W. N. Holmes

Representation for Complex Numbers

Abstract: This communication suggests the feasibility of a single-component scheme for representing complex numbers with real bases. Several advantages are pointed out, including the very simple extraction of the real and imaginary parts of a complex number.

Introduction

Complex numbers are more awkward to compute than are real numbers. In conventional use an arithmetic expression shows the complex number as the *sum* or *difference* of two components, one real and one imaginary, the imaginary component being distinguished by an *i* or *j* affix.

Algorithms for manipulating complex numbers treat the real and imaginary components separately so that, for instance, multiplication of two complex numbers involves four real multiplications, a real addition, and a real subtraction.

Sporadic proposals have been made for single-component representation of complex numbers [1]. These proposals use real digits and an imaginary or complex base.

This communication notes that schemes for representing single-component complex numbers with real bases are feasible, and suggests advantages and uses of such schemes.

Base-3 complex number representation

If a set of complex digits is properly chosen, with for example four digits for a base-2 scheme or nine for a base-3, a real base can be used to represent complex numbers in single component form.

Consider the following allocation of complex meaning to the otherwise familiar decimal digits 0 to 8.

DIGIT	0	1	2	3	4	5	6	7	8
real part	0	1	−1	0	1	−1	0	1	−1
imaginary part	0	0	0	1	1	1	−1	−1	−1

This allocation is arbitrary in the sense that any nine different symbols could be allocated to the nine digits needed for a complex base-3 system, and further, a different allocation of meaning might be used for the same nine familiar symbols to obtain advantages other than the ones suggested below.

For ordinary arithmetic operations, simple digital algorithms can be used that are quite similar to the algorithms for arithmetic with real numbers. A multiplication table and a combined addition and subtraction table follow for the base-3 complex digits given previously.

Addition/subtraction:

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	
1	12	0	4	15	3	7	18	6	**−2**
2	0	21	5	3	24	8	6	27	**−1**
3	4	5	36	37	38	0	1	2	**−6**
4	15	3	37	48	36	1	12	0	**−8**
5	3	24	38	36	57	2	0	21	**−7**
6	7	8	0	1	2	63	64	65	**−3**
7	18	6	1	12	0	64	75	63	**−5**
8	6	27	2	0	21	65	63	84	**−4**
	−2	**−1**	**−6**	**−8**	**−7**	**−3**	**−5**	**−4**	

Multiplication:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
2	1	6	8	7	3	5	4
3	6	2	5	8	1	4	7
4	8	5	36	21	7	12	63
5	7	8	21	63	4	36	12
6	3	1	7	4	2	8	5
7	5	4	12	36	8	63	21
8	4	7	63	12	5	21	36

Advantages of single-component representation

Representing complex numbers in single-component form allows the use of algorithms and procedures much more closely related to the much simpler algorithms and procedures applicable to real numbers. By the use of such algorithms, arithmetic with complex numbers can be implemented in computing machinery without the introduction of extra features such as two component operands, and in computational procedures without separation of real and imaginary components in algorithmic expressions. Further, when a real base is used, primitive operations such as extraction of the real or imaginary part of a complex number can be carried out digit-by-digit independently of position.

As suggested above, use of the digits 0 to 8 in a base-3 system raises the possibility of direct use of single-component complex number representation by students and professionals who must carry out calculations in the complex domain. Because the system allows very simple extraction of the real and imaginary parts of a complex number, practice would enable a user to immediately comprehend the nature of a complex number, whether it were represented in fixed-point or floating-point form. Because the system does not require the use of arithmetic signs for real and imaginary components of a complex number, use of a "complex sign" (the symbol _ springs to mind) would allow complex numbers to be clearly and unambiguously mixed with real numbers in programs and their output, in algebraic (APL) formulas and systems, and even in pocket and desk calculator operation.

Acknowledgment

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Reference

1. (a) D. E. Knuth, "An Imaginary Number System," *Commun. ACM* **3,** 245 (1960). Proposes imaginary base, real digits; concentrates on "quater-imaginary" system with base $2i$ but also mentions base i root 2.
(b) W. Penney, "A 'Binary' System for Complex Numbers," *J. ACM* **12,** 247 (1965). Proposes a complex base $i - 1$ to give a binary system.
(c) T. J. O'Reilly, "A Positional Notation for Complex Numbers," *IEEE Computer Society Repository, R74-169*, 12 pp. (1974), IEEE Computer Society Publications Office, Long Beach, CA 90803. Proposes an imaginary base, real digits; looks at base i root 2 for binary system.

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The author is located at IBM Australia Limited, 80 Northbourne Avenue, Canberra, A.C.T. 2601, Australia.