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Extrapolation of Seismic Waveforms by Fourier Methods

Abstract: The problem of constructing a cross section of reflectivity from the wave field recorded at the surface of the medium is discussed with particular reference to migration of seismic records. The numerical procedures are formulated in the frequency and wavenumber domain. The operations are defined in a fixed coordinate system, whereas finite difference methods require a downward-moving reference frame. The numerical algorithms in the frequency wavenumber domain are simpler and give more accurate results than finite difference methods. This is particularly true when the lateral velocity variation in the medium can be neglected. In this case the downward wave extrapolation is accomplished by implementing a phase change in the Fourier coefficients. Numerically, this is equivalent to a multiplication by a complex number of unit modulus. There is no stability condition associated with this operation. This means that the source and recorder positions can be lowered by any amount within one computational step.

Introduction

The processing and analysis of data obtained from seismic reflection procedures is crucial to the exploration and discovery of new hydrocarbon deposits. The world energy crisis is, of course, now placing a higher premium on this discovery process. With the aid of a regularly spaced array of geophones (recording instruments), geophysicists are able to 1) induce some acoustic wave in the earth; 2) record the signals reflected by the subsurface; 3) process and manipulate these records in order to learn about the geological makeup of the subsurface.

Seismic data is the digitized (in time and space) version of the reflected seismic wavefronts recorded at the surface of the earth. However, these recordings of seismic sections do not correspond exactly to the geological formations of the subterrain. This can be seen by considering that the amplitude of a trace is the result of a superposition of many wavefronts, propagating from all possible directions about the location of the geophone. The information about the size, the shape, and the location of the reflecting geological formations is contained in the amplitude and phase relationship of the wavefront at the surface as recorded by an array of geophones.

One of the most important procedures of seismic data processing is the construction of a seismic reflectivity display from the seismic records. This is a mapping from image space to object space, whereby the recorded images of the reflectors are mapped into their correct positions. This mapping process is referred to as migration. In order to produce a cross section of reflectivity, the waves observed at the surface must be extrapolated numerically, downward into the earth. This can be achieved by using wave equation techniques for migration. In recent years, J. F. Claerbout and his coworkers developed migration techniques based on the numerical approximation of the wave equation by finite difference methods. By treating the problem in a downward-moving coordinate system, Claerbout [1, 2] derived simplified equations that lend themselves conveniently to numerical treatment.

Migration using Fourier transform methods was studied by Stolt [3], Claerbout [4], and Lynn [5]. In these studies Fourier transforms were used to obtain a direct solution to the wave equation. Therefore, migration with these methods is limited to homogeneous media in which the velocity of the wave propagation can be regarded as constant.

Numerical solution methods based on finite Fourier transforms (FFT) for the migration of seismic data in layered media were reported by Gazdag [6, 7]. These works, together with earlier results [8], demonstrate conclusively that the effective use of FFT methods need not be restricted to linear partial differential equations with constant coefficients only. These methods are based on the equations derived by Claerbout [2] and are formulated in a downward-moving coordinate system.

In the present work we discuss numerical methods for wave extrapolation in inhomogeneous media. These solution methods are defined in the frequency and wavenumber domain, and the numerical procedures are based

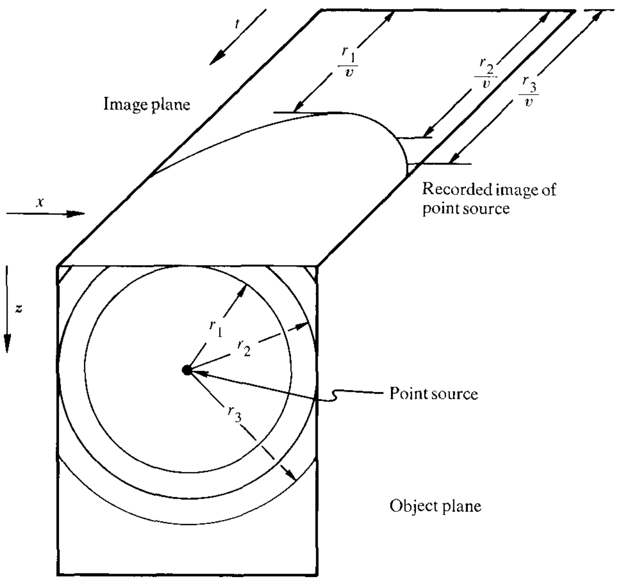

Figure 1 A coordinate system for wave extrapolation.

on FFT techniques. The present approach differs from previously reported works in that both the derivations as well as the actual computations are performed in a stationary rather than a moving reference frame. The derivations are tractable because they are based on analytic results derived for simple idealized models, which are generalized for realistic situations. In the following section we discuss the differential equation for downward-propagating waves in an inhomogeneous medium. The third section deals with the migration of zero offset seismic sections. The numerical results are discussed in the fourth section, which is followed by the concluding remarks.

Equation for downward-propagating waves

Acoustic waves are described by the equation

$$\frac{\partial^2 p}{\partial t^2} = v^2\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2}\right), \tag{1}$$

where p is pressure, t is time, and x and z represent horizontal and vertical distances, respectively. For constant velocity v, the solution to (1) can be expressed as

$$p(x, z, t) = \sum_{k_x} \sum_{k_z} P\,(k_x, k_z, t = 0) \exp\,[i(k_x x + k_z z + \omega t)], \tag{2}$$

where k_x and k_z are the wavenumbers, or spatial frequencies, whereas ω is the temporal frequency. For each wave vector (k_x, k_z), there are two solutions, corresponding to the two ω values given by the dispersion relation

$$\omega = \pm v(k_x{}^2 + k_z{}^2)^{1/2}. \tag{3}$$

These two solutions represent waves propagating in opposite directions. In seismic data processing it is often essential to differentiate between upcoming and downgoing waves. An excellent treatment is given by Claerbout [2] on this topic.

For the sake of definiteness we carry out all derivations with reference to the coordinate system shown in Fig. 1, in which the variable z is directed downward and represents depth. In this representation the (x, z) plane is the object space and the (x, t) plane is the image space. The true physical reality is described by Eqs. (2) and (3), according to which any source in the (x, z) plane radiates in all directions. An example for this is shown in Fig. 2(a). However, if the solution is restricted to temporal frequencies

$$\omega = vk_z[1 + (k_x/k_z)^2]^{1/2}, \tag{4}$$

Eq. (2) represents only upward-going waves. Downward-propagating waves, whose wave vector is directed within plus or minus 90° of the positive z axis, are characterized by the dispersion relation

$$\omega = -vk_z[1 + (k_x/k_z)^2]^{1/2}. \tag{5}$$

For a homogeneous medium, in which v is constant, we can immediately write the differential equation which describes downward-propagating waves as

$$\frac{\partial P}{\partial t} = -ivk_z[1 + (k_x/k_z)^2]^{1/2}\, P. \tag{6}$$

The physical interpretation of (6) is simple. Each wave component with $k_z > 0$ is displaced in the direction of its wave vector (k_x, k_z). On the other hand, plane waves with $k_z < 0$ are convected in the direction opposite to that of their wave vector. All the waves move with velocity v measured in the (x, z) space.

The physical picture is not as simple in the more general case, when v is some function of the space variables x and z. In this case the multiplication by v is replaced by the convolution operation, which is denoted by the symbol $\oplus$. Thus the general form of (6) reads

$$\frac{\partial P}{\partial t} = -iv \oplus k_z[1 + (k_x/k_z)^2]^{1/2}\, P. \tag{7}$$

The convolution operation in the (k_x, k_z) domain corresponds to a multiplication by $v(x, z)$ in the (x, z) domain. Within regions of constant velocity Eqs. (6) and (7) are essentially equivalent. At velocity discontinuities (7) does not have the same reflection and transmission coefficients as the full wave equation. The fact that (7) does not describe an actual physical phenomenon is of no consequence to us in this application. We are interested only in some descriptive information regarding the composition of the subterrain, which is provided by the solution of (7).

We have tested (7) by means of numerical experiments. The aim was to show that the extension of (6) to the variable velocity case, as given by (7), is valid for downward-propagating waves in inhomogeneous media. The numerical simulation results shown in Figs. 2 and 3 provide a contrast between the solution of the full wave equation (1) and that of (7). In these examples we find a good qualitative agreement between the downward-moving parts of the waves.

Reconstruction of wave fields from measurements

We consider now the problem of determining the field existing in a propagation medium from the record of the wave field taken at $z = 0$ depth, as shown in Fig. 1. In exploration geophysics, the application of this process is the construction of cross sections of reflectivity within the earth from seismic sections. We begin with the uniform velocity case, for which the solution can be expressed in analytic form. We then generalize this result to inhomogeneous media.

We shall assume that the wave $p\,(x, z, t)$ induced over the x, z plane is propagating upward. Such a wave field can be expressed in the form of (2), in which ω is given by (4). The recorded wavefront is obtained by setting $z = 0$ in (2); i.e.,

$$p(x, 0, t) = \sum_{k_x} \sum_{k_z} P\,(k_x, k_z, t = 0) \exp\,[i(k_x x + \omega t)]. \tag{8}$$

From this record we wish to reconstruct the field that existed in the propagation medium at $t = 0$, which can be written from (2) as

$$p(x, z, 0) = \sum_{k_x} \sum_{k_z} P(k_x, k_z, t = 0) \exp\,[i(k_x x + k_z z)]. \tag{9}$$

In order to obtain the desired expression relating (8) and (9) we obtain a Fourier transformation of (8) with respect to x and t, which gives

$$P(k_x, z = 0, \omega) = P(k_x, k_z, t = 0). \tag{10}$$

The correspondence between k_z and ω is given by the dispersion relation (4). The substitution of (10) and (4) into (9) gives

$$p(x, z, 0) = \sum_{k_x} \sum_{k_z} P(k_x, z = 0, \omega) \exp\left(i\{k_x x + m[1 - (k_x/m)^2]^{1/2} z\}\right), \tag{11}$$

in which

$$m = \omega/v. \tag{12}$$

We can easily verify that (11) is the solution to

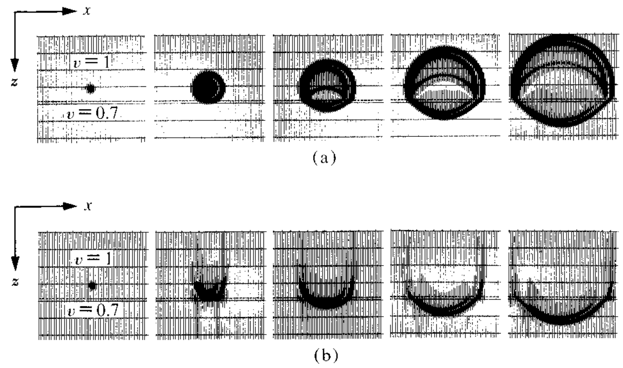

Figure 2 Time sequences showing the evolution of an acoustic wave from a point source: (a) solution of the full wave equation (1), and (b) solution of the equation for downward-propagating waves (7).

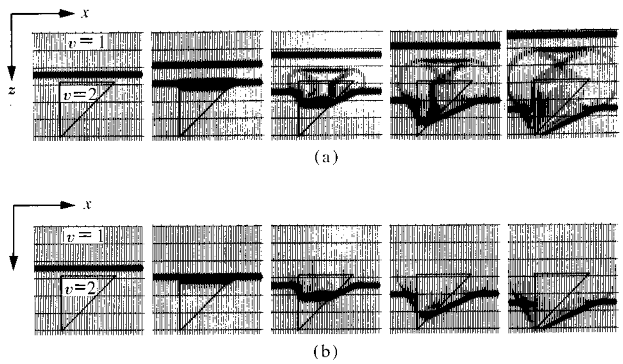

Figure 3 Time sequences showing the evolution of an acoustic wave from a line source: (a) solution of the full wave equation (1), and (b) solution of the equation for downward-propagating waves (7).

$$\frac{\partial P(k_x, z, \omega)}{\partial z} = \mathrm{i}m\,[1 - (k_x/m)^2]^{1/2}\, P(k_x, z, \omega) \tag{13}$$

at $t = 0$. Thus, when solving (13) for P over the (k_x, ω) domain, we obtain the correctly migrated section at some depth z by implementing the equation

$$p(x, z, 0) = \sum_{k_x} \sum_{\omega} P(k_x, z, \omega) \exp\,(i\, k_x x). \tag{14}$$

We consider here the laterally homogeneous case, in which the velocity is specified as a function of the depth; i.e., $v = v(z)$. Let the z axis be subdivided into N_z increments of Δz length. The jth increment (layer) is denoted as ξ_j, where

$$\xi_j = (z \mid z_j \le z < z_{j+1}). \tag{15}$$

Within each layer ξ_j the velocity is assumed as constant. Thus, (13) can be solved for P at z_{j+1} by using P given at z_j as the initial conditions. This solution can be expressed as

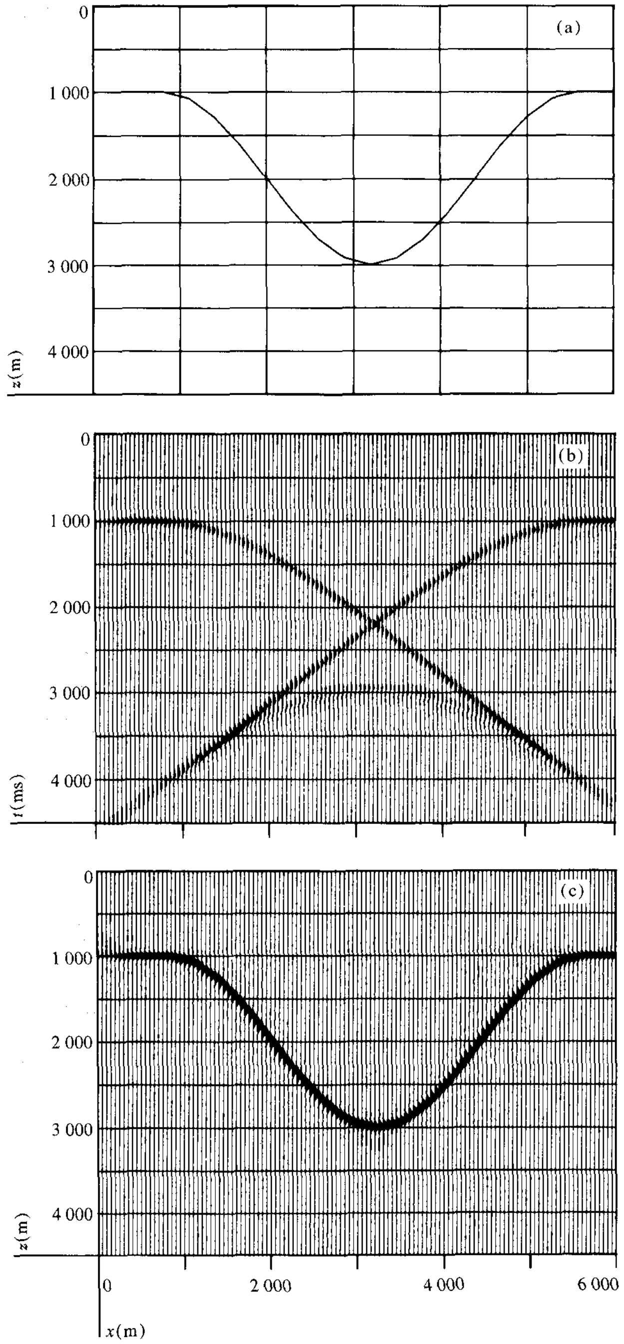

Figure 4 Modeling and migration of a dipping reflector. (a) Model of dipping reflector. (b) Synthetic zero offset section of (a). (c) Section (b) after migration by means of (13).

$$P(k_x, z_{j+1}, \omega) = P(k_x, z_j, \omega) \exp (i\phi_j \Delta z), \tag{16}$$

in which

$$\phi_j = m_j [1 - (k_x/m_j)^2]^{1/2}, \tag{17}$$

or more explicitly,

$$\phi_j = (\omega/v_j) [1 - (v_j k_x/\omega)^2]^{1/2}, \tag{18}$$

where v_j is the velocity of the jth layer (15).

It is evident from (16) that P is advanced to greater depths by implementing a phase shift in its Fourier coefficients. The rate of phase change per unit depth is ϕ_j (18). One of the noteworthy properties of this solution method is that the phase shifts expressed in (16) are accumulative. This also means that if we are interested in the migrated section between some specified depths, it is not necessary to solve (13) for other than the region of interest. We can immediately write P at z_{j+1} in terms of P at zero depth z_0 as

$$P(k_x, z_{j+1}, \omega) = P(k_x, z_0, \omega) \exp (i\psi_j z), \tag{19}$$

where

$$\psi_j = \sum_{n=0}^{j} \phi_n. \tag{20}$$

The remarkable simplicity of this solution method for laterally homogeneous media is possible only because $v = v(z)$ is independent of both k_x and ω, which appear under the square root sign in (13). Unfortunately, we cannot use (13) for problems with horizontal velocity variations. We can, however, approximate the square root by series expansion, which gives

$$\frac{\partial P(k_x, z, \omega)}{\partial z} = i \left(\frac{\omega}{v} \right) \left(1 - \frac{v^2 k_x^{\,2}}{2\omega^2} - \frac{v^4 k_x^{\,4}}{8\omega^4} - \cdots \right) \times P(k_x, z, \omega). \tag{21}$$

The truncation of the fourth order and higher order terms results in a second order approximation to (13). The second order equation can be used for migrating dips up to 30° [7]. The fourth order term in (21) increases this angle to 50°, which is adequate in most practical applications. We can express (21) in the (x, ω) domain as

$$\frac{\partial p(x, z, \omega)}{\partial z} = i \left(\frac{\omega}{v} \right) \times \left(p + \frac{v^2}{2\omega^2} \frac{\partial^2 p}{\partial x^2} - \frac{v^4}{8\omega^4} \frac{\partial^4 p}{\partial x^4} + \cdots \right). \tag{22}$$

In this equation the dependence of v need not be restricted to the depth variable alone, but it can also include lateral variations; i.e., $v = v(x, z)$. The solution of (22) at $t = 0$ gives an approximation to the wave field corresponding to the migrated section.

Numerical results

We tested Eq. (7) for downward-moving waves and compared the results with those of the full wave equation (1).

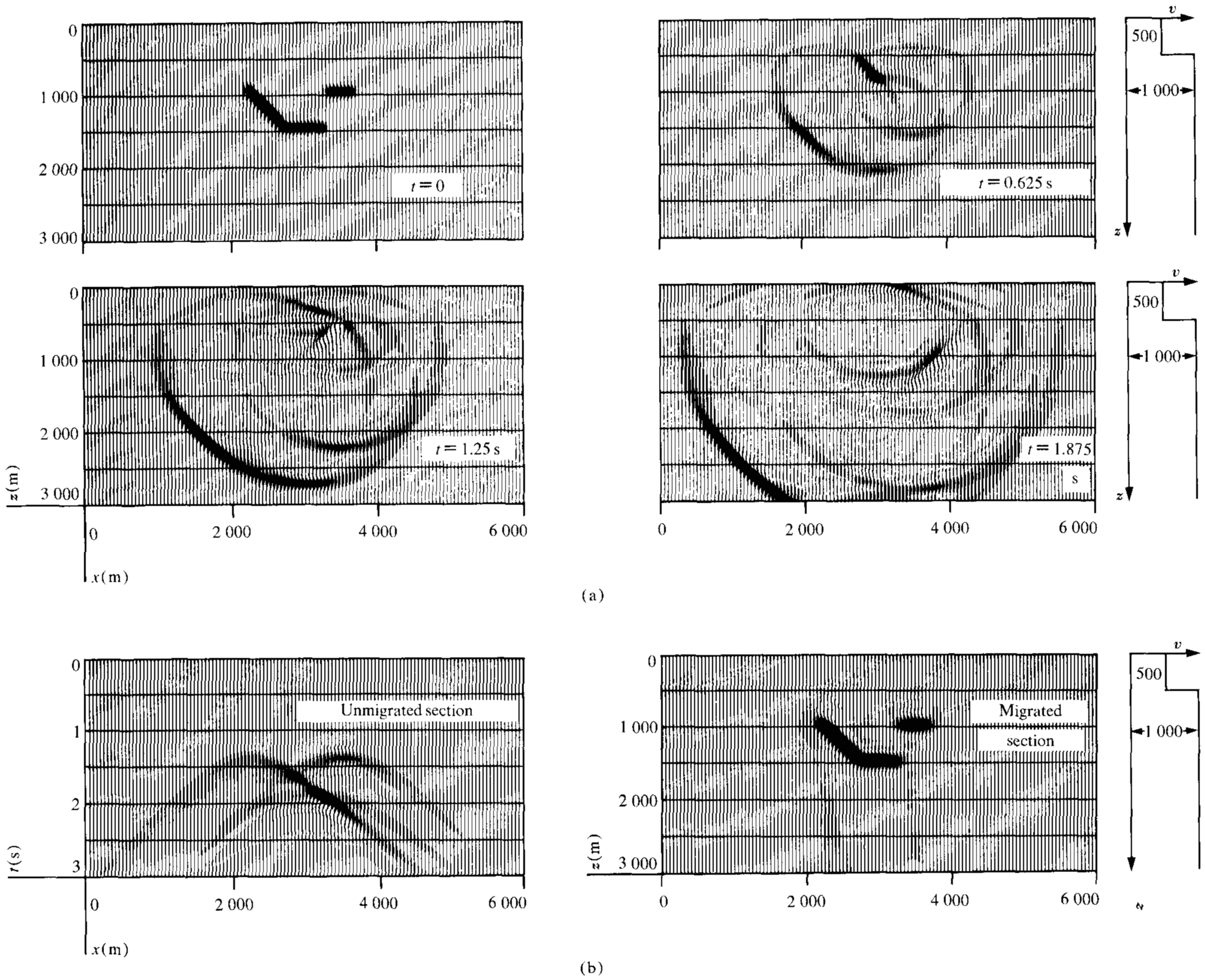

Figure 5 Modeling and migration in a medium with vertical velocity variations. (a) Time sequence showing the evolution of the wave in the (x, z) plane. (b) Zero offset section before and after migration.

In both examples shown in Figs. 2 and 3 we used a 64×64 computational grid. The initial conditions can be seen in the first frame shown at $t = 0$. The evolution of the waves is shown by the sequence of frames taken at equal time intervals.

In the first example shown in Fig. 2, the wave is initiated in a medium with $v_1 = 1.0$, from which it enters a lower velocity medium with $v_2 = 0.707$. We can observe very clearly the wave reflected from the interface with opposite sign, indicating a negative reflection coefficient. This is entirely absent in Fig. 2(b), which shows a remarkable qualitative agreement with the downward-moving portion of the full wave equation.

The initial conditions in Fig. 3 are set up to initiate a plane wave in the medium with $v_1 = 1.0$. The downward-propagating part of the wave impinges upon a prism in which $v_2 = 2.0$. In Fig. 3(a) we observe the wave reflected from the top of the prism. We can also remark the circular diffraction patterns developed about the upper corners of the prism. Only a small fraction of these circular diffraction patterns are noticeable in Fig. 3(b), which does not include upward-moving waves. The downward-traveling waves, including those which exit the prism at the left and the bottom edges, are seen very clearly in both Figs. 3(a) and 3(b).

The numerical example for the reconstruction of the wave field from a synthetic seismic record is shown in Fig. 4. The synthetic record shown in Fig. 4(b) was generated by implementing the theory of Trorey [9] for seismic diffractions. This is a zero offset record section in which the source and the recorder positions are the same at the $z = 0$ level. The same record would be obtained from a line source shaped as the earth model shown in Fig. 4(a). In such an experiment one would have to use one-half of the actual velocity in order to compensate for the two-way travel of the waves in the zero offset sections. The computations were performed over a 128 × 128 grid, with $\Delta x = \Delta z = 50$ m, and $\Delta t = 50$ ms. The migration velocity was constant, $v = 2000$ m/s. The migrated section shown in Fig. 4(c) is the result of the numerical solution of (13) and (14). It is in excellent agreement with the model in spite of its steep dip of 52°.

The last example deals with the generation of a synthetic time section and its migration. The velocity varies with the depth variable as shown in Fig. 5. The first frame of Fig. 5(a) shows the initial conditions for p at $t = 0$, while its time derivation $\partial p/\partial t$ is set uniformly to zero at $t = 0$. A sequence of four frames shows the time evolution of the wave. The wave field sampled at $z = 0$ depth gives the desired zero offset section. Figure 5(b) shows this section before and after migration using (19) and (20). Apart from numerical effects, the migrated section corresponds to the initial conditions shown in Fig. 5(a) at $t = 0$.

Concluding remarks

It has been shown [8] that numerical solution methods based on the use of FFT techniques offer certain advantages over finite difference methods. This is particularly true when dealing with the simulation of wave phenomena, the subject of the present paper, in which case the solution is expressed most naturally by means of Fourier series. Moreover, the operators used to formulate wave extrapolation processes are expressed more readily as the function of the temporal and spatial frequencies, rather than in terms of time and distance. Therefore, the solution methods are simple, and the results are more accurate than those obtained with finite difference methods. The first reason for accuracy is that we are solving equations which describe the desired process exactly. For (7) and (13) we can only write approximate expressions in the space-time domain. The second important factor is that our methods are practically free from truncation errors.

The last but by no means the least significant characteristic of the present solution methods is that they lend themselves conveniently to parallel processing. For example, there is no interdependence among the Fourier coefficients of (13) corresponding to different k_x values. Consequently, (13) can be solved for all depths over one column (with a fixed k_x) of the (k_x, ω) plane. This demands little storage for a considerable number of computations. This property can be particularly advantageous in the effective utilization of array processors.

Acknowledgments

The author thanks J. F. Claerbout for the opportunity to participate in the seminars of the Stanford Exploration Project, whose stimulating discussions had an inspiring influence on this work.

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Received September 12, 1977

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