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Boundary Layer Around a Liquid Jet

Abstract: The phenomenon of air wake caused by a train of liquid drops is studied in this paper by approximating the train to a cylindrical jet emerging from a nozzle. The boundary layer equations are derived by applying continuity of jet mass and matching the loss of jet momentum with the air drag on the jet. These equations are solved numerically and compared with experiments in which the velocity change of the jet along the stream is carefully measured through measurements of drop distances. Good agreement is obtained between experimental results and the analysis.

Introduction

This paper deals with boundary layers of air induced by a continuous liquid jet emerging from a nozzle. Similar boundary layers have been studied for cases in which air of initially uniform velocity flows over a cylindrical rod [1–4] and in which a solid cylinder issues out of a wall with a constant velocity, as in wire extrusions [5]. The above two cases are different from each other in that, although both boundaries grow in the direction of air flow, the relative air motions with respect to the rods are opposite to each other. The present study, although closely related to the latter case, differs from both cases in that the rate of momentum transfer from the cylinder to the boundary layer through skin friction varies along the axial coordinate because the transfer causes the jet to slow down, resulting in growth of the cross section of the jet. Recently, Hendriks treated such problems assuming a linearly varying line source of momentum. The resulting similarity equation is solved numerically for the air flow [6]. In the present analysis, the momentum variation is derived by matching it with the momentum gain in the air flow and allowing the growth of the sectional area as the jet loses its momentum.

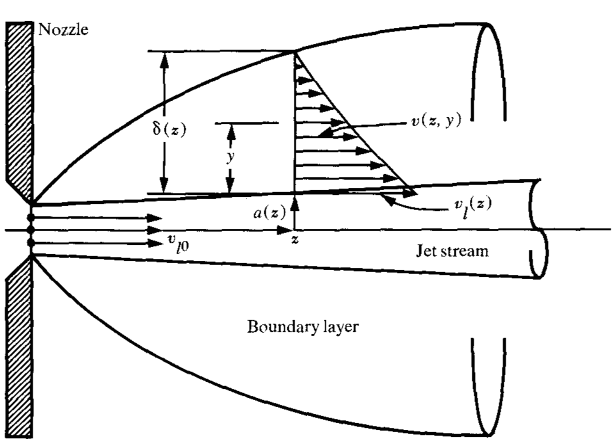

Figure 1 Boundary layer induced by a jet emerging from a nozzle.

An important well known characteristic should be recalled about a liquid jet. A liquid cylinder is extremely unstable, and the jet has a great tendency to break up into drops soon after leaving the nozzle. A slight excitation at a fixed frequency turns the jet into a stream of regularly spaced drops. When the drops are observed visually using stroboscopic lighting, the momentum loss of the drops to the air is easily seen from the gradual change in drop spacing downstream along the axis.

The wakes or boundary layers induced by such trains of drops are important in influencing the flight patterns of the drops in the vicinity of the stream. Because analysis of the stream of drops would be unreasonably complex, we present a method that approximates the problem by replacing the drops with an equivalent section of cylindrical shapes. It is shown that the analytically obtained momentum losses of liquid jets match experimental data very well.

Analysis

With reference to Fig. 1, the momentum and continuity equations applied to a boundary layer of thickness δ and a liquid jet of radius a are

$$2\pi\rho_a \int_0^{\delta(z)} [a(z) + y]\, v^2(z, y)dy + \rho_l\, \pi a^2(z) v_l^2(z) = \rho_l \pi a_0^2 v_{l0}^2, \qquad (1)$$

$$\frac{d}{dx}\,[\pi\rho_1 a^2(z)v_1^2(\bar{z})] = 2\pi a(z)\mu_a\frac{\partial v}{\partial y}\bigg|_{y=0}, \tag{2}$$

$$\pi a^2(z)v_1(z) = \pi a_0^2 v_{10}. \tag{3}$$

Here, the coordinate z is in the jet direction, and y is measured radially from the surface of the liquid jet. Velocity, density, and viscosity are denoted by v, ρ, and μ, respectively, with subscripts a and l for air and liquid, respectively. The subscript 0 denotes the initial values at the nozzle. Thus, Eq. (1) represents the conservation of axial momentum, which is given in the right-hand side, whereas (2) matches the rate of momentum loss to the skin friction on the jet. Mass conservation, or the continuity of the liquid jet, is expressed in Eq. (3).

An approximate solution for a boundary layer equation in integral form such as in Eq. (1) is normally obtained by assuming a velocity profile of the air flow with unknown parameters that are determined by satisfying certain boundary conditions in addition to the continuity and momentum equations, Eqs. (1–3). For boundary layer analyses of flat plates, the most popular form of profile is a polynomial of y. However, for cylindrical objects, because of the diverging flux characteristics in the radial direction, a logarithmic profile would be most appropriate [3–5]. Thus, the velocity profile of the air flow induced by the liquid jet is assumed to be [5]

$$v(z, y) = v_1(z)\left\{1 - \frac{1}{\beta(z)}\ln\left[1 + \frac{y}{a(z)}\right]\right\}, \tag{4}$$

where

$$\beta(z) = \ln\left[1 + \frac{\delta(z)}{a(z)}\right] \text{ or} \tag{5}$$

$$\frac{\delta(z)}{a(z)} = e^{\beta z} - 1. \tag{5}$$

This velocity profile satisfies the following boundary conditions:

$$v(z, 0) = v_1(z), \tag{6}$$

$$\left[\frac{\partial^2 v}{\partial y^2} + \frac{1}{a}\frac{\partial v}{\partial y}\right]_{y=0} = 0, \tag{7}$$

$$v(z, \delta) = 0. \tag{8}$$

The conditions of Eqs. (6) and (8) are obvious; i.e., the air velocity is zero at the outer boundary and is equal to the liquid velocity at the interface. Equation (7) is the consequence of applying the differential momentum relationship of the air at the interface boundary but away from the nozzle [5]. Another boundary condition usually required is the vanishing slope at the outer boundary. This condition is not satisfied exactly by the profile of Eq. (4). However, we do not regard this as important for the conditions at the interface used here [5]. An effort

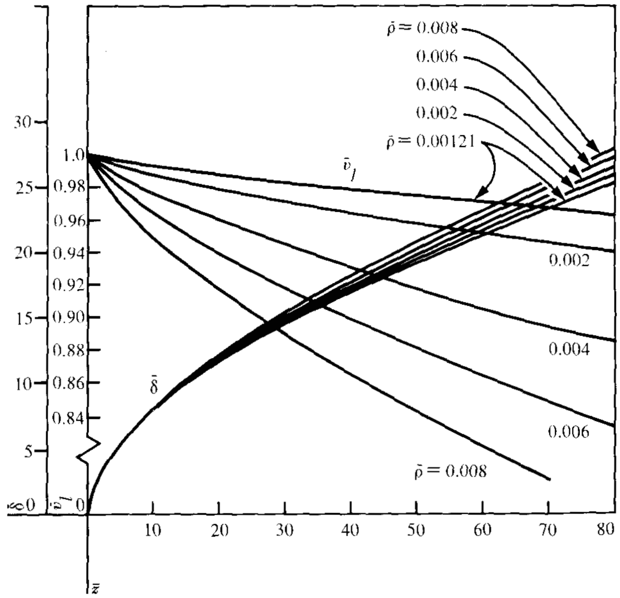

Figure 2 Stream velocity and boundary layer thickness for various values of density ratio.

to satisfy this extra condition by introducing another unknown parameter results in extreme complexity.

Substitution of Eq. (4) into Eqs. (1–3) with elimination of $a(z)$ results in

$$\bar{v}_1(z) = \frac{v_1(\bar{z})}{v_{10}} = \frac{1}{1 - \bar{\rho}\left[1 + \frac{1}{\beta} - \frac{1}{2\beta^2}(e^{2\beta} - 1)\right]}, \tag{9}$$

$$\frac{d\bar{v}_1}{d\bar{z}} = -\bar{\rho}\,\frac{\bar{v}_1(\bar{z})}{\beta(\bar{z})} \tag{10}$$

or, by substituting Eq. (9) into (10),

$$\frac{d\beta(\bar{z})}{d\bar{z}} = \frac{\beta^2 - \bar{\rho}\,[\beta^2 + \beta - \frac{1}{2}(e^{2\beta} - 1)]}{\beta(1 + e^{2\beta}) - (e^{2\beta} - 1)}, \text{ or} \tag{11}$$

$$\bar{z} = \int_0^{\beta} \frac{\beta(1 + e^{2\beta}) - (e^{2\beta} - 1)}{\beta^2 - \bar{\rho}[\beta^2 + \beta - \frac{1}{2}(e^{2\beta} - 1)]}\,d\beta, \tag{12}$$

where

$$\bar{z} = \frac{4}{R_e}\frac{z}{a_0},\ R_e = \frac{2a_0\rho_a v_{10}}{\mu_a},\ \text{and } \bar{\rho} = \frac{\rho_a}{\rho_1}. \tag{13}$$

Thus, for a given value of ρ, numerical integration of Eq. (11) or (12) yields β as a function of $\bar{z}$ or $\bar{z}$ as a function of β. Then, the stream velocity $v_1(\bar{z})$, stream radius $a(z)$, boundary layer thickness $\delta(z)$, and air flow profile $v(z, y)$ are obtained from Eqs. (9), (3), (5), and (4), respectively. Since the initial values of ρ and δ are zero, if Eq. (11) is to be integrated numerically using a finite difference method, a difficulty arises since the right-hand side of Eq. (11) is singular. However, this inconvenience can be

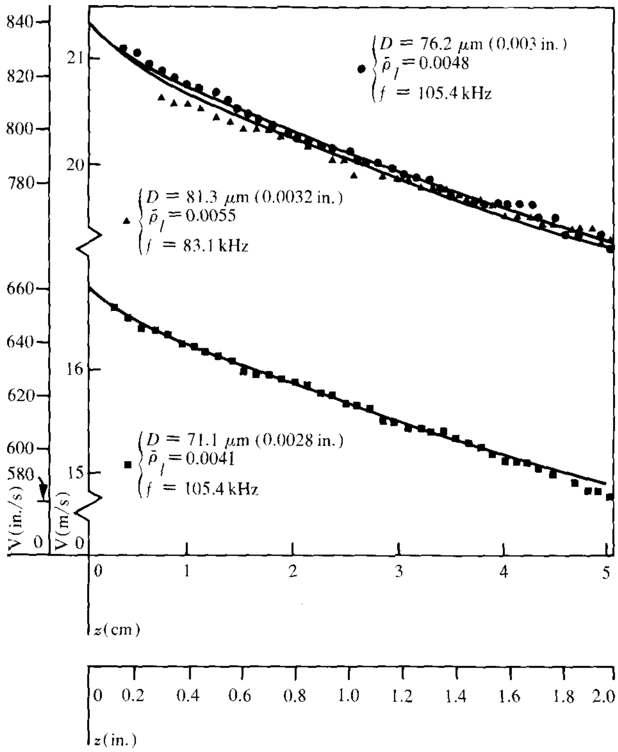

Figure 3 Comparisons of experimental and analytical results of stream velocities for a 38.1-μm (1.5-mil) nozzle.

easily circumvented if Eq. (12) is integrated to a finite distance away from the origin and the value of β at the new point is used as the initial condition. Figure 2 shows some results obtained by applying this method using Eqs. (11) and (12).

As seen in Eqs. (11) and (12) and in Fig. 2, the Reynolds number is combined into the dimensionless axial coordinate, $\tilde{z}$, Eq. (13), playing its role implicitly. This makes the density ratio, $\bar{\rho}$, the only parameter to be varied in numerical computations. For a jet of water in air, this value is 0.00121. However, even in this case, if the above analysis is to be applied, using the value 0.00121 directly would not be realistic because the jet breaks into drops soon after leaving the nozzle. Therefore, some readjustment of this parameter value is necessary, which we now consider.

Application to streams of drops

For the foregoing analysis to be of any practical value, an approximate method must be found to apply it to a stream of drops. Of course, any alternative, more precise analysis of streams of drops is a very difficult task.

When the drops are generated in a regular pattern starting at a certain break-off distance, the first approximation is to ignore this break-off distance and assume that the drops are generated right at the nozzle. Thus the initial jet diameter is replaced by the drop diameter, which is larger than the nozzle diameter. Now the stream of drops is treated as if it were a continuous jet emerging from a nozzle of drop diameter with a uniform velocity profile. This step implies that the drag on streams of drops, which is a very complex phenomenon, is about equal to the drag on cylinders of that same diameter. In other words, spheres are replaced by cylindrical slugs of the same diameter and of length equal to drop distance. In this conversion, the shortening of drop distance due to slowdown is equivalent to an increase in jet diameter. Finally, since the cylindrical slugs have larger volumes, for the initial momentum of the jet to be maintained, the liquid density must be reduced and the value of the density parameter to be used in the analysis must be modified to $\bar{\rho} = \rho_a D^2 / (\rho_l d_0^2)$, where D and d_0 denote diameter of drop and nozzle, respectively.

As discussed in the next section, the rationale of such a model is that, unlike a single drop moving in still air, a continuous stream of drops, when spaced closely enough, will trap the air between the drops and move it with the train. Thus, it is visualized that there exists a cylindrical surface on which radial movement of air is negligible.

Experiments and discussion

To investigate the usefulness of the approximation model presented in the preceding section, simple tests were made on streams of drops generated with 38.1-μm (1.5-mil) and 48.3-μm (1.9-mil) nozzles. In these tests drop spacing was carefully measured along the axis of the jet, and drop velocities were obtained from the product of drop spacing and drop rate. The diameters of drops were obtained indirectly from the flow rate measured by collecting the fluid in a given time period. The results are summarized in Figs. 3 and 4. The solid lines about the experimental data points are the analytical results obtained by using the approximate model presented earlier.

It is seen from these figures that, even with the drastic assumptions used in approximating trains of droplets by cylinders, reasonably good agreement on momentum transfer is obtained between experiments and the analysis. With momentum transfer into the surrounding air known, the air flow pattern around the stream is easily estimated from the boundary layer equations.

The results in Figs. 2, 3, and 4 show that momentum variation of the stream is almost linear except near the nozzle, where the theory is not accurate because Eq. (7) is not satisfied and experimental data are not available. This linear momentum variation supports the hypothesis Hendriks used in his similarity solution on air flow induced by a line source of momentum [6].

For a drop diameter of 91.4 μm (3.6 mils), the case of a single drop moving in still air is presented in Fig. 4 by a dashed line. The velocity retardation is computed by using the empirical drag formula obtained by Beard and Pruppacher [7]. The drastic velocity retardation as compared to a stream of drops is due to the pressure gradient, or form drag, developed by the drop. In a continuous stream of drops, the trapped air between closely spaced drops moves with the drops and inhibits the development of an axial pressure gradient, making the skin friction more significant than the form drag. This would explain the good agreement between the results of the present analysis and experiments.

As the drop space is lengthened, the radial activity of air flow in the drop spaces increases, making form drag the more dominant effect. No study has been made to investigate this transitional behavior from skin friction to form drag, because of the difficulties of generating largely spaced drops or effectively eliminating intermediate drops. The present study seems to indicate that, for drop spacings applicable to ink jet technology (spacings of two to four drop diameters), the skin friction as used here is the dominant drag mechanism.

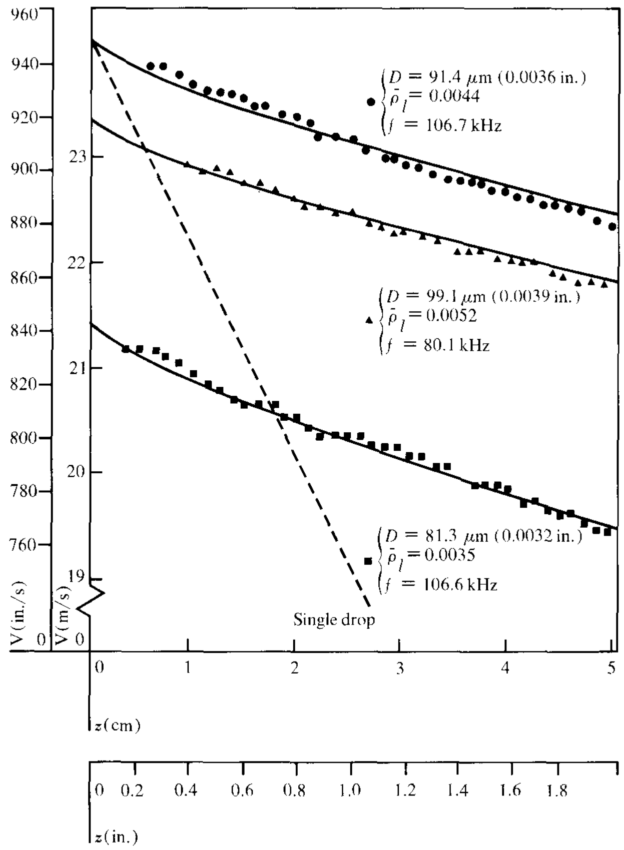

Figure 4 Comparison of experimental and analytical results of stream velocities for a 48.3-μm (1.9-mil) nozzle.

Summary

An understanding and quantitative analysis of air flow induced by a stream of drops is very important because of its influence on drop trajectories within the boundary layer. An exact solution for drops is very complex and not known. This paper has presented an approximate technique to study the boundary layer. Experimental evidence has been presented to show that the relatively simple analytical model can be used to investigate aerodynamic effects on drops passing near the stream of drops.

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