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Worst-Case Pattern Evaluation of Baseband Channels

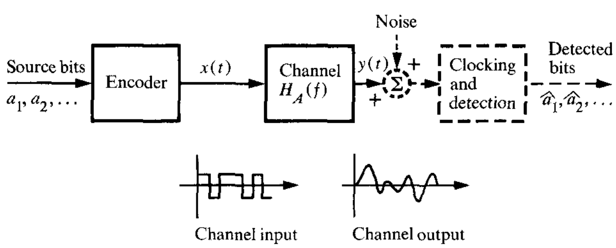
Abstract: This paper presents a new method for evaluating the performance of communication systems that use binary-valued signaling formats. This technique permits selection of the best overall code for a particular channel or, alternatively, provides a method for comparing different channels that use the same transmission code. Encoded waveforms at the channel input are presumed to be generated by a constrained-coding procedure, which ensures, for example, that the resulting binary waveform has a certain minimum number of transitions per unit time, limited digital sum variations (DSV), among other characteristics. With this method, one first determines that particular sequence (within the given code constraints) that produces the maximum amount of intersymbol-interference when transmitted through the given channel. A dynamic-programming procedure is used to compute this sequence. Overall channel performance is evaluated by calculating the probability of error and the minimum eye-pattern opening. Numerical examples illustrating the procedure for a synthetic channel are presented and analyzed.

Introduction

This paper presents a method for evaluating the performance characteristics of baseband, binary transmission channels, such as those found in magnetic recording applications. The channels of interest consist of the basic elements shown in Fig. 1. Each sequence of source bits $a_1, a_2, \cdots, a_N$ is encoded as a binary-valued, continuous waveform $x(t)$ which has transitions between levels only at multiples of the bit period T. In general the code contains redundancy, and more than N bits in $x(t)$ are used to encode the N source bits. This redundancy may be used, for example, to ensure that every encoded waveform has a minimum number of transitions per unit time to allow better clocking signal extraction at the channel output. Codes which achieve this and other constraints have been proposed [1, 2].

The overall error-rate performance of the channel shown in Fig. 1 depends on a variety of complex, interrelated factors. These include the output noise characteristics, the procedure used to extract a clock signal from the data, and the method of detection. Performance also depends largely on the particular input sequence that drives the channel. Specifically, certain input sequences result in channel outputs that have a large amount of intersymbol interference (ISI), and at least one input sequence exists that maximizes ISI levels. This encoded waveform then necessarily results in the poorest overall performance, all other factors being fixed, and is termed here the worst-case pattern (WCP).

Figure 1 Basic elements of baseband transmission system.



The objective of this paper is to present a simple method for finding the WCP and its corresponding ISI level, given a fixed encoder/channel combination. As a result, we are concerned only with the encoder and channel portion of the model shown in Fig. 1; the effects of noise, clocking accuracy, and detection do not enter into our analysis. Our results can be applied to compare quantitatively several different constrained codes (i.e., encoding procedures) for the same channel by evaluating their maximum ISI levels. The channel having the minimum level is preferred. Alternatively, we could compare different channels that use the same encoder. Although our primary concern is with magnetic recording channels, our approach is applicable to any baseband communication system.

The analysis is simplified here by assuming that the input/output characteristics of the channel are linear. Tools such as transfer function analysis and convolution are used to derive the basic results and to evaluate the overall performance of the system. Although the relationship between write current and recorded flux density in the magnetic environment is nonlinear, measurements have demonstrated that linear models can successfully predict the waveforms observed at the read head of magnetic recording channels that use binary-valued inputs. Thus, our worst-case pattern design procedure applies directly to magnetic recording systems.

It is shown that the problem of finding a WCP is relatively simple for encoders that operate without constraints as occurs, for example, if the source bits are converted to a binary waveform using the non-return-to-zero index (NRZI) encoding method. With NRZI, each transition in the continuous waveform represents a 1, and a 0 is indicated by the lack of a transition. The WCP for this encoder can then be expressed in closed form as a simple function of the channel impulse response.

For codes with constraints, however, the WCP is significantly more difficult to obtain because not all possible input waveforms are allowed. It is readily shown that the length of the WCP is determined by the time over which the impulse response of the channel is significantly different from zero. This period is generally long when measured in incoming bit periods and may be as long as several hundred bits for typical magnetic recording applications. It is therefore impractical, if not impossible, to apply an exhaustive search technique which measures the ISI level of all permissible sequences of this length within the code.

The approach taken here is based on the method of dynamic programming [3, 4]. We first divide the original channel into the sum of two other channels such that the impulse response of the sum is identical to the original. This decomposition is illustrated in Fig. 2. The upper "ideal" channel, fixed and independent of the original, is ideal in the sense that its output is ISI free at selected points, regardless of the input sequence. Nyquist [5] studied such channels in detail, and has shown that the so-called raised-cosine channel is ISI free at both mid-bit and bit-boundary points. Since many magnetic recording systems employ clock extraction based on zero crossings as well as mid-bit sampling detectors, the raised-cosine channel is a logical choice for the ideal and is used here. Generalizations to other choices will be apparent as we proceed. Kobayashi [6] uses a similar channel decomposition with a different choice of ideal to discuss code constraints.

The lower channel, termed "error" channel here, is then simply the difference between the ideal and actual channels. Evaluation of its output at mid-bit and bit-boundary points leads to a direct determination of ISI, resulting from our choice of ideal. Dynamic programming is used to construct a WCP by determining the input sequence which maximizes error channel output at these points. The technique requires only that the constrained code have a finite-state description, as defined by Franaszek [1] and others [7].

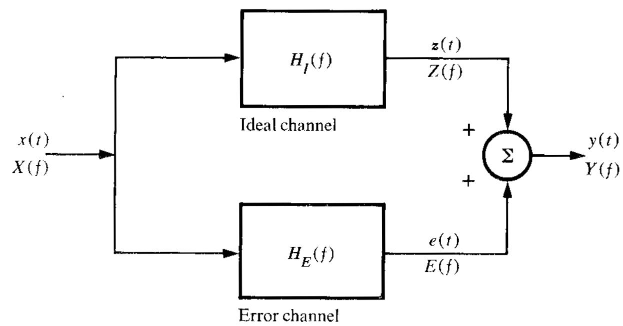


Figure 2 Decomposition model for original channel: $H_A(f) = H_1(f) + H_E(f)$.

In the following sections, the proposed method is developed and a numerical example illustrating its use is presented. The example includes a computation of the worst-case sequence, its resulting probability of error, and an experimentally derived eye pattern. In addition, the corresponding parameters for a pseudo-random input waveform are presented and compared with those for the WCP. As expected, the WCP produces significantly higher error probability and a reduced eye opening.

Mathematical channel model

The transfer function of the channel shown in Fig. 1 is denoted by the complex frequency response $H_A(f)$. The decomposition in Fig. 2 is then

$$H_A(f) = H_I(f) + H_E(f). \tag{1}$$

We assume that the actual channel has a usable maximum frequency of f_s Hz and that a bit period $T = 1/2f_s$ is employed. The ideal raised-cosine channel for this case then has an amplitude spectrum given by [8],

$$\begin{aligned} |H_I(f)| &= \frac{(1/2f_s)(1 + \cos \pi f/f_s)}{(\sin \pi f/f_s)/(f/f_s)}, \quad |f| \le f_s; \\ &= 0, \qquad\qquad\qquad\qquad\quad |f| > f_s. \end{aligned} \tag{2}$$

The phase function of this channel is assumed to be linear with slope τ and to have a value of zero at dc.

An error channel transfer function can then be computed using Eq. (2) and the known actual channel response $H_A(f)$. A direct difference operation as suggested by Eq. (1), however, cannot be used without first applying appropriate gain and bulk time-delay corrections. If this is not done, gain and linear phase differences between H_A and H_I will be falsely interpreted as ISI components in the error channel output.

To account for the effects of gain and bulk time-delay, the impulse response of the error channel is computed as

$$h_E(t) = Kh_A(t - \tau) - h_I(t), \tag{3}$$

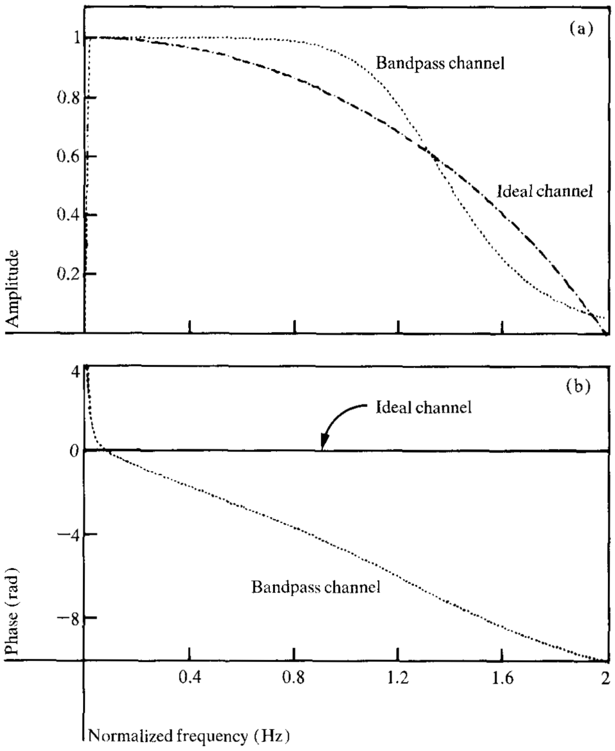


Figure 3 Transfer function response for ideal and bandpass filter channels.

with K and τ chosen to minimize the total energy content ρ of $h_E(t)$, where

$$\rho = \int_{-\infty}^{\infty} h_E^2(t)\,dt. \tag{4}$$

This ensures that any nonzero contributions to ρ are produced by true wave-shape differences between the ideal and error-channel outputs. By substituting Eq. (3) into Eq. (4) and collecting terms, we obtain

$$\rho = K^2 \int_{-\infty}^{\infty} h_A^2(t-\tau)\,dt + \int_{-\infty}^{\infty} h_I^2(t)\,dt - 2K \int_{-\infty}^{\infty} h_I(t)\; h_A(t-\tau)\,dt. \tag{5}$$

Because the first two terms are independent of τ, minimization of ρ with respect to τ for any value of K is achieved by selecting ρ such that the last integral is maximum. This term is recognized as the cross-correlation between the ideal and actual channels. The optimum time-delay correction τ_0 is simply that which provides a maximum cross-correlation value, independent of the gain-correction term.

Once τ_0 has been computed, the optimum gain correction K_0 can be determined using derivatives. Since ρ is quadratic in K and the coefficient of K^2 is positive, it has a unique minimum which may be computed from

$$\frac{d\rho}{dK} = 0 = -2 \int_{-\infty}^{\infty} [K_0 h_A(t-\tau_0) - h_I(t)]\; h_A(t-\tau_0)\,dt; \tag{6}$$

or

$$K_0 = \frac{\int_{-\infty}^{\infty} h_I(t)\; h_A(t-\tau_0)\,dt}{\int_{-\infty}^{\infty} h_A^2(t-\tau_0)\,dt}. \tag{7}$$

With the choice of ideal channel given Eq. (2) and this procedure, the error-channel output signal $e(t)$ evaluated at bit-boundary and mid-bit points indicates the ISI content of $y(t)$ at these points. In the time domain, $e(t)$ is the convolution of the encoded waveform $x(t)$ and the computed error-channel impulse impulse response $h_E(t)$, i.e.,

$$e(t) = \int_{-\infty}^{\infty} h_E(\alpha)\; x(t-\alpha)\,d\alpha. \tag{8}$$

Since $x(t)$ is a binary-valued sequence with transitions occurring only at multiples of T seconds, the integral in Eq. (8) can be replaced by a sum of integrals taken over the bit periods. Thus if α_n denotes the location of the start of the nth bit and $a_n = \pm 1$ is the bit value,

$$e(t) = \sum_{n=-\infty}^{\infty} a_n \int_{\alpha_n}^{\alpha_{n+1}} h_E(t-\alpha)\,d\alpha. \tag{9}$$

This formulation illustrates the procedure for finding the WCP when $x(t)$ is unconstrained. To maximize $e(t)$ at any particular point t_0, we simply choose

$$a_n = +1, \quad \text{if} \int_{\alpha_n}^{\alpha_{n+1}} h_E(t_0-\alpha)\,d\alpha \geq 0, \text{ and}$$

$$a_n = -1, \quad \text{if} \int_{\alpha_n}^{\alpha_{n+1}} h_E(t_0-\alpha)\,d\alpha < 0. \tag{10}$$

Thus, each element of the sum in Eq. (9) is nonnegative and the resultant is maximum. Worst-case patterns for codes with constraints, however, cannot be generated in this fashion because the a_n cannot be chosen independently of each other, as required by the solution in Eq. (10).

Computational considerations

For those cases in which the actual channel-impulse response is represented in sampled-data format, it may be necessary to provide an optimum time-delay correction τ_0 which is not an integer number of sampling times. For example, the results presented later use a sampling period equal to one-half of the data-bit period T, and time-delay corrections which are not multiples of the sampling period are applied to the actual channel. Non-multiple time shifts are readily obtained utilizing Fourier transform techniques. If we define $R(\tau)$ as the cross-correlation term used to find τ_0, that is

$$R(\tau) = \int_{-\infty}^{\infty} h_I(t)\; h_A(t-\tau)\,dt \tag{11}$$

and

$$\tau_0 = \text{value of } \tau \text{ maximizing } \mathrm{R}(\tau), \tag{12}$$

then $R(\tau)$ can be represented in the transform domain utilizing Parseval's theorem

$$\int_{-\infty}^{\infty} a(t)\; b(t)dt = \int_{-\infty}^{\infty} A(\omega)\; B^*(\omega)\frac{d\omega}{2\pi}$$

as

$$R(\tau) = \int_{-\infty}^{\infty} H_I(\omega)\; H_A^*(\omega)\; e^{j\omega\tau}\frac{d\omega}{2\pi}, \tag{13}$$

where $H_I(\omega)$ and $H_A(\omega)$ are the Fourier transforms of $h_I(t)$ and $h_A(t)$, respectively. The superscript asterisk in Eq. (13) denotes a complex conjugate. For sampled-data calculations, Eq. (13) is replaced by an appropriated summation taken over the discrete Fourier transform samples. These may be conveniently computed using the Fast Fourier Transform (FFT) method [9]. In the transform domain, the time-delay τ becomes a linear phase term $e^{j\omega\tau}$, and the continuous values of τ may be searched for a maximum, regardless of the sampling interval used to represent $h_I(t)$ and $h_A(t)$. The specific method used in this paper to find τ_0, the optimum value for τ, involved the use of an interval-halving iterative technique. Once τ_0 has been determined, an inverse transform is used to obtain the appropriately shifted impulse response $h_A(t-\tau_0)$.

Bandpass filter example

For purposes of experimentally demonstrating the WCP method suggested in this paper, a series connection of two bandpass filters was used to approximate a baseband transmission system having low-frequency roll-off. In effect, the low end roll-off characteristics were those of a single four-pole Butterworth filter with a cutoff frequency of 0.005 times the data rate. We achieved the upper cutoff frequency of 0.7 times the data rate by using two cascaded four-pole Butterworth filters. [The actual filters used to implement this transmission system were two (Kronhite Model 3103) active filters.] The magnitude and phase characteristics of the resulting channel are presented as a function of frequency in Fig. 3. A normalized frequency scale corresponding to a one second bit duration has been used. The corresponding values for the ideal raised-cosine channel defined by Eq. (2) are also included for comparison. Optimum gain and time-delay corrections, K_0 and τ_0 respectively, were computed for the bandpass channel using the procedures outlined above. Figure 4 shows the magnitude and phase response curves after these corrections were applied. An error-channel response was then computed as the difference between the transfer function plots shown in Fig. 4. This calculation was carried out in the frequency domain using appropriate values of both magnitude and phase at each frequency point. The resulting error-channel transfer function is shown in Fig. 5. As expected, the largest magnitudes are concentrated at the low frequency end corresponding to frequencies at which the bandpass channel differs markedly from the ideal. The time-domain impulse response of the error channel presented in Fig. 6 has a total time duration of 300 sample points or 150 bits. Based on these results, one would expect that the encoded waveform having the greatest error-channel output level would be of length no greater than 150 bits. The numerical example presented in the next section describes the dynamic-programming method used to compute the WCP for this case.

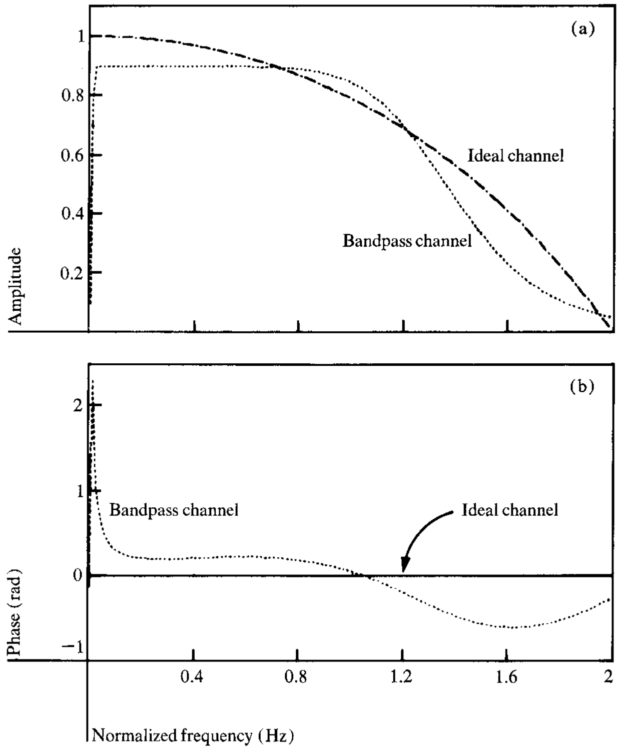


Figure 4 Transfer function response for ideal and bandpass filter channels after gain and phase correction.

Dynamic programming solution

The dynamic programming method proposed here to find the WCP for a constrained code is based on a finite-state description of the code. This description assumes that the code consists of a set of equal-length encoded

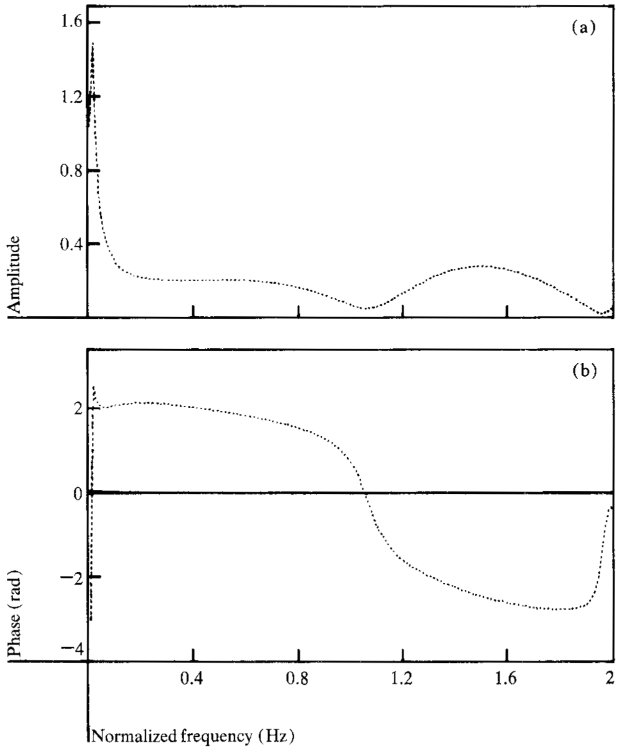


Figure 5 Transfer function response of errror channel computed for bandpass filter example.

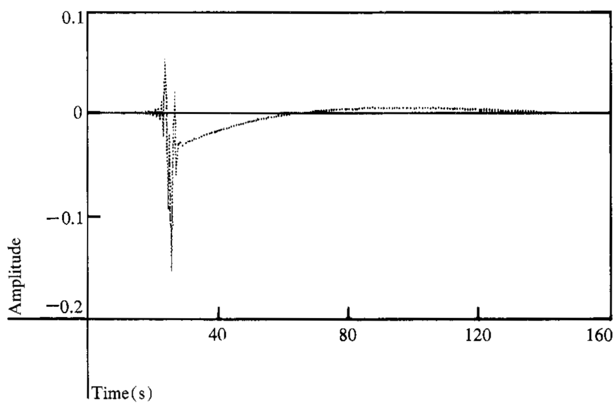


Figure 6 Impulse response of error channel computed for bandpass filter example.

waveforms, termed codewords here, each containing M bits and having a duration of MT seconds. The codewords are grouped into N subsets labeled by the integers $i = 1, 2, \cdots, N$. In general, the subsets have differing numbers of codewords, and m_i is used to indicate the number in subset i. Thus, the maximum total number of code words is $m_1 + m_2 + \cdots + m_N$. However, the code may have the same codeword in one or more subsets, and the number of unique codewords may be smaller than this total.

The code is described by labeling the subsets as states and defining the rules for moving from one state to another. When the encoder is in state i, one of the m_i codewords in that subset is generated; the particular one selected is determined by the source input. The next state of the encoder, in turn, is determined by the codeword produced by the previous state. Thus, the encoder moves from state to state, generating a continuous waveform consisting of a sequence of codewords, each MT seconds in duration. The only restriction we place on the code is that all codewords within state i must be reachable by selecting an appropriate source input. This is a minor restriction, however, and does not significantly limit the application of our technique. For example, in addition to fixed-length codes, variable-length synchronous codes can also be used [2].

We denote the jth codeword in state i as $w_{i,j}(t)$ and assume that this waveform is zero outside the range $0 \leq t \leq MT$. The resulting channel input $x(t)$ produced by the encoder (Fig. 1) can then be expressed as

$$x(t) = \sum_{k} w_{i(k),j(k)}\,(t - kMT). \tag{14}$$

The problem of finding the WCP is that of finding the sequence of codewords $(w_{i(k),j(k)}(t - MT))$ which maximizes the error channel output $e(t)$ in Eq. (8). Given that the impulse response $h_E(t)$ is significantly different from zero for a length of time L seconds, the number of codewords entering into the integral that determines $e(t)$ is L/MT. Thus, if R is the smallest integer that exceeds L/MT, then the number of codewords in the WCP is not greater than $R + 1$. [One extra word is required for those cases in which $h_E(t - \alpha)$ is not perfectly aligned with the codeword boundaries.]

The method of finding the WCP proceeds by partitioning the integral calculation in Eq. (8) into $R + 1$ stages, each taken over exactly one codeword. The contribution to the kth stage, $C_{i,j,k}(t)$, is defined as

$$C_{i,j,k}(t) = \int_{kMT}^{(k+1)MT} w_{i(k),j(k)}\,(\alpha - kMT)\; h_E(t - \alpha)\,d\alpha, \tag{15}$$

and the error channel output is

$$e(t) = \sum_{k=0}^{R} C_{i,j,k}(t). \tag{16}$$

The problem of maximizing this quantity by choosing the sequence of codewords is a classical dynamic programming problem.

The maximization problem is iterative and follows these steps:

1. Assume the system is in stage R (the final stage) and in state i. Find

$$C'_{i,R}(t) = \max_j C_{i,j,R}(t) \tag{17}$$

for each $i = 1, 2, \cdots, N$. Equivalently, find a codeword in the ith state which produces the maximum contribution. (Note that since we have assumed that all codewords within a state are reachable by appropriate choice of input source bits, this maximization also defines that input sequence.)

2. Assume that the system is in stage $R - 1$ and state i. The maximum contribution over the last two stages is found by considering the contributions for each permissible codeword $C_{i,j,R-1}(t)$, $j = 1, 2, \cdots, m_i$, summed with the maximum obtained for its successor state at stage R. Thus, we calculate

$$C'_{i,R-1}(t) = \max_j \, [C_{i,j,R-1}(t) + C'_{f_i(w_{i,j}),R}(t)] \tag{18}$$

for each $i + 1, 2, \cdots, N$, where $w_{i,j}$ denotes the jth code word in state i, and $f_i(w_{i,j})$ is the next state of the system when the codeword $w_{i,j}$ is used at state i.

3. Repeat the procedure for all stages down to the first stage:

$$C'_{i,k}(t) = \max_j \, [C_{i,j,k}(t) + C'_{f_i(w_{i,j}),k+1}(t)], \tag{19}$$

for $i = 1, 2, \cdots, N$ and $k = R - 2, R - 3, \cdots, 0$.

The maximum over all i states, of the contribution $C'_{i,0}(t)$ gives the maximum total value of the integral in Eq. (8) that can be achieved using permissible code patterns and the assumed time shift t. This is the maximum value that can be achieved for $e(t)$ when all possible permissible code sequences are considered, and we denote it by

$$e'(t) = \max_i C'_{i,0}(t). \tag{20}$$

After finding $e'(t)$, the steps through the procedure are retraced. The sequence of states and codewords used at each stage to produce the overall maximum value is thus obtained and defines both the entire coded waveform and the corresponding sequence of source bits required to generate this waveform.

At this point we have found a pattern which gives maximum output from the error channel at time t. In general, when the time shift in Eq. (8) is changed from t to some other value t', the WCP sequence resulting from application of the above procedure changes, as does the correlation value $e'(t')$. This suggests that to find the final WCP, we need to solve the above problem for a continuum of values $0 \leq t \leq L$. However, the primary points in the waveform which are of interest for the ISI model used in this paper are those at the bit boundaries and centers. As a result, the search for t over the continuum mentioned above reduces to a search of those discrete values of t corresponding to these points within one codeword. Searches over more than one codeword are not required because no restrictions are placed on the starting state for the system. (Equivalently, the codeword selected from the current state depends only on the input, not on the previous codeword.) In summary, the final WCP is determined by finding the maximum value achieved by $e'(t)$ for $t = n\ (T/2)$, $n = 0, 1, \cdots, 2M - 1$. The waveform associated with this maximum value is the overall WCP for the code under consideration.

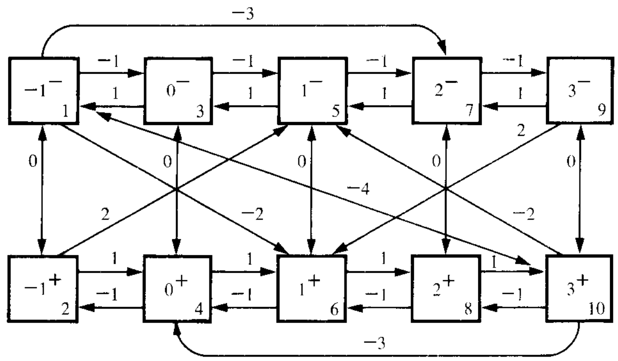


Figure 7 State diagram description of 4/5 rate code.

Practical example

The bandpass filter example described above can be used to exhibit the results one might obtain in practical channel applications. We choose a code having $(d,k) = (0,6)$ (see [1]). The finite-state description of this code is given in terms of its accumulated digital sum (DS) [10]. The DS of an encoded waveform at time t is simply the integral of that waveform up to time t. In this example, DS is computed only at the end of each codeword. It is expressed in bit periods, and $\pm$ superscripts are used to denote the actual signal level at the end of the last codeword. For example, 1^+ represents an accumulated DS of unity and a positive signal level. The digital sum of a codeword CDS is the integral of the associated waveform assuming that the starting level is +. Figure 7 shows the state diagram for the code of interest.

The states are numbered from 1 through 10 in the lower right-hand corner of each block, and the accumulated DS is presented in the center. This code, therefore, can attain only 10 distinct DS values at codeword boundaries due to the nature of its specific constraints. State transitions are determined by the CDS of the codeword to be used next from each state. These numbers appear on the lines that denote transitions from state to state. As a specific example, given that the encoder is in state 1, the next state of the system will be 6 if the output codeword has a CDS of −2.

Table 1 gives the codewords for each state and their associated CDS values. Sixteen codewords are available

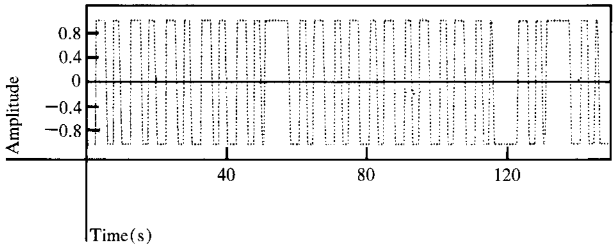


Figure 8 Worst-case pattern encoded waveform for bandpass filter example.

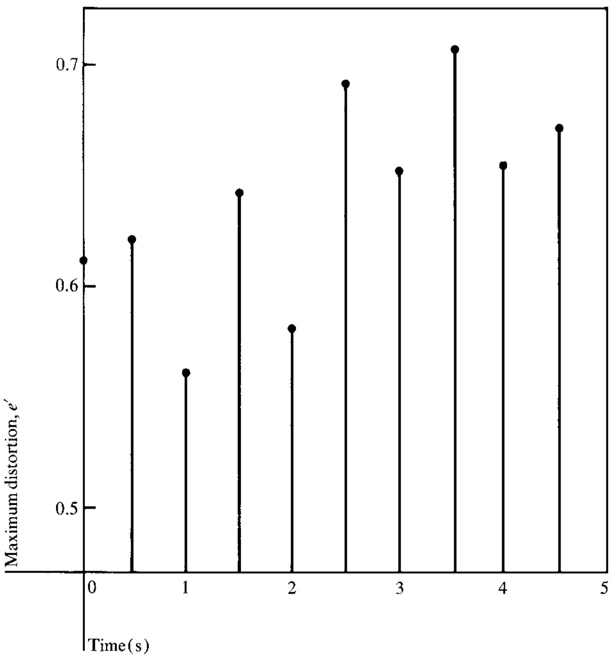
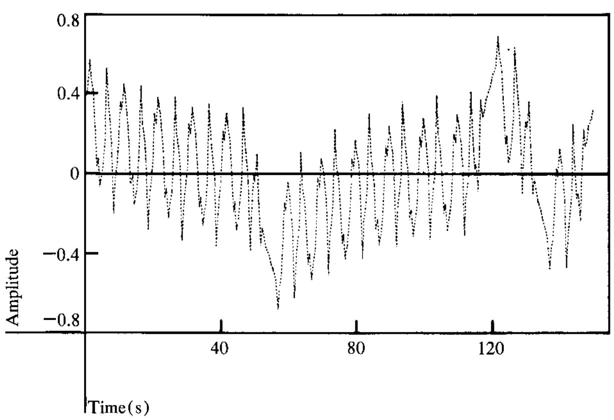


Figure 9 Maximum distortion level values computed for bandpass filter example.

Figure 10 Error-channel output waveform computed for bandpass filter example with WCP input waveform.



from each state, which allows transmission of four bits of source information. Each codeword has five binary digits, so the resulting rate of the code is 4/5. In addition, it can be shown that this code has a maximum digital sum variation (DSV) [10] of 10.

Dynamic programming techniques were used to compute the WCP for this code when used with the bandpass filter example. (Figure 6 shows the impulse response $h_E(t)$ of the error channel for this example.) The resulting WCP waveform is presented in Fig. 8. In this example, the maximum distortion value e' was 0.72 and occurred at a relative time shift of $3\frac{1}{2}$ bit periods. The maximum distortion levels for other initial time shifts are shown in Fig. 9.

As discussed previously, the error-channel output signal represents the ISI level in the channel. Figure 10 shows this signal when the WCP is applied to the input. A peak negative level is seen at bit number 57 and an equally large positive peak value occurs at bit 122. It is not too surprising that peak probability of error for the overall channel also occurs at these points, as illustrated below.

Performance evaluation

A variety of methods are available for studying the properties of any particular encoded-signal waveforms. Among the most commonly used are the eye pattern and the probability of error observed when the waveform is transmitted through the channel of interest. We now review these basic techniques and show their application to the WCP derived for the 4/5 rate code example. For comparison, the same techniques are used to measure the properties of the 4/5 rate code when the encoded waveform is generated using a pseudo-random source. In both cases, the channel of interest is our bandpass filter example described.

• *Eye pattern representation*

Eye patterns [11] for encoded signal waveforms have been used for many years to evaluate the performance of a given signal at the output of a known channel. Eye patterns are easily generated using an oscilloscope in which the horizontal sweep rate is synchronized with the pulse-repetition rate of the code. The horizontal sweep rate is selected such that only two or three bit periods are presented on the oscilloscope screen. The resulting display is then a superposition of all possible waveforms over these few bit periods. Figure 11 shows a typical experimental eye pattern. The vertical eye opening is defined as the distance between the least negative voltage sweep and the lowest positive voltage sweep as measured at the mid-bit period point. This opening is an effective measure of detection performance because output deci-

Table 1 4/5 Rate Code Table.

States 1, 10			*States 2, 9*			*States 3-8*		
Code Word Number	*Code Word*	*CDS*	*Code Word Number*	*Code Word*	*CDS*	*Code Word Number*	*Code Word*	*CDS*
1	0 0 1 0 0	0	1			1		
2	0 1 1 1 0	0	2			2		
3	0 1 0 1 1	0	3	Same as		3	Same as	
4	1 1 0 1 0	0	4	States 1, 10		4	States 1, 10	
5	1 1 1 1 1	0	5			5		
6	1 0 1 0 1	0	6			6		
7	1 0 0 0 0	−4	17	0 1 1 1 1	1	17		
8	1 0 0 0 1	−3	18	0 0 1 0 1	1	18	Same as	
9	0 1 0 0 0	−2	19	0 1 0 1 0	1	19	States 2, 9	
10	1 1 1 0 0	−2	20	1 1 1 1 0	1	20		
11	1 0 1 1 0	−2	21	1 0 1 0 0	1	21		
12	1 0 0 1 1	−2	22	1 1 0 1 1	1	22		
13	1 0 0 1 0	−1	23	0 0 0 1 0	2	13	Same as	
14	1 0 1 1 1	−1	24	0 0 1 1 1	2	14	States 1, 10	
15	1 1 1 0 1	−1	25	0 1 1 0 1	2	15		
16	0 1 0 0 1	−1	26	1 1 0 0 1	2	16		

sions are generated by sampling at the mid-bit point and interpreting positive samples as $+1$ and negative values as -1. Small vertical eye openings infer a decreased tolerance for the effects of noise in the output signal. Similarly, horizontal eye opening is defined along the zero-output voltage reference line and is a measure of the sensitivity of the received waveform to errors in the output clock which is used to locate the sampling point.

The eye pattern shown in Fig. 11 was generated using the experimental bandpass filter channel and the 4/5 rate WCP encoded waveform shown in Fig. 8. The experimental WCP used to obtain these measurements is shown in Fig. 12. All measurements were taken under conditions of high output signal-to-noise and ideal output clocking. As a result, the eye pattern is a direct consequence of intersymbol interferences in the output waveform.

The eye pattern observed for this channel using a pseudo-random data source is illustrated in Fig. 13. A nine-bit shift register with appropriate feedback taps [12] was used to produce a pseudo-random source sequence of length $2^9 - 1 = 511$ bits. This sequence was repeated continuously and used as an input to the channel encoder. Both the vertical and horizontal eye openings for this pattern are larger than those shown for the worst-case waveform. Although the pseudo-random source cycles through all possible nine-bit patterns, with a repetition rate of 511 bits, it will not necessarily generate an encoded pattern identical to the 150-bit worst-case pattern derived in the section entitled "Dynamic programming solution." In fact, it can easily be verified that the pattern does not occur with this pseudo-random sequence. One would be assured of observing this WCP by using a 150-bit shift register to generate the pseudo-random sequence.

Figure 11 Experimental eye pattern obtained using bandpass channel and 4/5 rate WCP.

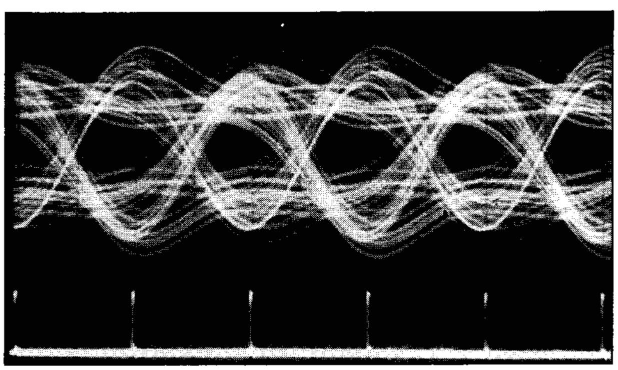
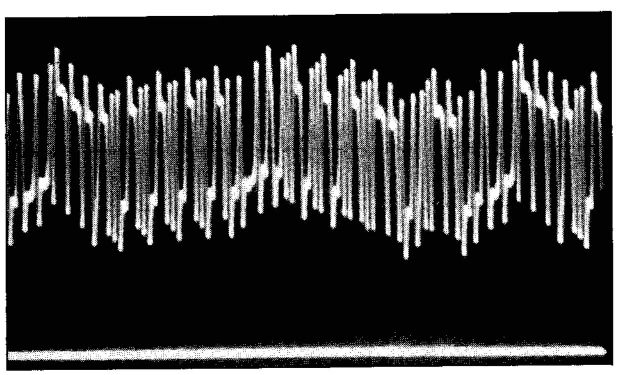


Figure 12 Experimental WCP waveform for 4/5 rate code.



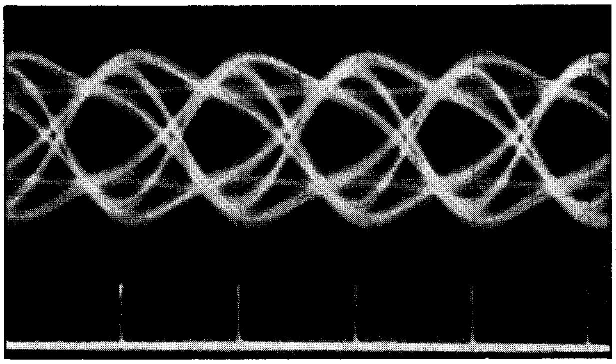


Figure 13 Experimental eye pattern obtained using bandpass channel and 4/5 rate code with pseudo-random source.

However, the repetition period for this generator would be prohibitively long. Although shorter register configurations may exist which yield this or an equivalent WCP, we are not aware of any procedure for finding such configurations.

- *Probability-of-error calculations*

The use of probability-of-error calculations to evaluate both transmission code performance and channel characteristics has received wide attention in the communications field [13]. This interest is a direct consequence of the fact that probability of error serves as an efficient reliability measure of communication systems. Unfortunately, although any given channel and transmission code can be evaluated using this measure, it is extremely difficult to derive analytically closed-form expressions for encoded waveforms which will lead to a maximum probability of error when one is dealing with a realistic channel and code constraint. The procedure used to derive the worst-case pattern presented in this paper, for example, is not guaranteed to produce a maximum probability-of-error waveform. However, since the pattern is designed on the basis of maximum intersymbol interference, which is known to be closely related to probability-of-error performance [14, 15], it does provide significantly higher error rates than those observed for randomly encoded data.

The method we used to compute probability of error is based on the model shown in Fig. 1. The known worst-case pattern $x(t)$ is presented as an input waveform to the bandpass filter channel. Since the transfer function $H_A(f)$ is known, the channel-output signal $y(t)$ can be computed numerically. Figure 14 shows a portion of the 4/5 rate worst-case pattern and the corresponding output waveform which is computed using ten samples per bit period. The input to the detection circuitry was presumed to consist of this signal plus additive white gaussian noise. A detector model which included the effects of clocking inaccuracies was used to compute probability of error for each bit. The bit probabilities of error were averaged over all 150-bit periods in the WCP to obtain an overall probability of error value. The details of this calculation are best described in terms of the samples of a particular bit.

We let $Y_i(1)$, $Y_i(2), \cdots, Y_i(10)$ represent the ten successive samples of the output waveform during the ith bit. Assuming that this bit was originally positive at the channel input, the average probability of error observed by an ideal detector which compares $Y_i(m)$ with zero would be given by $P_e(i,m)$:

$$P_e(i, m) = \int_{-\infty}^{0} \Phi[Y_i(m), \sigma_n]\, d\alpha. \tag{21}$$

In this expression, $\Phi[Y_i(m), \sigma_n]$ denotes a gaussian probability density with mean $Y_i(m)$ and standard deviation σ_n. In effect, Eq. (21) is the probability that the mth sample of the ith bit is negative when gaussian noise with standard deviation σ_n is added to the channel output. To account for the effects of clocking variation, we assume that the detector operates on only one sample during any given bit period and that the probability of the mth sample is $P_c(m)$. In our model, $P_c(m)$ was a truncated, gaussian probability density with mean 0.5 (mid-bit point) and standard deviation σ_c.

With this model, assuming independent clocking errors, the average probability of error observed during the ith output bit $[\bar{P}_e(i)]$ is given by

$$\bar{P}_e(i) = \sum_{m=1}^{10} P_e(i, m)\, P_c(m). \tag{22}$$

The overall average probability of error P_e' for the entire 150-bit WCP was then computed as

$$P_e' = \frac{1}{150} \sum_{i=1}^{150} \bar{P}_e(i). \tag{23}$$

Figure 15 summarizes the overall average probabilities of error P_e obtained using the WCP and a pattern obtained using a pseudo-random input. Results are presented as a function of output signal-to-noise ratio in dB, defined as 10 times $\log_{10}$ of the ratio of the average squared value of $y(t)$ to noise power σ_n^2. The standard deviation, σ_c, of the gaussian distribution used to describe clocking error was set at 0.1, one tenth of a bit period, so that truncation is done at ± 4 standard deviations. Examination of probability of error on a bit basis shows that peaks in the curve occur at bit numbers 57 and 122, which agrees exactly with the locations of peak output from the error-channel, as shown in Fig. 10. This behavior was also observed with a clocking error value of $\sigma_c = 0.01$ corresponding to one one-hundredth of a bit. This is not a surprising result, since large intersymbol interference values are likely to produce errors in the presence of noise.

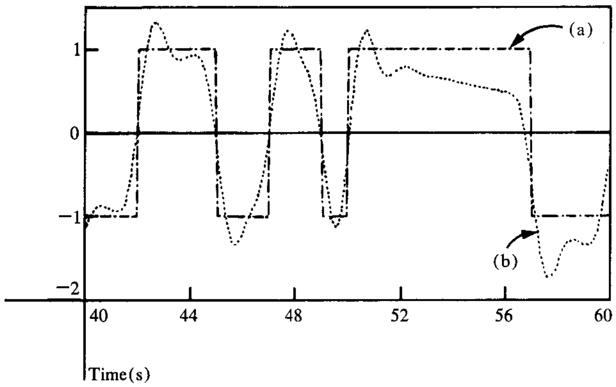


Figure 14 Encoded WCP waveform (a) and computed bandpass channel output waveform (b) for 4/5 rate code.

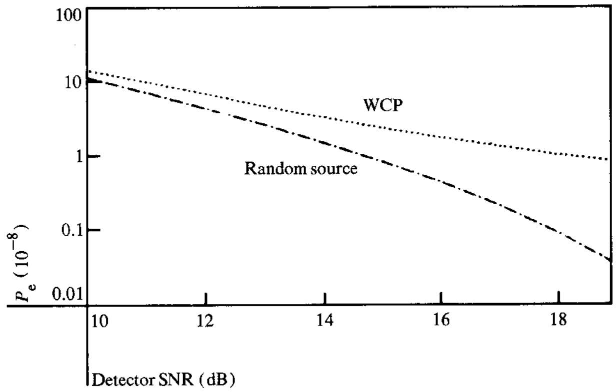


Figure 15 Average probability of error vs detector SNR for 4/5 rate WCP and random source, $\sigma_c = 0.1$.

Summary and conclusions

A procedure has been presented for generating worst-case patterns (WCP) for a wide class of transmission code and communication channel combinations. An error channel is constructed by taking the difference between the actual channel available for use and an ideal channel. Output of the error-channel for a given pattern input provides a measure of ISI for that pattern. The procedure selects patterns within a given set of code constraints, which maximizes ISI at sample time. The procedure uses dynamic programming techniques to select a WCP from the set of all possible code patterns of a given length. The number of members of such a set is usually so large that an exhaustive search is impractical. Two methods were used to evaluate the WCP obtained using this technique in a sample situation. The eye pattern for the WCP shows significant degradation over the eye pattern for a randomly generated sequence. Probability of error also shows a significant degradation for the WCP. In addition, the peak values for probability of error occur at exactly the points where error channel output is maximal. The WCP concept we have described is useful for evaluating the performance of a proposed transmission code since it isolates the worst than can happen, in terms of the amount of ISI energy. There may be, in fact, many equally bad patterns. Further study is needed to determine the number of such patterns compared to the total number of patterns to indicate the likelihood of their occurrence. Similarly, it would be useful to know how many patterns exceed a given ISI threshold for a particular code. The relationship between code parameters, such as *d*, *k*, DSV etc. and the probability of error needs to be established.

In summary, although many questions remain unanswered before a "best" code can be selected for a specified channel, the WCP provides significant insight into a code's potential performance.

References

1. P. A. Franaszek, "Sequence-state Methods for Run Length Limited Coding," *IBM J. Res. Develop.* **14,** 376 (1970).
2. P. A. Franaszek, "On Synchronous Variable Length Coding for Discrete Noiseless Channels," *Info. Control* **15,** 155 (1969).
3. R. Bellman, *Dynamic Programming*, Princeton University Press, Princeton, NJ, 1957.
4. M. H. Mickle and T. W. Sze, *Optimization in Systems Engineering*, Intext Ed. Pub., San Francisco, 1972, Ch. 8.
5. H. Nyquist, "Certain Factors Affecting Telegraph Speed," *Bell Syst. Tech. J.* **3,** 324 (1924).
6. H. Kobayashi, "Coding Schemes for Reduction of Intersymbol Interference in Data Transmission Systems," *IBM J. Res. Develop.* **14,** 343 (1970).
7. D. T. Tang and L. R. Ball, "Block Codes for a Class of Constrained Noiseless Channels," *Info. Control* **17,** 436 (1970).
8. W. R. Bennet, *Introduction to Signal Transmission*, McGraw-Hill Book Co., Inc., New York, 1970.
9. A. V. Oppenheim and R. W. Schafer, *Digital Signal Processing*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1975.
10. A. Croisier, "Introduction to Pseudoternary Transmission Codes," *IBM J. Res. Develop.* **14,** 354 (1970).
11. R. W. Lucky, J. Salz, and E. J. Weldon, Jr., *Principles of Data Communication*, McGraw-Hill Book Co., Inc., New York, 1968.
12. S. W. Golomb, *Shift Register Sequences*, Holden-Day Publishing Co., Inc., San Francisco, 1967.
13. R. W. Lucky, "A Survey of the Communication Theory Literature: 1968-1973," *IEEE Trans. Info. Theory* **IT-19,** No. 6, 725 (1973).
14. R. Lugannani, "Intersymbol Interference and Probability of Error in Digital Systems," *IEEE Trans. Info. Theory* **IT-15,** 682 (1969).
15. F. E. Glave, "An Upper Bound on the Probability of Error due to Intersymbol Interference for Correlated Digital Signals," *IEEE Trans. Info. Theory* **IT-19,** 440 (1973).

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