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Influence of Scattering and Ionization on RF Impedance in Glow Discharge Sheaths

The effects of scattering and ionization on the rf impedance of a glow discharge sheath are calculated using an equivalent dc sheath model. The effects of scattering are treated in terms of a drag force; equilibrium between the ion drift velocity and field is not required. The ratio of first ionization coefficient to pressure, α/p, is assumed to be constant, and the ion energy and ion current injected from the glow are assumed as initial parameters. In the limit of low pressures, the calculation agrees with the Child-Langmuir law. At intermediate pressures, the results agree with the mobility limited solution. At high pressures, the product of pressure and sheath dimension, pd, becomes constant because of ionization effects. The results of this calculation, obtained by numeric integration, can be accurately approximated by an interpolation formula. This formula provides a simple means for calculating the rf impedance of a sheath.

Introduction

In the preceding paper, Keller and Pennebaker [1], in their calculation of the electrical properties of the rf sputtering system, use an equivalent dc sheath model to calculate the impedance of the rf sheath. The simplest form of this model was first proposed by Koenig [2], who used the Child-Langmuir law [3] to relate the sheath thickness d to the average potential across the sheath, and to the ion current injected into the sheath from the glow. The sheath capacitance was assumed to be given by the thickness of this sheath and the real component of the sheath impedance was assumed to be small. As shown in the preceding paper [1], the equivalent parallel resistance can be calculated from the sheath voltage and ion current density; this was not done in Koenig's work.

Aside from neglecting the time dependence of the potential (apparently justified by the excellent results obtained with the dc approximation), the major drawback to the use of the Child-Langmuir law is that the law is valid only at very low pressures; it assumes that the ions "free fall" from glow to boundary. At higher pressures, both scattering (primarily due to charge exchange [4]) and ionization should substantially increase the positive charge density in the sheath, and consequently decrease the distance across the sheath. This, of course, will affect the sheath impedance.

This paper describes a calculation of the influence of gas pressure on the dc sheath. Both scattering and ionization are included in the calculation, although ionization effects are treated approximately. Several calculations of the cathode sheath, also incorporating scattering and ionization, have been given by Ward [5, 6]. However, Ward's treatments start with the assumption that the ion drift velocity $\bar{v}$ is a function of the field divided by the pressure, E/p. For values of E/p greater than 0.75 V/Pa-cm (100 V/torr-cm),

$$\bar{v} = k(E/p)^{1/2}. \tag{1}$$

While this relationship is derived both from theory and experiment [7] the experiment is always done with a constant field. Warren [8] has noted that the field in the cathode fall region varies so rapidly with distance that the assumption of a drift velocity in equilibrium with the field—that is, the use of Eq. (1)—is highly questionable. This is a particularly important point in a calculation of the dc

equivalent sheath, for in this problem the interest is mainly in that pressure region where the ions *almost* free fall through the sheath. When ions almost free fall, their velocity is determined uniquely by neither the field nor the potential.

In this paper the scattering effects are included through the use of an equivalent drag force. The neutral gas is treated as a viscous medium, and the drag force caused by this medium is derived from Eq. (1). The ionization effects are treated qualitatively, in that α/p (the first ionization coefficient divided by pressure) is assumed to be constant. There are no data on α/p for $E/p > 7.5$ V/Pa-cm (10^3 V/torr-cm), but the best extrapolation of existing data [7] indicates $\alpha/p \approx 0.1$ (Pa-cm)$^{-1}$ [14 (torr-cm)$^{-1}$] for argon. Even if the variation in α/p with E/p were completely known, its use would not be justified. Just as the ion is not in equilibrium with the field, neither is the electron. Further arguments will be presented that show the assumption of constant α/p to be reasonably consistent with the results of this paper.

No attempt is made to calculate the rate of ion injection from the glow. This is assumed as an initial condition, as is the injection energy. In Ward's analysis these two parameters were calculated from a knowledge of ionization caused by secondary electrons passing through the glow region. From purely empirical observations, the rf discharge can be sustained at pressures substantially lower than those required for the equivalent dc discharge. In the preceding paper we note that this is due to the large displacement current flowing through the sheaths. Therefore, it is more reasonable to assume ion injection parameters than to calculate them from secondary-electron ionization.

General theory

If the first Townsend ionization constant α is a constant, the ion current density at any point x is related to the ion current density at the boundary J_{ob} by

$$J_o(x) = J_{ob}\{1 - \gamma[\exp(\alpha x) - 1]\}, \tag{2}$$

where γ is the second Townsend coefficient. This follows from the fact that the total current is constant, given by $J_{ob}(1 + \gamma)$; the electron current is given by $\gamma J_{ob} \exp(\alpha x)$. Therefore, Eq. (2) follows. The coordinates are defined such that x is zero at the boundary. The point $x = d$ is defined to be the transition point between the sheath and the negative glow. When $x > d$, the electron density can no longer be neglected. The potential change, in going from $x = d$ to a point where the electron and ion densities are approximately equal, is assumed to be negligible. Certainly it is considerably less than the wall voltage, and the wall voltage in turn is usually considerably less than the total voltage across the sheath.

The change in the ion current density in an interval dx is, from Eq. (2),

$$J_o(x) = J_{ob}\{1 - \gamma[\exp(\alpha x) - 1]\}, \tag{3}$$

The ion density at a point x' due to ions injected from the glow is

$$N_{+i}(x') = \frac{J_o}{ev(d, x')}, \tag{4}$$

where v is the velocity the ion attains in traveling from d to x', and $J_o \equiv J_o(d)$.

The ion density due to ionization within the sheath is given by

$$N_{+i}(x') = \int_{x'}^{d} \frac{-dJ_o(x)}{ev(x, x')}, \tag{5}$$

where $v(x, x')$ is the velocity at point x' of the ion created at x.

Since the electrons are quickly removed from the sheath by the large fields, their effect on the space charge can be safely neglected. (Their neglect may not be justified in the transition region between sheath and glow, but by assumption the potential drop across that region can be neglected.) The total space charge is thus the sum of Eqs. (4) and (5).

The velocity of the ion is obtained as follows. In the limit of low pressures the velocity is given by

$$v = \left[\frac{2e}{M}(V + \phi_o)\right]^{1/2}, \tag{6}$$

where ϕ_o is the initial energy at the point where $V = 0$, and V is the potential relative to that point. The force on the ion in order for Eq. (6) to be valid is just $F = eE$.

As noted earlier, the ion drift velocity $\bar{v}$ in a constant field in a gas at pressure p is given by Eq. (1) ($E/p > 0.75$ V/Pa-cm or 100 V/torr-cm).

For this velocity to be constant, the force due to the field must be exactly balanced by a drag force due to scattering. If we assume that the gas can be regarded as a viscous medium insofar as scattering effects are concerned, the drag force is given by

$$F_d = \frac{ep}{k^2}\bar{v}^2, \tag{7}$$

and the equation of motion by

$$\bar{v} = \left(\frac{1}{2}\right)\frac{d(\bar{v}^2)}{dx} = \frac{eE}{M} + \frac{ep}{k^2M}\bar{v}^2. \tag{8}$$

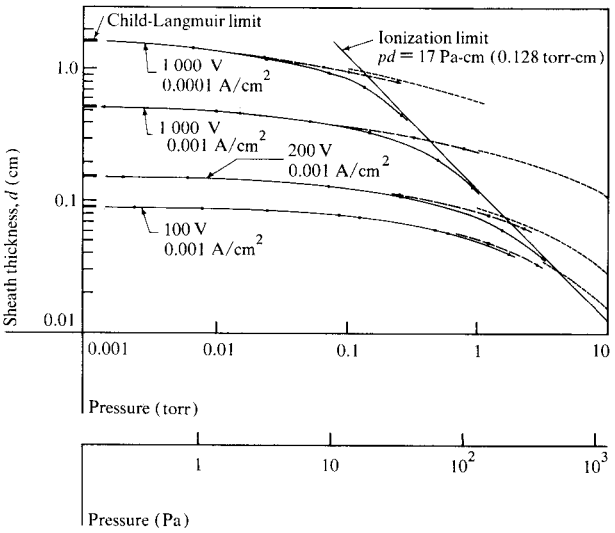

Figure 1 Sheath dimension as a function of pressure for various voltages and injected ion current densities. The curves are calculated for argon assuming $\alpha/p = 0.1$ (Pa-cm)$^{-1}$ [14 (torr-cm)$^{-1}$] (where applicable), $\gamma = 0.2$, and a 1-eV ion-injection energy. Dashed lines without data points give the curves for the scattering only, mobility limited solution. Dashed lines with data points are for the scattering only, numerically calculated solution. Solid lines represent the numerically calculated solution involving both scattering and ionization.

Equation (8) is a linear, first-order differential equation with the solution [9]

$$[\bar{v}(x')]^2 = \frac{\int_x^{x'} \left(\frac{2eE}{M}\right) \exp\left[(x - x')\frac{2ep}{k^2M}\right] dx'}{\exp\left[(x - x')\frac{2ep}{k^2M}\right]} + [\bar{v}(x)]^2. \quad (9)$$

If the field is known, this equation can be used with Eqs. (4) and (5) to obtain the positive space charge in the sheath. When these equations are combined with Poisson's equation the problem is, in principle, solved.

The boundary conditions on the voltage required for the solution of Poisson's equation are $V = 0$ at $x = d$, and $V = V_M$ (the maximum voltage) at $x = 0$. In addition we must determine the field at $x = d$. If the Child-Langmuir law is assumed to be the limiting low pressure solution, the field at $x = d$ would be zero at low pressure. However, the Child-Langmuir law does not account for the nonzero injection energy. When the injection energy ϕ_0 is properly incorporated into this low pressure solution (a modification of the Child-Langmuir law that is discussed in more detail in the following section), the field at $x = d$ is given by

$$E_{x=d} = -\frac{4}{3}\left(\frac{\phi_0}{e}\right)^{1/4}\left[\left(\frac{9}{4\epsilon_0}\right)^2 \frac{M}{2e}\right]^{-1/4} J_0^{1/2}. \quad (10)$$

At higher pressures there is no good guide to the field value at the edge of the glow. However, from Eq. (8), if the field is less than

$$E_{x=d} = -\frac{p}{k^2}\bar{v} = -\frac{2ep}{k^2M}\left(\frac{\phi_0}{e}\right), \quad (11)$$

the ion injected from the glow will slow down initially. This is physically unreasonable. Consequently, we define the edge of the glow to be that point where the field is equal to the sum of Eqs. (10) and (11);

$$E_{x=d} = -\left\{\frac{4}{3}\left(\frac{\phi_0}{e}\right)^{1/4}\left[\left(\frac{9}{4\epsilon_0}\right)^2 \frac{M}{2e}\right]^{1/4} J_0^{1/2}\right\} - \frac{2ep}{k^2M}\left(\frac{\phi_0}{e}\right). \quad (12)$$

This expression ensures that the field at the edge of the sheath is correct in the limit of low pressures, and sufficiently large at high pressures that the ion does not slow down after injection. The two terms in the above expression are of comparable magnitude at 133 Pa in argon.

In specifying that $V = V_M$ at $x = 0$, and in requiring Eq. (12) to hold, we have uniquely specified the solution. The additional boundary condition, $V = 0$ at $x = d$, is not redundant, however. This third condition relates the distance across the sheath d to the voltage across the sheath. The relationship is given in the Appendix, along with a description of the numeric techniques used to obtain a solution of the equations given in this section.

Low and high pressure approximations

As has been noted, at low pressures the solutions described in the preceding section should be identical with the Child-Langmuir law. When corrected for nonzero injection energy, the Child-Langmuir law is

$$-V + \frac{\phi_0}{e} = \left[\left(\frac{9J_0}{4\epsilon_0}\right)^2 \frac{M}{2e}\right]^{1/3} (d - x + a)^{4/3}, \quad (13)$$

where a is a distance defined by

$$a = \left[\frac{2e}{M}\left(\frac{4\epsilon_0}{9J_0}\right)^2\right]^{1/4}\left(\frac{\phi_0}{e}\right)^{3/4}, \quad (14)$$

and ϕ_0 is the injection energy.

At higher pressures, when ionization can be ignored, the field varies so little over a scattering distance that Eq. (1) holds. Poisson's equation can then be solved explicitly, and the solution (known as the mobility limited solution [10]) is

$$V = -\frac{3}{5}\left[\frac{3J_0(p)^{1/2}}{2\epsilon_0 k}\right]^{2/3}\left[(d - x + a)^{5/3} - a^{5/3}\right], \quad (15)$$

where

$$a = \frac{2}{3}\left(\frac{2e}{M}\right)^{3/2} \frac{p\epsilon_0}{k^2 J_0}\left(\frac{\phi_0}{e}\right)^{3/2}. \tag{16}$$

In this high pressure limit the field at the sheath-glow boundary is

$$E|_{x=d} = \frac{2ep}{k^2 M}\left(\frac{\phi_0}{e}\right), \tag{17}$$

in agreement with Eq. (1).

If the ionization coefficient is not zero, ionization effects must dominate the charge density in the sheath in the limit of high pressures. This occurs when the total ion current to the boundary is much larger than the ion current injected from the glow. From Eq. (2) the ion current at the boundary becomes infinite when the product of pressure and distance across the sheath, pd, is given by

$$pd = \frac{1}{\alpha/p} \ln\left(\frac{1+\gamma}{\gamma}\right). \tag{18}$$

Thus, in the limit of very high pressures, the distance across the sheath is inversely proportional to the pressure, and dependent on the other discharge parameters only to the extent that α and γ are [11].

Numeric results—application to an argon discharge

In the Appendix it is shown that the various equations describing the space charge in the sheath can be expressed in terms of the relative coordinate x/d. When this is done, the parameters which must be specified in order to arrive at a charge distribution are the pressure-distance product, pd; the ion current density injected from the glow, J_0; the voltage across the sheath, V_M; the ion injection energy, ϕ_0; the first ionization coefficient divided by pressure, α/p; and the second ionization coefficient, γ. In addition, a parameter V_L, a voltage interval in the vicinity of ionization, is required but V_L must simply be made small enough that the charge density is not dependent on it.

Referring again to the Appendix, Eq. (A10) gives the value of d appropriate to the specific charge distribution calculated. Since pd is specified, the pressure is thus determined.

The numeric results for an injection energy of 1 eV and a secondary-electron emission coefficient of 0.2 are shown in Figs. 1 and 2 for several sheath voltages and ion injection current densities. The ratio of ionization coefficient to pressure α/p was assumed to be either 0.1 $(\text{Pa-cm})^{-1}$ [14 $(\text{torr-cm})^{-1}$] or zero, as noted. The Child-Langmuir low pressure limit, the mobility limited solution, and the limiting pd product (which occurs when ionization effects dominate) are all shown.

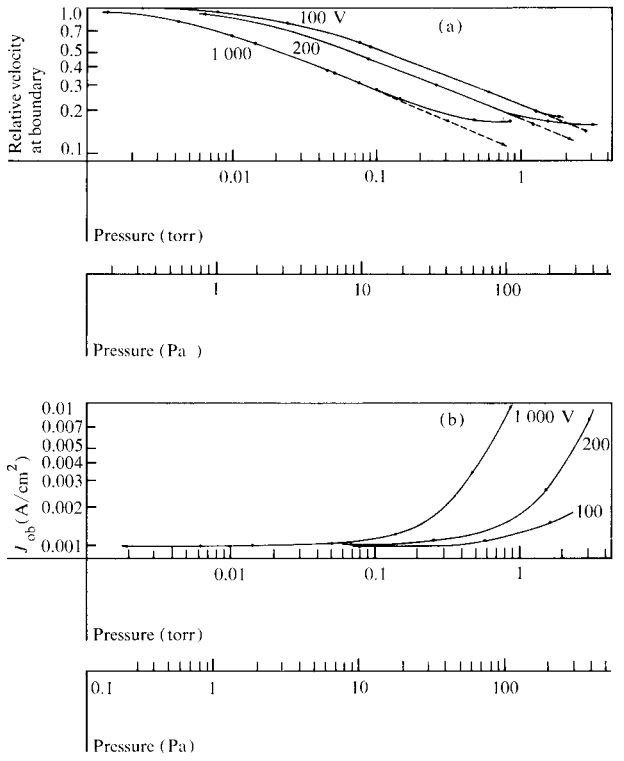

Figure 2 (a) Relative velocity of injected ions striking the boundary; same assumed parameters as in Fig. 1. Dashed lines give the scattering only cases; solid lines, the scattering and ionization cases. (b) Ion current density at the boundary as a function of pressure.

It is apparent that the numerically calculated curves approach the various limits quite reasonably. The somewhat greater difference at lower voltages between the numeric results for no ionization and the high pressure limit may be attributed to the difference in field assumed at the edge of the glow [compare Eq. (12) with Eq. (17)]. The curves would have to be extended to somewhat higher pressures before they would merge.

One test of the calculation is that if ionization is allowed, the field should be essentially linear at higher pressures. A typical result is shown in Fig. 3.

RF sheath impedance in an argon discharge

The main purpose of the preceding calculations was to obtain information on the rf impedance of the sheath as a function of pressure. Unfortunately, the calculations are too lengthy to be of much use in calculating rf sputtering system operation. Therefore, an approximate interpolation formula was devised by trial and error, which describes the curves in Fig. 1 quite well:

$$d = \left[\left(\frac{1}{d_L}\right)^4 + \left(\frac{1}{d_M}\right)^4 + \left(\frac{1}{d_I}\right)^4\right]^{-1/4}, \tag{19}$$

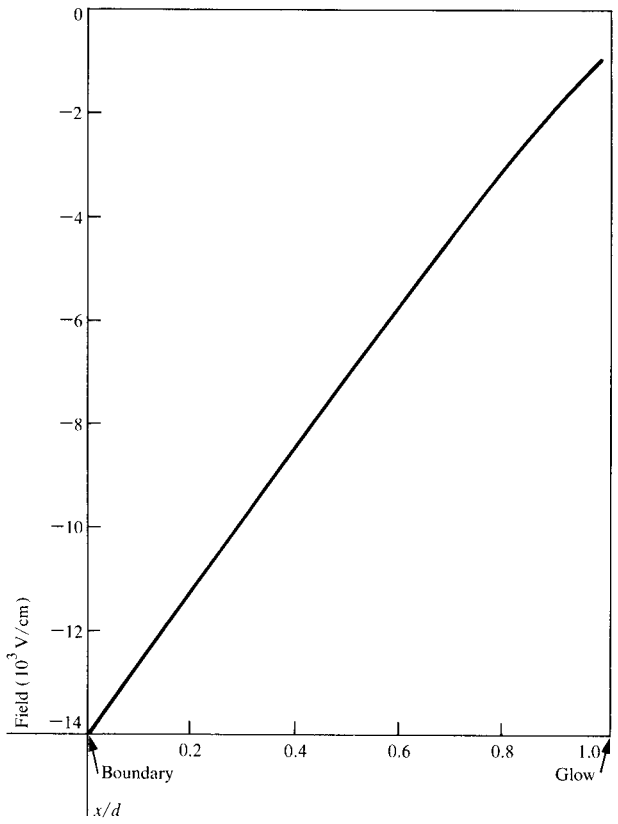

Figure 3 Electric field as a function of position in the sheath, illustrating the linear field typically encountered when ionization effects dominate. The following parameters apply: $\alpha/p = 0.1$ (Pa-cm)$^{-1}$ [14 (torr-cm)$^{-1}$], $\gamma = 0.2$, $\phi_o = 1$ eV, $J_o = 0.001$ A/cm^2, $p_{calc} = 118$ Pa (0.887 torr), and $d_{calc} = 0.137$ cm.

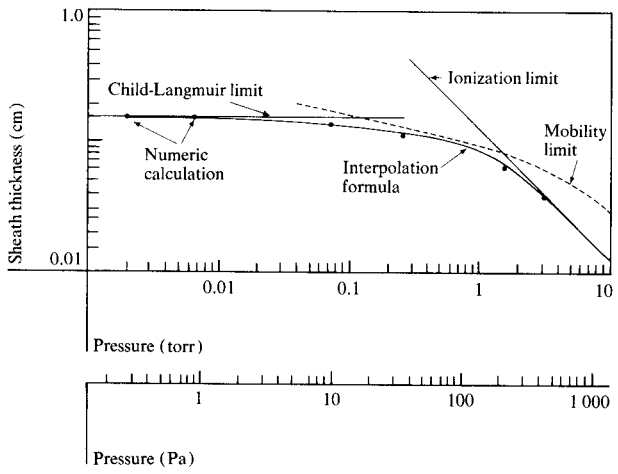

Figure 4 Comparison of numeric calculation with interpolation formula. The following parameters apply: $V_M = 200$ V, $J_o = 0.001$ A/cm^2, $\alpha/p = 0.1$ (Pa-cm)$^{-1}$, and $\gamma = 0.2$.

where d_L, d_M, and d_i are the sheath dimensions calculated using Eqs. (13), (15), and (18), respectively. That is, d_L is the low pressure sheath dimension, d_M is the mobility limited sheath dimension, and d_i is the sheath dimension when ionization effects dominate. This interpolation formula is compared with the 200-V curves from Fig. 2 in Fig. 4. While the agreement is generally excellent, the numerically integrated curves fall below the interpolation curves in the transition region betwen low and high pressure limits. It should be noted that the discrepancy may be attributed to the numeric curves rather than the interpolation formula. The field [see Eq. (12)] at the sheath-glow boundary may be over-estimated in this transition region, and this would cause d to be too small there.

Following Koenig [2], the sheath capacitance is given by

$$C = \frac{\epsilon_o A}{d}. \tag{20}$$

The parallel resistance R, calculated from the power dissipation in the sheath, is given by

$$R = \frac{V_o^2\{1 - \gamma[\exp(\alpha d) - 1]\}}{2J_o A(1 + \gamma)(V_M - V_f)}, \tag{21}$$

which is the same as the equation for the parallel resistance R given in the preceding paper [Eq. (13)], except for the factor describing the increase in ion current due to ionization. V_f is the floating potential in the absence of an rf voltage across the sheath. The relationship between V_o, the rf voltage across the sheath, and $(V_M - V_f)$ is given in Ref. [1]. We note that if $V_o >> kT_e/e$, $V_o \approx V_M$.

The admittance per unit sheath area, given by

$$\frac{G}{A} = \frac{1}{RA} + \frac{j\omega C}{A}, \tag{22}$$

is displayed as a function of pressure in Fig. 5 for a frequency of 13.56 MHz.

For pressures below 13 Pa (0.1 torr) the reactive component is more sensitive to pressure than is the real component. This is a scattering effect—scattering causes the sheath dimension to decrease, but does not increase the power dissipation. Above 13 Pa both components are quite pressure sensitive, but the real component changes more rapidly. This is primarily an ionization effect, for ionization both decreases the sheath dimension and increases the power dissipation.

Conclusions

The purpose of this work has been to extend the equivalent dc sheath model of the rf sheath to higher pressures. A calculation has been given which shows the effects of

scattering and ionization on the sheath dimension for pressures throughout the range of interest in rf sputtering.

Two of the assumptions made in the calculation may introduce small, but perhaps significant, errors. First, a field was assumed at the sheath-glow interface. The only guide in assuming this field was that it should have the correct low and high pressure limits. Further work is needed to clarify this choice. Second, the scattering effects were treated in terms of a drag force. This implies a very well-defined ion velocity. If, as is almost certainly the case, there is a distribution of velocities, errors are introduced in using Eq. (9) in Eqs. (4) and (5). Without knowing the velocity distribution it is difficult to estimate this error. It is also possible that ionization mechanisms other than direct electron impact contribute to ionization in the sheath.

With the basic behavior of the sheath now known, the assumption of a constant α/p can be better justified. Equation (18) gives an upper limit to the product αd; if γ is 0.2, αd is less than 2. Therefore, only a few ionizing collisions can occur during the electron's transit of the sheath. The efficiency of ionization—the ratio of energy lost in ionizing collisions to total energy expended—is known to be 0.46 for argon [12]. Therefore, few collisions occur in the sheath, little energy is lost to collisions, and most electrons (even those created by ionization) are able to attain relatively high energies. The probability of ionization ψ_i, which is equal to α at 133 Pa (1 torr) if the electron velocity distribution is narrow [7], has a maximum of 11 ions/cm at 50 eV [13], and decreases slowly to about 5 ions/cm at 500 eV. Therefore, for the most probable electron energies, α/p is a slowly varying function which can be approximated as a constant.

The value of α/p [0.1 (Pa-cm)$^{-1}$, 14 (torr-cm)$^{-1}$] used in the numeric calculations was obtained from extrapolation of low energy measurements. Judging from the values of ψ_i given above, this estimate of α/p was slightly high; a more reasonable value would have been about 0.04 (Pa-cm)$^{-1}$, [10 (torr-cm)$^{-1}$]; perhaps slightly smaller values would have been appropriate at higher sheath voltages. The effect of a smaller α/p would be to raise the limiting pd value [Eq. (18)], and thus increase the sheath dimension at higher pressures. On the other hand, the data given in Ref. [1] seem to require a value of 0.1 (Pa-cm) [14 (torr-cm)$^{-1}$].

The two approximate equations developed for the sheath capacitance and parallel resistance are perhaps the most useful results of this work. It is now possible to calculate the electrical behavior of rf sputtering systems throughout the range of useful pressures. Another interesting result, which is a by-product of the calculation, is the values obtained for the drift velocity of injected ions when they strike the substrate and target. Figure 2 shows the drift velocity obtained for various pressures relative to that which would occur in the absence of scattering. The decrease in energy of bombardment is surprising. However, if the velocities were distributed around the average value, the average energy would be higher than Fig. 2 indicates.

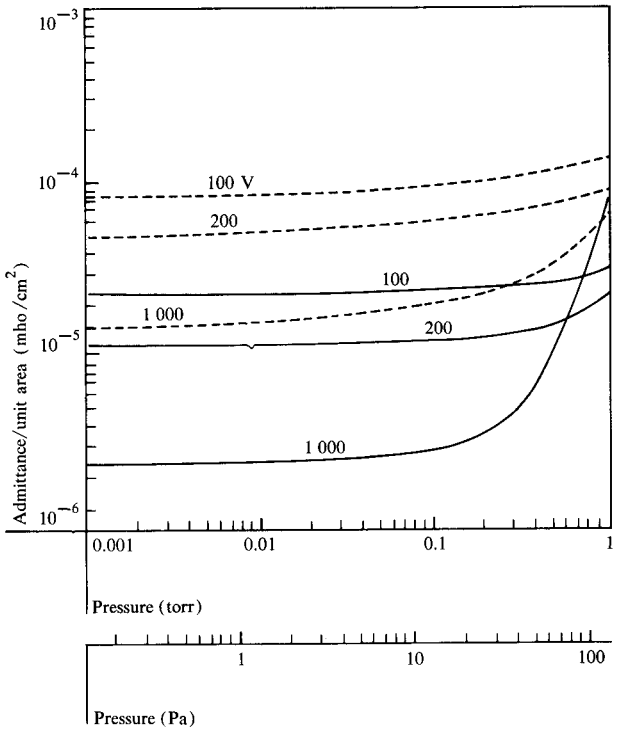

Figure 5 Admittance per unit argon sheath area as a function of pressure, assuming $J_0 = 0.001$ A/cm^2, $\alpha/p = 0.1$ (Pa-cm)$^{-1}$, $\gamma = 0.2$, a 1-eV ion-injection energy, a wall potential of 10 V, and an electron temperature of 1.7×10^4 K (2 eV). Solid lines represent the real component; dashed lines, the reactive component.

Acknowledgments

John Keller is acknowledged for many useful conversations. This work is, in fact, a direct outgrowth of our joint calculations on rf sputtering system operation. H. Koenig is also acknowledged for his ideas on the effect of charge exchange. The results of this calculation appear to support his qualitative conclusions completely.

Appendix: Numeric calculations

If the potential distribution can be expressed as a power series, the equations developed in the second section can be expressed as follows.

If the potential distribution is

$$\frac{V}{V_M} = \sum_n P_n \left(\frac{x}{d}\right)^n, \tag{A1}$$

where V_M is the potential at the target, d is the distance from target to glow, and $V = 0$ at $x = d$, then Eq. (9) can be reduced to

$$v = \left(\frac{2e}{M}\right)^{1/2} V_M^{1/2} \left[\frac{\phi_0/eV_M + \sum_{n=0}^{\infty} I_n}{\exp\left[a(x - x_0)/d\right]}\right]^{1/2}, \tag{A2}$$

where x_0 is the starting point, the initial energy is ϕ_0, and $a = 2ep/k^2Md$.

Note that $(2e/M)^{1/2}\,(|V_M|)^{1/2}$ is the velocity of an ion of energy $e\,|V_M|$. I_n in the above equation is given by

$$I_n = nP_n \exp\left(-\frac{ax_0}{d}\right) S_n, \tag{A3}$$

where

$$S_0 \equiv 0;$$

$$S_n = \frac{(x/d)^{n-1} \exp(ax/d) - (x_0/d)^{n-1} \exp(ax_0/d)}{a} - \left(\frac{n-1}{a}\right) S_{n-1}. \tag{A4}$$

We define a relative field E_r by the equation

$$E_r = \frac{Ed}{|V_M|} = \sum_{n=0} n\,P_n \left(\frac{x}{d}\right)^{n-1}. \tag{A5}$$

The integral giving the charge due to ionization, Eq. (5), is not, as it stands, integrable numerically. The velocity of the ion at the point of creation will be essentially the thermal velocity of the atom just prior to ionization. Thus, on the average, this velocity will be zero. Consequently, the integrand diverges at the lower limit, and near that limit, the integral must be evaluated explicitly.

Fortunately, it is always possible to define a distance Δx considerably smaller than a scattering length ($\approx d/a$), and sufficiently small that the field can be regarded as constant. Then Eq. (5) becomes

$$N_{+i}\left(\frac{x'}{d}\right) = \frac{2J_{ob}\gamma\alpha d}{(2e^3/M)^{1/2}V_M}\left[\frac{\exp(d\alpha x'/d)}{E_r}\right] V_L^{1/2} + \int_{x'+\Delta x}^{d} \frac{-dJ_0(x)}{ev(x, x')}, \tag{A6}$$

where

$$\Delta x = \frac{dV_L}{V_M E_r},$$

and V_L is the voltage interval over which the field is regarded constant. The integral in the above equation can be evaluated numerically; V_L must be sufficiently small that N_{+i} has no significant dependence on it.

Referring to Eqs. (A1) and (A5), it is clear that if a power series in x/d is assumed for the voltage and field, that power series must be fitted to the charge distribution obtained from Eqs. (A6) and (A3). Least square fitting of the calculated charge distribution to a power series is a convenient way of doing this, provided a large number of points are calculated near $x/d = 1$. The charge density is usually quite large in that region, and thus must be fitted carefully if the correct field is to be obtained. Once the charge distribution is obtained, the potential and field functions can be calculated using the boundary conditions from the second section. This will be described shortly.

In practice, a potential function is assumed (a linear field approximation seems to be a good first guess), and the charge distribution appropriate to that potential is calculated. This charge distribution is used to obtain a new potential function, and the process is repeated. Usually, three to four iterations are sufficient to obtain a self consistent charge distribution and potential function.

At low pressures the ionized charge density is very low, and to first order can be neglected. This makes the calculation of the charge distribution far easier, in that a rather accurate potential function can be obtained without numeric integration. This is fortunate, for at very low pressures many points must be calculated and a power series of at least eighth order must be used in order to accurately describe the rapidly varying charge distribution. If the point density is insufficient, one of two things may happen. Either the least square fit to the charge distribution is sufficiently poor that the calculated distance across the sheath is inaccurate, or the potential function calculated from the least square fit oscillates sufficiently to allow a positive potential, causing the velocity [Eq. (A2)] to be imaginary. Needless to say, the computer will error stop if that happens.

Once a charge distribution has been determined, the various potential function coefficients must be determined. The following equations are used for this.

Given that

$$N_+ = \sum A_n \left(\frac{x}{d}\right)^n, \tag{A7}$$

the field is given by

$$E = E_{x=d} - \left(\frac{ed}{\epsilon_0}\sum \frac{A_n}{n+1}\right) + \frac{ed}{\epsilon_0}\sum \frac{A_n(x/d)^{n+1}}{n+1}, \tag{A8}$$

where $E_{x=d}$ is obtained from Eq. (12). The potential is given by

$$\frac{V}{V_M} = 1 + \left[\frac{e}{\epsilon_0}\left(\frac{d^2}{V_M}\right)\sum \frac{A_n}{n+1}\right]\frac{x}{d} - \left(\frac{E_{x=d}}{V_M}\right)\frac{x}{d} - \frac{e}{\epsilon_0}\left(\frac{d^2}{V_M}\right)\sum \frac{A_n(x/d)^{n+2}}{(n+1)(n+2)}. \quad \text{(A9)}$$

The third boundary condition, $V = 0$ at $x = d$, gives the relation

$$d = \frac{-\theta + \left\{\theta^2 + 4\frac{e}{\epsilon_0}\left[|V_M| - \left(\frac{2e}{k^2M}\right)\frac{\phi_0}{e}pd\right]\sum \frac{A_n}{n+2}\right\}^{1/2}}{2\frac{e}{\epsilon_0}\sum \frac{A_n}{n+2}}, \quad \text{(A10)}$$

where

$$\theta = \frac{4}{3}\left[\left(\frac{9}{4\epsilon_0}\right)^2\frac{M}{2e}\right]^{1/4}.$$

The computer programs used for the numeric calculations were written in APL. It should be noted that these programs made use of an APL library numeric integration program; they also used a matrix inversion program written by R. L. Anderson.

References

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11. The pressure intervals over which these various approximate solutions hold can be obtained as follows. The transition from "low pressure" to "mobility limited" behavior occurs when the sheath dimensions for these two regions have the same value. The transition pressure at which this occurs [from Eqs. (13) and (15)] is

$$p = \left(\frac{125}{8}\right)\left(\frac{M}{2e}\right)^{5/4}\left(\frac{k^2}{\epsilon_0^{1/2}}\right)\left(\frac{J_0^{1/2}}{V^{3/4}}\right).$$

Similarly, the transition from "mobility limited" to "ionization dominated" behavior occurs when the sheath dimensions for these two regions are equal; this occurs at a pressure given by

$$p = \left(\frac{243}{500}\right)^{1/4}\frac{\ln[(1+\gamma)/\gamma]}{\alpha/p}\left(\frac{1}{\epsilon_0 k}\right)^{1/2}\left(\frac{J_0^{1/2}}{V^{3/4}}\right).$$

Injection energy effects are ignored in deriving these expressions. Note that the functional dependencies on J_0 and V are the same for both transitions; all other quantities are constant, fixed by the choice of gas and target.
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