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Electric Motor Requirements for Positioning an Inertial Load

Abstract: This paper deals with the motor, the inertia ratio, and the power input requirements for moving an inertial load over a specified distance in a specified time. A linear speed-torque relationship is assumed, and selected motor parameters are normalized to the load to establish generally applicable solutions and characteristic curves. Emphasis is placed on the velocity-time diagrams and the relationship among the inertia ratio, the rated motor power, and the electrical input power. It is shown that optimization is possible for input power at stall, input power immediately following torque reversal, and average input power. Computer generated curves are presented for these three cases, and their relationships are discussed. Finally, it is shown that the motor time constant has a great influence on power requirements.

Introduction

This paper describes a general procedure for finding the optimal motor and the optimal gear ratio for moving a specified load a specified distance in a specified time. Optima are found for minimizing the rated power of the motor, the peak power input, and the average power input.

The movement of an inertial load from one position to another is a common engineering problem. The time-optimal control has been shown [1, 2] to be of the "bang-bang" type. For the second order system considered here, the solution consists of an initial period of maximum available acceleration torque followed by a period of maximum available deceleration torque. If the torque of the motor is constant and independent of speed, the optimum gear ratio renders the motor inertia reflected to the load equal to the load inertia [3]. However, this well-known result is not valid when the torque of the motor drops with speed, and this paper examines the special, yet frequent, case of a linear speed-torque relationship.

The differential equations for acceleration and deceleration are solved separately in conventional fashion, and then the two parts of the move are joined by stipulating that the velocity at the end of acceleration must equal the velocity at the beginning of deceleration. The resulting normalized equations describe the entire range of possible moves and permit the calculation of velocity diagrams.

The motor time constant, the rated motor power, and the inertia ratio (motor inertia to total inertia) are then introduced as parameters, facilitating optimization for power. The motor time constant and the rated power are normalized to the load to facilitate iterative computer solutions for optimum inertia ratios and powers. The results are presented as curves for all three cases of optimization.

The paper discusses the results of the optimization, the general equations for moves, the case of direct coupling between motor and load, the use of series resistance to boost motor power, and the significance of a short time constant of the motor.

Assumptions

This section presents the assumptions underlying our analysis for a linear speed-torque relationship. For positive torques, this relationship can be expressed as

$$Q_{\mathrm{m}} = Q_{\mathrm{mo}}\,(1 - \dot{N}_{\mathrm{m}}/\dot{N}_{\mathrm{m}\infty}), \tag{1}$$

where Q_{m} is the driving torque at motor velocity $\dot{N}_{\mathrm{m}}$, Q_{mo} is stall torque, and $\dot{N}_{\mathrm{m}\infty}$ is ultimate no-load speed. (Subscript m refers to the motor and subscript o to stall.) For negative torques,

$$Q_{\mathrm{m}} = -Q_{\mathrm{mo}}\,(1 + \dot{N}_{\mathrm{m}}/\dot{N}_{\mathrm{m}\infty}). \tag{2}$$

In words, Eq. (2) states that if a motor were electrically reversed while running at its ultimate speed, the braking torque would start at twice the stall torque, because of the additional braking torque from counter-emf.

Furthermore, throughout the analysis, we assume that

1. There is complete freedom in the choice of gear ratio between motor and load. However, we show that with a suitable choice of motor, the optimal gear ratio may be made equal to one.

2. The gear train has zero inertia. In cases in which the inertia of the gear train is significant, the estimated gear inertia should be added to that of the load.
3. The only losses in the system are the i^2R losses in the motor.
4. Inductive effects in the motor upon sudden application of a finite voltage are neglected.

These assumptions imply that there are no friction losses anywhere; that counter-emf is proportional to motor velocity; that current is zero at no-load speed; and that during acceleration, conversion efficiency from electrical to mechanical energy is equal to the ratio of actual speed to no-load speed.

Differential equations and normalized velocity diagram

The equations are solved in a piecewise linear manner by means of the known initial values for the acceleration period and the known terminal values for the deceleration period. These equations are then solved simultaneously to yield a necessary relationship between the total move time and the total move distance for given motor parameters. To begin, a motor with a linear speed-torque relationship is defined by Q_{mo}, $\dot{N}_{\mathrm{m}\infty}$, and its inertia I_{m}; and a move is defined by

N_{ls} = distance (in radians) over which load moves,

I_{l} = load inertia, and

T_{s} = move time.

Note that subscript l is associated with the load and subscript s with the sum of acceleration and deceleration distances and times.

During acceleration, the motor torque is positive, and the condition of equilibrium between motor torque and the inertial torques, all referred to the load axis, requires that

$$(I_{\mathrm{l}} + G^2 I_{\mathrm{m}})\ \ddot{N}_{\mathrm{l}} = Q_{\mathrm{mo}}\ (1 - \dot{N}_{\mathrm{l}} G / \dot{N}_{\mathrm{m}\infty}), \tag{3}$$

where N_{l} is the distance over which the load has moved and G is the gear ratio (motor distance/load distance).

If time T is normalized to move time T_{s} and load distance N_{l} to final move distance N_{ls} by setting

$$\alpha = T/T_{\mathrm{s}}, \tag{4}$$

and

$$\nu = N_{\mathrm{l}}/N_{\mathrm{ls}}, \tag{5}$$

the differential equation becomes

$$\nu'' + \frac{G^2 I_{\mathrm{m}}}{I_{\mathrm{l}} + G^2 I_{\mathrm{m}}}\ \frac{T_{\mathrm{s}} Q_{\mathrm{mo}}}{I_{\mathrm{m}} \dot{N}_{\mathrm{m}\infty}}\ \nu' = \frac{G^2 I_{\mathrm{m}}}{I_{\mathrm{l}} + G^2 I_{\mathrm{m}}}\ \frac{T_{\mathrm{s}}^2 Q_{\mathrm{mo}}}{I_{\mathrm{m}} \dot{N}_{\mathrm{m}\infty} N_{\mathrm{ls}}}, \tag{6}$$

where ν' and ν'' are derivatives of ν with respect to α.

From the gear ratio and the motor and load inertias, an inertia ratio may be defined as

$$\eta = G^2 I_{\mathrm{m}} / (I_{\mathrm{l}} + G^2 I_{\mathrm{m}})$$

$$\text{or } G = \eta^{1/2}\ (1-\eta)^{-1/2}\ I_{\mathrm{l}}^{1/2}\ I_{\mathrm{m}}^{-1/2}, \tag{7}$$

so that the differential equation becomes

$$\nu'' + \eta\ \frac{T_{\mathrm{s}} Q_{\mathrm{mo}}}{I_{\mathrm{m}} \dot{N}_{\mathrm{m}\infty}}\ \nu' = \frac{Q_{\mathrm{mo}}\ \eta^{1/2}\ (1-\eta)^{1/2}}{I_{\mathrm{m}}^{1/2}\ I_{\mathrm{l}}^{1/2}}\ \frac{T_{\mathrm{s}}^2}{N_{\mathrm{ls}}}. \tag{8}$$

This equation holds from the start of motion to the instant of torque reversal, which is identified by the subscript r. For convenience, the equation is rewritten as

$$\nu'' + 2\Gamma\ \nu' = 2\Delta, \tag{9}$$

where

$$\Gamma = \tfrac{1}{2} Q_{\mathrm{mo}} I_{\mathrm{m}}^{-1}\ \eta T_{\mathrm{s}}\ \dot{N}_{\mathrm{m}\infty}^{-1}, \tag{10}$$

and

$$\Delta = \tfrac{1}{2} Q_{\mathrm{mo}} I_{\mathrm{m}}^{-1/2}\ \eta^{1/2} (1-\eta)^{1/2}\ T_{\mathrm{s}}^2 I_{\mathrm{l}}^{-1/2}\ N_{\mathrm{ls}}^{-1}. \tag{11}$$

The solution now becomes

$$\nu' = \frac{\Delta}{\Gamma}\ (1 - e^{-2\Gamma\alpha}), \tag{12}$$

$$\nu_{\mathrm{r}}' = \frac{\Delta}{\Gamma}\ (1 - e^{-2\Gamma\alpha_{\mathrm{r}}}), \tag{13}$$

$$\nu = \frac{\Delta}{\Gamma}\left(\alpha - \frac{1 - e^{-2\Gamma\alpha}}{2\Gamma}\right), \tag{14}$$

and

$$\nu_{\mathrm{r}} = \frac{\Delta}{\Gamma}\left(\alpha_{\mathrm{r}} - \frac{1 - e^{-2\Gamma\alpha_{\mathrm{r}}}}{2\Gamma}\right) = \frac{\Delta}{\Gamma}\ \alpha_{\mathrm{r}} - \frac{\nu_{\mathrm{r}}'}{2\Gamma}. \tag{15}$$

During deceleration, the motor torque is negative, and by analogy to the case of acceleration, the differential equation becomes

$$\nu'' - 2\Gamma\nu' = -2\Delta. \tag{16}$$

This equation holds from the instant of torque reversal to the end of motion. Thus,

$$\nu' = \left(\nu_{\mathrm{r}}' + \frac{\Delta}{\Gamma}\right) e^{-2\Gamma(\alpha-\alpha_{\mathrm{r}})} - \frac{\Delta}{\Gamma}, \tag{17}$$

and at the final condition, $\alpha = 1$, the velocity must be zero, so that

$$\nu_{\mathrm{r}}' = \frac{\Delta}{\Gamma}\ (e^{2\Gamma(1-\alpha_{\mathrm{r}})} - 1). \tag{18}$$

The deceleration distance now becomes

$$\nu - \nu_{\mathrm{r}} = \left(\nu_{\mathrm{r}}' + \frac{\Delta}{\Gamma}\right) \frac{1 - e^{-2\Gamma(\alpha-\alpha_{\mathrm{r}})}}{2\Gamma} - \frac{\Delta}{\Gamma}\ (\alpha - \alpha_{\mathrm{r}}), \tag{19}$$

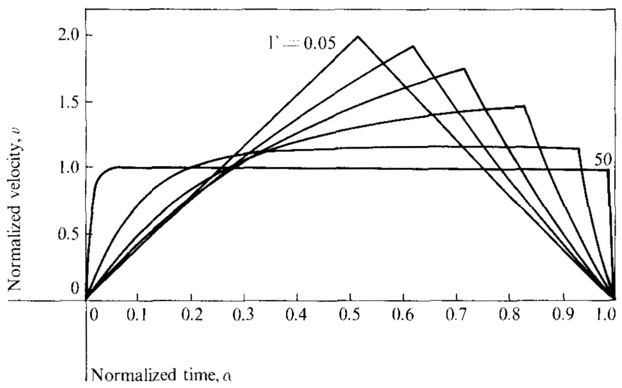

Figure 1 Velocity diagrams normalized to unity length of move.

and the total distance moved at $\alpha = 1$ is $\nu = 1$, so that with ν_r' from Eq. (18),

$$1 - \nu_r = \frac{\Delta}{\Gamma}\left(\alpha_r - 1 + \frac{e^{2\Gamma(1-\alpha_r)} - 1}{2\Gamma}\right)$$

$$= \frac{\Delta}{\Gamma}\left(\alpha_r - 1 + \frac{\nu_r'}{2\Gamma}\right). \tag{20}$$

It should be noted that the velocity ν_r' at the instant of torque reversal is the same for Eqs. (13) and (15), which refer to the end of acceleration, as for Eqs. (18) and (20), which refer to the beginning of deceleration. This equality is used to establish relationships among α_r, Γ, and Δ.

The time at reversal can be found from the simultaneous solutions of Eqs. (13) and (18) to be

$$\alpha_r = \frac{1}{2\Gamma}\ln\frac{e^{2\Gamma} + 1}{2} = \frac{1}{2} + \frac{1}{2\Gamma}\ln\cosh\Gamma. \tag{21}$$

Then, Eq. (13) yields

$$\nu_r' = \frac{\Delta}{\Gamma}\tanh\Gamma. \tag{22}$$

The ratio Δ/Γ may be expressed in terms of the basic parameters as

$$\Delta/\Gamma = \dot{N}_{l\infty}/(N_{ls}/T_s), \tag{23}$$

i.e., the ratio of the ultimate speed to average speed; this ratio is calculated next in terms of Γ. Addition of the normalized move distances for acceleration and deceleration, Eqs. (15) and (20), respectively, leads to

$$1 = \nu_r + (1 - \nu_r)$$

$$= \frac{\Delta}{\Gamma}[\alpha_r - (1 - \alpha_r)] = \frac{\Delta}{\Gamma}(2\alpha_r - 1), \tag{24}$$

from which Δ/Γ is obtained as

$$\Delta/\Gamma = \frac{1}{2\alpha_r - 1} = \frac{\Gamma}{\ln\cosh\Gamma}. \tag{25}$$

Equation (25), or the direct solution for Δ derived from it,

$$\Delta = \frac{\Gamma^2}{\ln\cosh\Gamma} \tag{26}$$

represents the basic relationship governing a move.

The velocity at the instant of torque reversal is now obtained from Eqs. (22) and (25) as

$$\nu_r' = \frac{\Gamma\tanh\Gamma}{\ln\cosh\Gamma}. \tag{27}$$

Thus, the normalized time and velocity at the instant of torque reversal have been expressed in terms of the single parameter Γ. For any value of Γ there is a single normalized velocity diagram that can be calculated from Eqs. (12) and (17); Fig. 1 shows a family of velocity diagrams for $\Gamma = 0.05$ to $\Gamma = 50$.

Convenient compound motor parameters and their normalization to the load

The next step in the investigation concerns the effect of the inertia ratio η on the power required to accomplish a specified move. Thus, η should be treated as an independent parameter, and the other parameters should be independent of η. Whereas the parameters Γ and Δ proved convenient for the solution of the differential equations and for establishing the velocity diagrams, they both contain η and we must revert to their respective definitions to separate η from the other components. Inasmuch as Δ and Γ are related by Eq. (26), it is necessary only to express either Γ or Δ in terms of η and parameters independent of η. The choice clearly is Γ because the definition of Γ in Eq. (10) involves fewer basic parameters than the definition of Δ in Eq. (11).

Although all subsequent equations could be written in terms of the basic motor and load parameters, certain combinations of these parameters are sufficient and reduce the numbers of parameters needed. For instance, possible compound parameters are

$$T_m = I_m\dot{N}_{m\infty}/Q_{mo}, \tag{28}$$

the time constant of the motor with no external load;

$$P_m = Q_{mo}\dot{N}_{m\infty}/4, \tag{29}$$

the rated power (peak mechanical power) of the motor; $\frac{1}{2}I_m\dot{N}_{m\infty}^2$, the kinetic energy of the motor at no-load speed; and Q_{mo}^2/I_m, the rate at which the kinetic energy of the motor with zero load increases at zero velocity (power rate). The latter is the only parameter of interest if the motor torque is independent of speed, and hence it is the dominant motor parameter for short moves.

Any two of these four parameters in conjunction with the inertia ratio are sufficient to define a motor's performance in moving an inertial load. The time constant is chosen as the first, because for a particular class of motors it is substantially independent of size and voltage input. It serves as a normalizing factor for move time T_s. The rated power of the motor is chosen as the second parameter, because we are concerned primarily with optimization of power.

The move time is normalized to the time constant of the motor by setting

$$\tau_s = T_s / T_m. \tag{30}$$

The peak power delivered to the load by a constant torque motor achieving a specified move is

$$P_l = 8N_{ls}^2 I_l / T_s^3, \tag{31}$$

and the rated power of the motor, P_m, is normalized to P_l by setting

$$\psi_m = P_m / P_l = \Delta^2 / 16(1-\eta)\Gamma \tag{32}$$

from the definitions of Γ and Δ in Eqs. (10) and (11). Now that the normalized move time τ_s and the normalized rated power ψ_m of the motor have been introduced, their relationship will be established.

If the move time is specified, τ_s can be calculated without knowledge of the motor power, and τ_s becomes the preferred independent parameter. Furthermore, Eqs. (10), (28), and (30) combine into

$$\Gamma = \eta\tau_s/2, \tag{33}$$

so that all parameters can be expressed in terms of η and τ_s. From Eq. (25),

$$\Delta/\Gamma = \frac{\eta\tau_s/2}{\ln\cosh(\eta\tau_s/2)}; \tag{34}$$

from Eq. (26)

$$\Delta = \frac{(\eta\tau_s/2)^2}{\ln\cosh(\eta\tau_s/2)}; \tag{35}$$

and from Eq. (32)

$$\psi_m = \frac{(\eta\tau_s/2)^3}{16(1-\eta)\ln^2\cosh(\eta\tau_s/2)}. \tag{36}$$

Thus, the rated motor power required to complete a move has been found as a function of the normalized move time τ_s and the inertia ratio η. Figure 2 illustrates this relationship and shows that for any particular value of τ_s, there is an associated inertia ratio η that minimizes the normalized rated power of the motor. The above selection of parameters has the advantage that for any given τ_s, the inertia ratio can be chosen and the gear ratio established so as to minimize power.

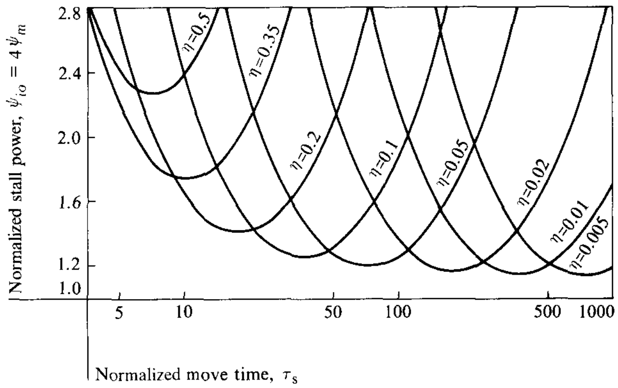

Figure 2 Normalized stall power vs normalized move time for several inertia ratios.

Consideration of electrical input power

In addition to the rated power of the motor, the electrical input powers (stall, peak, and average) are of interest. The electrical input powers, P_i, are also normalized to the load, so that

$$\psi_i = P_i / P_l = P_i T_s^3 / 8N_{ls}^2 I_l, \tag{37}$$

where the subscript i stands for input.

A motor with a linear speed-torque relationship develops its peak power at a speed ratio of one half. The efficiency of conversion of electrical power to mechanical power is also one half at this point, so that power input is twice power output. The stall input power is twice the input power at half speed; hence, the input power at stall is

$$P_{io} = 4P_m, \tag{38}$$

and, by means of Eq. (36), the normalized input power at stall is

$$\psi_{io} = 4\psi_m = \frac{(\eta\tau_s/2)^3}{4(1-\eta)\ln^2\cosh(\eta\tau_s/2)}. \tag{39}$$

The peak input torque, and hence the peak input power, occur immediately following torque reversal. The voltage is fixed, and the current during deceleration is proportional to $(1 + \dot{N}_m/\dot{N}_{m\infty})$, so that the normalized peak input power is, from Eq. (39),

$$\psi_{ir} = \psi_{io}\,(1 + \dot{N}_{mr}/\dot{N}_{m\infty}) = \frac{(\eta\tau_s/2)^3(1 + \dot{N}_{mr}/\dot{N}_{m\infty})}{4(1-\eta)\ln^2\cosh(\eta\tau_s/2)}. \tag{40}$$

The ratio $\dot{N}_{mr}/\dot{N}_{m\infty}$ is obtained from Eqs. (12), (22), and (33) as

$$\dot{N}_{mr}/\dot{N}_{m\infty} = \nu_r'/\nu_\infty' = \frac{\nu_r'}{\Delta/\Gamma} = \tanh\Gamma = \tanh\eta\,\tau_s/2, \tag{41}$$

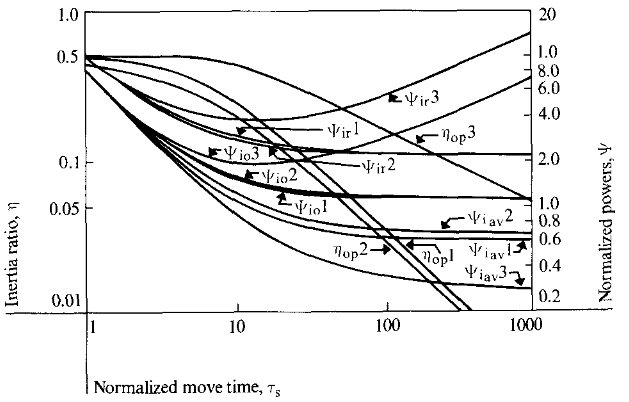

Figure 3 Optimized inertia ratios and normalized powers vs normalized move times. Rated power of motor is $\Psi_m = \frac{1}{4}\Psi_{io}$; 1 is optimization for Ψ_{io}; 2 is optimization for Ψ_{ir}; and 3 is optimization for $\Psi_{i\,av}$.

so that the normalized input power drawn at the instant of torque reversal becomes

$$\psi_{ir} = \frac{(\eta\tau_s/2)^3[1 + \tanh(\eta\tau_s/2)]}{4(1-\eta)\ln^2\cosh(\eta\tau_s/2)}. \tag{42}$$

The input energy during acceleration is found as the time integral of the power drawn from the supply between times 0 and α_r. The voltage is fixed, and the current is proportional to one minus the speed ratio, so that the normalized power input is, from Eqs. (38) and (12),

$$\psi_{io}\,(1 - \dot{N}_m/\dot{N}_{m\infty}) = 4\psi_m\, e^{-2\Gamma\alpha}. \tag{43}$$

The normalized energy input during acceleration thus becomes

$$4\psi_m \int_0^{\alpha_r} e^{-2\Gamma\alpha}d\alpha = \frac{2\psi_m}{\Gamma}\,(1 - e^{-2\Gamma\alpha}). \tag{44}$$

The energy input during deceleration is the same as the energy input during acceleration, because the change in momentum is the same, and therefore the current time integral must also be the same. The normalized total move time for the above equation is $\alpha = 1$, and therefore the normalized average input power for the complete move is numerically equal to the normalized total energy input:

$$\psi_{i\,av} = \frac{4\psi_m}{\Gamma}\,(1 - e^{-2\Gamma\alpha_r}). \tag{45}$$

With Γ, α_r, and ψ_m replaced by means of Eqs. (33), (21), and (36), respectively, the normalized average input power becomes

$$\psi_{i\,av} = \psi_m \frac{4\tanh(\eta\tau_s/2)}{\eta\tau_s/2} = \frac{(\eta\tau_s/2)^2\tanh(\eta\tau_s/2)}{4(1-\eta)\ln^2\cosh(\eta\tau_s/2)}. \tag{46}$$

Optimization and selection of motors and gear ratios

For any value of τ_s, the inertia ratio η can be optimized to result in a minimum of rated motor power, peak input power, or average input power.

Minimization of ψ_m in Eq. (39) results in the smallest motor of the class under consideration, or for a specified motor it leads to the minimum move time.

The normalized input power at reversal of Eq. (42) should be minimized if the instantaneous power drawn from the supply is of primary interest.

When heat dissipation of the motor is of greatest concern, the normalized average energy input of Eq. (46) should be minimized.

All three optimizations have been carried out by computer, and the results are presented in Fig. 3. Curves are shown for optimum η, $\psi_{io} = 4\psi_m$, ψ_{ir}, and $\psi_{i\,av}$ as functions of τ_s for all three optimizations. This permits examination of the sacrifice entailed in the remaining powers when η is optimized for any one of the three powers.

Generally, the move is specified and one has to select a motor that can accomplish it with either minimum rated power, minimum peak power, or minimum average power.

For any particular class of motors, $\tau_s = T_s/T_m$ can be calculated, and then Fig. 3 shows the optimum η and ψ_{io} for the three cases of optimization. The rated power of the motor is then obtained from Eqs. (32) and (36) as

$$P_m = P_1\,\psi_m = \tfrac{1}{4}P_1\,\psi_{io}. \tag{47}$$

Further, the stall input power is $4P_m = P_1\,\psi_{io}$, the peak input power from Eq. (42) is

$$P_{ir} = P_1\,\psi_{ir}, \tag{48}$$

and the average input power from Eq. (46) is

$$P_{i\,av} = P_1\,\psi_{i\,av}. \tag{49}$$

Where a compromise is indicated between optimized rated motor power (i.e., minimum motor size) and optimized average input power (i.e., minimum heat dissipation in the motor), a value of η between $\eta_{op\,1}$ and $\eta_{op\,3}$ of Fig. 3 may be considered for the particular τ_s, and then $\psi_m = \psi_{io}/4$ and $\psi_{i\,av}$ should be calculated from Eqs. (39) and (46).

Once η has been determined and a motor has been selected on the basis of $P_m = \psi_m P_1$, the motor inertia becomes known and the gear ratio can be calculated from Eq. (7). This completes the definition of the system.

Discussion

• *Optimization*

The asymptotic values for $\tau_s \to \infty$ are shown in Table 1. The framed values represent the minima for the respective optimized powers and may be approached with very long moves.

Figure 3 shows that there is relatively little difference between optimization for input power at stall and optimization for input power upon torque reversal. Their η curves and their normalized average power curves are very similar; their normalized stall power curves and their curves of normalized power upon torque reversal are practically identical.

On the other hand, the results of optimization for average power are quite different. Only the normalized average power has a finite asymptotic value as τ_s increases, whereas the normalized powers at stall and upon torque reversal have a broad minimum in the range of τ_s from 8 to 25, and then increase toward infinity. The physical explanation for this is that optimization for average power leads to a greater η for a given τ_s, hence to a greater $\Gamma = \eta\tau_s/2$ and a more nearly rectangular velocity diagram in Fig. 1. Thus, some energy is used to bring the system up to speed at the start of the move and to stop it at the end, but very little energy is consumed while the system runs at substantially no-load speed. The power drain upon torque reversal is high, of course, in this case and approaches twice the stall power for values of τ_s above 8.

A word of caution is in order here: This entire analysis is based on the only losses being i^2R losses in the motor. This assumption does not hold well in the case of long moves at high motor speed, where bearing friction and wind resistance are consuming some energy. The expressions derived are therefore minima and allowance should be made for the other losses.

• *Moves in general*

The foregoing selection of a motor and gear ratio was based on optimization for a particular move time. One may now wish to examine how well the system performs for other move times, where it is no longer optimized. Because η, $\dot N_{1\infty} = \dot N_{m\infty}/G$, and T_m have been defined, the examination may proceed by means of Eqs. (23) and (34):

$$\frac{N_{1s}}{T_s} = \Gamma \dot N_{1\infty}/\Delta = \dot N_{1\infty}\,\frac{\ln\cosh(\eta T_s/2T_m)}{\eta T_s/2T_m}, \qquad (50)$$

from which

$$N_{1s} = \dot N_{1\infty}\,\frac{2T_m}{\eta}\ln\cosh(\eta T_s/2T_m), \qquad (51)$$

or conversely,

$$T_s = \frac{2T_m}{\eta}\,\text{arc}\cosh e^{\eta N_{1s}/2T_s \dot N_{1\infty}}. \qquad (52)$$

The fully generalized equations are obtained from Eqs. (51) and (52) by replacing $\dot N_{1\infty}$ by $\dot N_{m\infty}/G$ and introducing G from Eq. (7):

Table 1 Asymptotic values as $\tau_s \to \infty$.

	Optimization for: input power at stall (Ψ_{io})	input power upon torque reversal (Ψ_{ir})	average input power ($\Psi_{i_{av}}$)
$\eta\tau_s/2$	1.813	1.756	∞
Ψ_{io}	$\boxed{1.134}$	1.144	∞
Ψ_{ir}	2.209	$\boxed{2.195}$	∞
$\Psi_{i_{av}}$	0.5935	0.6668	$\boxed{0.25}$

$$\dot N_{1\infty} = (1-\eta)^{1/2}\,\eta^{-1/2}\,I_m^{1/2} I_l^{-1/2}\,\dot N_{m\infty}. \qquad (53)$$

• *Direct coupling between motor and load*

So far, the gear ratio could be used to bring about the desired inertia ratio. When the motor is directly coupled to the load, this convenience disappears: the actual gear ratio is 1.0, and therefore the optimum gear ratio should also be in the vicinity of 1.0. With $G_{op} = 1$, Eq. (7) requires that

$$I_{m\,op} = I_l\eta_{op}/(1-\eta_{op}). \qquad (54)$$

Because η_{op} is a function of only $\tau_s = T_s/T_m$ for any of the three optimization procedures, the optimum motor inertia depends only on the load inertia, the time constant of the motor, and the move time, but not on the move distance.

As in the case of unrestricted gear ratio, the values of η_{op} and $\psi_m = \psi_{io}/4$ may be obtained from Fig. 3 and then $I_{m\,op}$ can be calculated from Eq. (54). However, the ψ_m and $I_{m\,op}$ so obtained may be mutually exclusive, particularly for long moves (large values of τ_s) in that a motor with inertia $I_{m\,op}$ may just not be able to develop sufficient torque. To keep inertia down, the rotor diameter must be small and, to develop the required torque, a small diameter rotor becomes very long. However, there is a practical limit to the slenderness of motor rotors, and beyond this limit optimization becomes impossible.

In such cases, one has to work with an actually attainable inertia ratio. Figure 2 and the equations relating η, ψ, and τ_s hold generally and are applicable whether or not η is optimized. Therefore, they all apply in the case of direct coupling between motor and load, but the ψ curves of Fig. 3 do not apply when the actual η differs from η_{op}.

• *Series resistance to boost motor power*

A series resistance may be used to boost motor power output at the expense of increased total power consumption. If

Table 2 Effects of adding series resistance.

	No series resistance	*Series resistance*
Stall torque	Q_{mo}	$Q^*_{\mathrm{mo}} = Q_{\mathrm{mo}}$
Inertia	I_{m}	$I^*_{\mathrm{m}} = I_{\mathrm{m}}$
No-load speed	$\dot{N}_{\mathrm{m}\infty}$	$\dot{N}^*_{\mathrm{m}\infty} = (1+\rho)\dot{N}_{\mathrm{m}\infty}$
Peak power output	$\frac{1}{4}Q_{\mathrm{mo}}\dot{N}_{\mathrm{m}\infty}$	$\frac{1}{4}Q_{\mathrm{mo}}\dot{N}_{\mathrm{m}\infty}(1+\rho)$
Rated power	$\frac{1}{4}Q_{\mathrm{mo}}\dot{N}_{\mathrm{m}\infty}$	
Time constant	T_{m}	$T^*_{\mathrm{m}} = T_{\mathrm{m}}(1+\rho)$
Normalized move time	$\tau_{\mathrm{s}} = T_{\mathrm{s}}/T_{\mathrm{m}}$	$\tau^*_{\mathrm{s}} = T_{\mathrm{s}}/T_{\mathrm{m}}(1+\rho)$

$$\rho = R_{\mathrm{e}}/R_{\mathrm{m}}, \tag{55}$$

where R_{e} is series resistance and R_{m} is motor resistance, and if the supply voltage is increased by $(1+\rho)$ to maintain the same stall current, the motor parameters and τ_{s} compare as shown in Table 2, where the asterisk identifies parameters modified by the addition of the series resistance.

Figure 3 holds for the parameters with asterisks as well, and η^*_{op}, $\psi^*_{\mathrm{io}} = 4\psi^*_{\mathrm{m}}$, and $\psi^*_{\mathrm{i\,av}}$ may be read off as functions of τ^*_{s} for any particular optimization. The respective powers consumed by the system are then obtained as the products $\psi^*_{\mathrm{i}} P_{\mathrm{l}}$, and the power dissipated within the motor as $\psi^*_{\mathrm{i\,av}} P_{\mathrm{l}}/(1+\rho)$. The rated power of the motor (without series resistance) is found as $\psi^*_{\mathrm{io}} P_{\mathrm{l}}/4(1+\rho)$.

For any specified move, a series resistance reduces the required power rating of a motor (of the same class). The reduction is greater for large values of τ_{s}, where ψ_{m} changes less for a given change in τ_{s}. The power dissipated within the motor actually decreases when a series resistance is used, and again the decrease improves as τ_{s} increases.

Thus, for large values of τ_{s}, the use of a series resistor becomes attractive when motor size or heat dissipation within the motor are limiting factors. In particular, series resistance should be considered when the load is directly coupled to the motor and η cannot be chosen at will. Here, the series resistance provides a most useful additional degree of design flexibility.

• *Effect of motor time constant on power consumption*

It has been shown [3] that an important measure of motor performance is the power rate at stall, which was discussed briefly under "Convenient compound motor parameters and their normalization to the load." In terms of the compound motor parameters used here for optimization, the power rate is $4P_{\mathrm{m}}T_{\mathrm{m}}^{-1}$. The ratio of power rate to power dissipation at stall is a factor of merit, which has been shown [3] to be very high for reluctance-type motors. Because power dissipation at stall was found here to be $4P_{\mathrm{m}}$, this ratio simply becomes T_{m}^{-1}. Hence, if the only losses in the motor are the i^2R losses and the speed-torque relationship is linear, the factor of merit is the reciprocal of the time constant of the motor. This interesting result implies that the time constant (a dynamic term usually found by measurement of steady state velocity or from frequency response data) can be determined without any dynamic measurements from input power at stall, stall torque, and inertia. Figure 3 shows that a move can be made with less power if τ_{s} is large (i.e., T_{m} is small), which is the same as a large factor of merit.

Hence, it is clear that a large factor of merit can improve power consumption, particularly in the case of short moves. The potential for improvements through higher factors of merit decreases as τ_{s} becomes larger, and Fig. 3 shows that with τ_{s} at 30, the theoretically possible improvement with the factor of merit becoming very large is below 20%.

In the region of τ_{s} above 20, an increase in the factor of merit in combination with a series resistance may result in a smaller motor with less internal heat dissipation and little increase in power consumption of the system.

Summary

If the speed-torque characteristic of the motor is linear, then the power required is proportional to the load inertia, the square of the move distance, and the inverse of the cube of the move time. All powers are normalized to these load-dependent parameters as given by Eq. (31). The normalized stall power, peak power, and average power are given by Eqs. (39), (42), and (46), respectively, in terms of the inertia ratio and the normalized move time. These equations are generally applicable and can be used for detailed evaluation of power for specific motors and inertia ratios.

However, each of these equations has an optimum inertia ratio for each value of move time, and the power required at the optimum inertia ratio is illustrated in Fig. 3 along with the optimum inertia ratio. By using Eq. (31) and Fig. 3, the power necessary to achieve a specified move can readily be found. Also, Fig. 3 or Table 1 can be useful in showing how near to an optimum design any operating design is.

In designing a system, Fig. 3 can be used to select the best gear ratio or motor inertia from the optimum η curves. The power curves show that for short moves (i.e., a ratio of move time to motor time constant of under 20), the motor time constant has a substantial influence on power requirements; the shorter the time constant, the lower the power required. This is very useful in designing or selecting a motor for a specified move.

If motor size or heat dissipation within the motor is important and if a motor can be found that makes the ratio of move time to motor time constant large (>20), then a series resistor for the motor was shown to be desirable.

References

1. B. Szabados, N. K. Sinha, and C. D. diCenzo, "Practical Switching Characteristics for Minimum-Time Position Control Using a Permanent Magnet Motor," *IEEE Trans. Ind. Electron. Control Instrum.* **IECI-19,** 69 (1972).
2. C. J. Brown and J. T. Ma, "Time-Optimal Control of a Moving-Coil Linear Actuator," *IBM J. Res. Develop.* **12,** 372 (1968).
3. G. C. Newton, Jr., and R. W. Rasche, "Can Electric Actuators Meet Missile Requirements," *AIEE Trans. Appli. Ind.* **80,** 306 (1961).

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