental results for these parameters gave forward merging satellites for $z_B < 185$.) It was anticipated that the one-dimensional Cosserat theory used here, which includes radial inertia, would show better agreement with experiment than Pimbley and Lee were able to obtain using Lee's one-dimensional equation. After careful study and recalculation of their results it can be concluded that Pimbley and Lee's forward merging satellite predictions are a result of the way they formulated the problem. As mentioned earlier, they set four boundary conditions at the nozzle, whereas the proper formulation (on the basis of the uniqueness theorem) allows only two conditions to be set at the nozzle, and the other two conditions must be imposed downstream. As explained in [2] we satisfied a downstream-radiation-type condition by the exclusion of two of the four branches of the frequency spectra. The significant effect of the additional two boundary conditions enters Pimbley and Lee's [4] second perturbation solution through their equations (22) and (23). Their $\delta_{2g}$ should be a two-term summation rather than four. Then the $\tilde{C}_j$ in their (23) would not depend on $R_3$ and $R_4$ since, from their (19), these quantities arise from the two extraneous boundary conditions at the nozzle.

There is reason to believe that at least a third perturbation solution is required to adequately describe the satellite drop behavior. This is the case in the treatment of the Rayleigh infinite jet problem by use of three-dimensional theory as done by Lafrance [9], where the third perturbation is required to predict correct size ratios of satellite to main drops.

## Conclusions

The second perturbation solution obtained here predicts satellites but always in the backward merging condition. The forward merging satellites predicted by Pimbley and Lee [4] in their second-order perturbation solution using Lee's [7] one-dimensional equations are a result of their formulation of the problem in which they set four boundary conditions at the nozzle rather than two.

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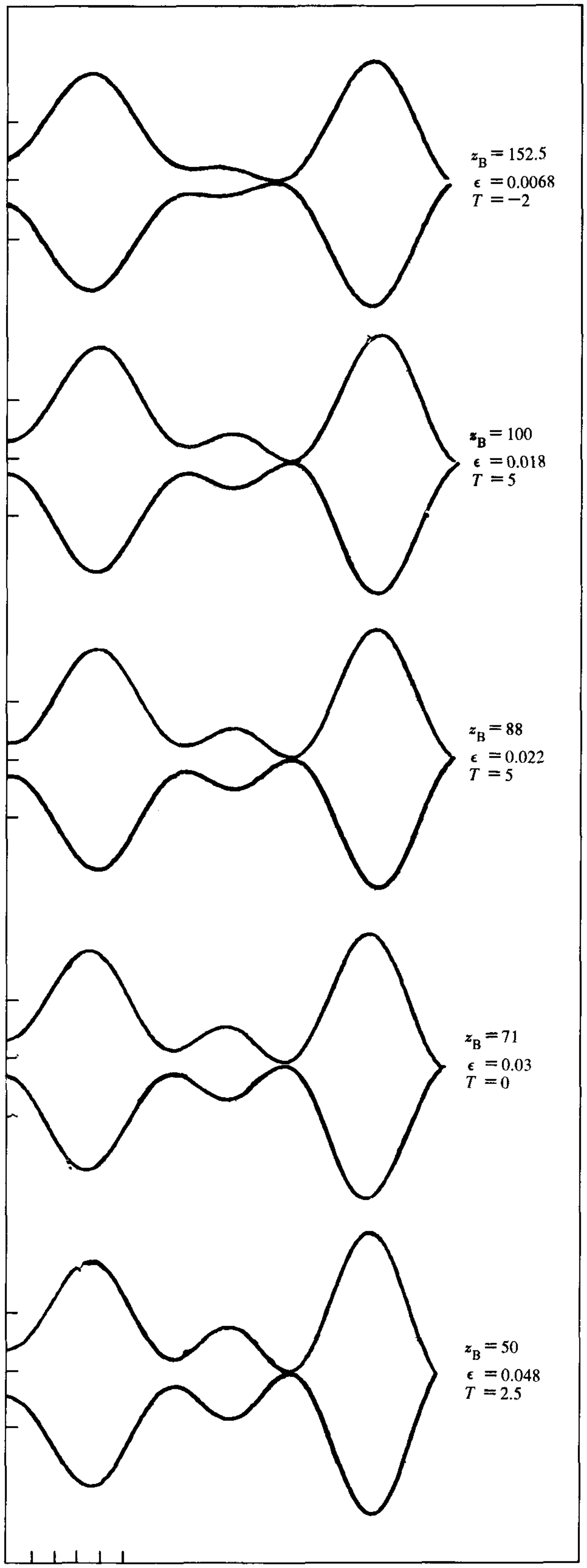


**Figure 2** Second perturbation drop formation shapes near break-off point for various $\epsilon$: $\beta^2 = 250$, $\omega = 0.525$.

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