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On Some Relations between the Laplace and Mellin Transforms

Abstract: Several earlier papers have applied some identities relating the Laplace and Mellin transforms; this note develops certain such identities, achieving greater generality and rigor. Specifically each of these transforms is expressed as a contour integral involving the other, and an expansion of the Laplace transform is derived in terms of the functions $(s \cdot d/ds)^n(1+s)^{-1}$ with coefficients defined by the Mellin transform.

1. Introduction

Given a complex-valued locally integrable function f on $[0, +\infty)$, we define its Laplace transform by

$$Lf(s) = \int_0^\infty \exp(-st)\, f(t)\, dt \tag{1.1}$$

and its Mellin transform by

$$M[f; z] = \int_0^\infty t^{z-1} f(t)\, dt \tag{1.2}$$

whenever and wherever these integrals exist. Typically $M[f; z]$ is absolutely convergent on some strip $a < \mathrm{Re}(z) < b$, and $Lf(s)$ is absolutely convergent on some half-plane $\mathrm{Re}(s) > c$ [1]. However, both integral transforms may admit analytic continuations beyond these convergence domains.

Certain relations between these transforms have been found useful in approximating density functions for sums and products of random variables [2-6], and in generating asymptotic series for these and other integral transforms [7-11]. Several such relations from the former context are proved here with greater generality and rigor, from more explicit assumptions on f. These results are summarized in three theorems. Various authors have discussed similar applications, either without transform results [12-15] or only for special densities [16-20]. We hope that more careful proofs of these relations will facilitate their application in such contexts.

2. Integral relations

In this section we consider a locally integrable function f as before, and extend some integral relations between its two transforms. Our first theorem reviews two connected identities, the former noted implicitly by Handelsman and Lew [21] and derived explicitly by Prasad [22], and the latter gradually refined by Prasad [2, 4, 5] and proved as stated by Handelsman and Lew [9, 10]. For functions with suitable expansions near $+\infty$, further results of this type can be obtained by analytic continuation, even though the integral for the Mellin transform may converge in no strip [23].

Theorem 1 If $M[f; z]$ is absolutely convergent on $a < \mathrm{Re}(z) < b$ with $a < 1$, then

$$M[f; z] = M[Lf; 1-z]/\Gamma(1-z) \tag{2.1}$$

on $a < \mathrm{Re}(z) < b$, and

$$Lf(s) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} M[f; z]\Gamma(1-z)s^{z-1}dz \tag{2.2}$$

on $\mathrm{Re}(s) > 0$, for any c with $a < c < \min(1, b)$.

Proof If $a < \mathrm{Re}(z) < \min(1, b)$ and if

$$g(u) = u^{1-z}e^{-u},\ h(u) = u^{-z}f(u^{-1}) \text{ on } (0, +\infty), \tag{2.3}$$

then $u^{-1}g(u)$ and $u^{-1}h(u)$ are in $L^1(0, +\infty)$. By hypothesis $Lf(s)$ exists for $\mathrm{Re}(s) > 0$; by the Mellin convolution theorem [24]

$$\begin{aligned} M[Lf; 1-z] &= \int_0^\infty s^{-z}ds \int_0^\infty e^{-st}f(t)\,dt \\ &= \int_0^\infty s^{-1}ds \int_0^\infty g(s/u)h(u)\,du/u \\ &= M[g; 0]M[h; 0] \\ &= \Gamma(1-z)M[f; z]; \end{aligned} \tag{2.4}$$

and by analytic continuation, (2.4) holds for $a < \mathrm{Re}(z) < b$. If $z = c + iy$ and if $y \to \pm\infty$, then $M[Lf; z]$ decreases exponentially by a gamma function estimate [25, 26], so that (2.2) follows immediately by the Mellin inversion theorem [27].

Our next theorem has been obtained previously by Prasad [2, 4, 5] but is derived here from sharper assumptions. It is proved first on $\mathrm{Re}(z) > 1$ for quite general f, and is extended then to a larger domain for more restricted f. A preliminary lemma provides an important estimate for the latter part. This result, together with additional hypotheses, yields an expansion of $M[f; z]$ in powers and logarithms.

Lemma 1 For all positive t, any real y, and any positive c, p, q, x, let

$$g(t) = t^{z-1}e^{-ct} \text{ with } z = x + iy,$$

$$g(p, q; t) = (2\pi i)^{-1}\Gamma(z) \int_{-ip}^{iq} (c + s)^{-z}e^{st}ds. \tag{2.5}$$

Then for all positive t,

$$2\pi t|g(p, q; t) - g(t)| \leq \Gamma(x)e^{\pi|y|}(p^{-x} + q^{-x}). \tag{2.6}$$

Proof By a simple deformation, the inversion integral for the Laplace transform [28] yields

$$2\pi i g(t) = \Gamma(z) \int_C (c + s)^{-z}e^{st}ds \tag{2.7}$$

for all positive t, where the contour C runs from $-\infty$ to $-ip$ to iq to $-\infty$. If we define

$$I(r) = \int_{ir-\infty}^{ir} (c + s)^{-z}e^{st}ds \tag{2.8}$$

for any nonzero real r, then we obtain

$$2\pi it[g(p, q; t) - g(t)] = t\Gamma(z)[I(q) - I(-p)] \tag{2.9}$$

by subtraction. We now make direct estimates of these two remainder integrals, though we might express them both as incomplete gamma functions.

Indeed if $-\infty < s \leq 0$ and if $\phi = \arg(s + c + ir)$ then $|\phi| \leq \pi$, so that

$$|(s + c + ir)^{-z}| = |s + c + ir|^{-x}e^{-\phi y} \leq |r|^{-x}e^{\pi|y|}. \tag{2.10}$$

The result now follows by substitution from $|\Gamma(x)| \leq \Gamma(x)$ [29], and

$$|I(r)| = |\int_{-\infty}^{0} (s + c + ir)^{-z}e^{st+irt}ds|$$

$$\leq |r|^{-x}e^{\pi|y|} \int_{-\infty}^{0} e^{st}ds = t^{-1}|r|^{-x}e^{\pi|y|}. \tag{2.11}$$

Theorem 2 If $Lf(s)$ converges absolutely on $\mathrm{Re}(s) \geq -c$ for some $c > 0$, then $M[f; z]$ converges absolutely on $\mathrm{Re}(z) > a$ for some $a \leq 1$, and

$$M[f; z] = (2\pi i)^{-1}\Gamma(z) \int_{-c-i\infty}^{-c+i\infty} (-s)^{-z}Lf(s)\,ds,$$

at least on $\mathrm{Re}(z) > 1$. This identity extends to any larger half-plane $\mathrm{Re}(z) > b$ for which the integral converges uniformly in z on compact subsets. It extends in particular at least to $\mathrm{Re}(z) > 0$ whenever $f(t)$ has bounded variation on some neighborhood of the origin.

Proof For any complex z, let

$$g(t) = e^{-ct}t^{z-1},\ h(t) = e^{ct}f(t) \text{ on } (0, +\infty). \tag{2.13}$$

Then h is in $L^1(0, +\infty)$ by hypothesis, so that

$$t^{z-1}f(t) = t^{z-1}e^{-ct}h(t) \tag{2.14}$$

is in $L^1(0, +\infty)$, at least for $\mathrm{Re}(z) \geq 1$. For any real w, let

$$\hat{g}(w) = \int_0^\infty \exp(iwt)g(t)\,dt = \Gamma(z)(c - iw)^{-z},$$

$$\hat{h}(w) = \int_0^\infty \exp(iwt)h(t)\,dt = Lf(-c - iw). \tag{2.15}$$

Then $\hat{g}$ is in $L^1(-\infty, +\infty)$ for $\mathrm{Re}(z) > 1$, so that

$$M[f; z] = \int_0^\infty g(t)h(t)\,dt$$

$$= (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{g}(-w)\hat{h}(w)\,dw \tag{2.16}$$

by the Parseval theorem [30], which is the desired relation.

If the integral converges uniformly in z on some open set, then it defines a holomorphic function of z on that set [31], so that the result extends by analytic continuation to any half plane with the stated property. In particular if f has bounded variation on $[0, t_0)$ and is zero on $[t_0, +\infty)$ then

$$Lf(s) = \int_0^\infty e^{-st}f(t)\,dt = [-s^{-1}e^{-st}f(t)]_0^{t_0}$$

$$+ s^{-1} \int_0^{t_0} e^{-st}df(t) = 0(s^{-1}) \text{ as } |s| \to \infty \tag{2.17}$$

through Riemann-Stieltjes integration by parts [32], so that (2.12) holds for $\mathrm{Re}(z) > 0$ by this remark. Thus, in proving the last statement, we may suppose that f vanishes on $[0, t_0)$, in which case $M[f; z]$ converges absolutely for all z.

If $z = x + iy$ with x positive and y real, then g and h are in $L^1(0, +\infty)$, so that $\hat{g}$ and $\hat{h}$ are bounded. If $g(p, q; t)$ is given by (2.5) with p and q positive, then

$$(2\pi)^{-1} \int_{-p}^{q} \hat{g}(-w)\hat{h}(w)\,dw = \int_{t_0}^{\infty} g(p, q; t)h(t)\,dt \tag{2.18}$$

by Fubini's theorem. As p and q approach $+\infty$ separately, $g(p, q; t)$ approaches $g(t)$ uniformly on $[t_0, +\infty)$ by Lemma 1, so that (2.12) holds for $x > 0$ in the sense of this improper integral.

3. Convergent expansion

For certain transforms $M[f; z]$ analytic at $z = 1$, and the and the expansion coefficients $c_n(f)$ defined by

$$M[f; z]/\Gamma(z) = \sum_{n=0}^{\infty} c_n(f)(z-1)^n, \tag{3.1}$$

Prasad [5] has introduced and applied the identity

$$Lf(s) = \sum_{n=0}^{\infty} c_n(f)(s \cdot d/ds)^n(1+s)^{-1}. \tag{3.2}$$

However, he obtains this result rather formally, under analyticity assumptions on $M[f; z]$, whereas we now recover this identity more rigorously for an explicit class of functions. The functions f of interest in this development are essentially the Laplace transforms of certain distributions [33] with compact support. To simplify the final proof we establish two preliminary lemmas.

Lemma 2 For any $s \neq -1$ there exists $R(s) \geqslant |\log|s||$ such that

$$P(s, t) = 1/[1 + s \cdot \exp(t)] \tag{3.3}$$

is analytic in the disc $|t| < R(s)$ and

$$P(s, t) = \sum_{n=0}^{\infty} (t^n/n!)(s \cdot d/ds)^n(1+s)^{-1}. \tag{3.4}$$

Proof For any complex s, the points

$$t = (2n+1)i\pi - \log|s| - i \cdot \arg(s) \text{ with } n = 0, \pm 1, \pm 2, \cdots \tag{3.5}$$

are the singularities of $P(s, t)$, and their distances from the origin are not less than $|\log|s||$. For $s \neq -1$, none of these points is the origin, so that $P(s, t)$ has some expansion

$$P(s, t) = \sum_{n=0}^{\infty} p_n(s) t^n/n! \tag{3.6}$$

The stated form of these $p_n(s)$ follows by induction from

$$p_0(s) = P(s, 0) = 1/(1+s),$$

$$p_{n+1}(s) = (s \cdot d/ds)p_n(s) \text{ for } n = 0, 1, 2, \cdots; \tag{3.7}$$

and the latter of relations (3.7) follows by equating coefficients in

$$(\partial/\partial t)P(s, t) = (s \cdot \partial/\partial s)P(s, t). \tag{3.8}$$

Lemma 3 Given $0 < a < b < +\infty$, let g be a complex-valued continuous function on $[0, +\infty)$ with support in $[a, b]$ and let $g^{(m)}$ be its mth derivative, a distribution with support in $[a, b]$. Then the integrals

$$f(s) = Lg^{(m)}(s) = \int_0^{\infty} e^{-st} g^{(m)}(t)\,dt,$$

$$M[g^{(m)}; -s] = \int_0^{\infty} t^{-s-1} g^{(m)}(t)\,dt \tag{3.9}$$

are analytic for all complex s, while the integral

$$G(s) = \int_0^{\infty} g(t)(-\partial/\partial t)^m t^{-1}(s + \log t)^{-1} dt \tag{3.10}$$

is analytic for $s = \infty$, and all complex s, except perhaps $-\log b \leq s \leq -\log a$. Moreover

$$M[g^{(m)}; -s] = M[f; 1+s]/\Gamma(1+s),$$

$$G(s) = \sum_{n=0}^{\infty} n! c_n(f) s^{-n-1}, \tag{3.11}$$

where the $c_n(f)$ are defined by (3.1).

Proof Integrating by parts, we obtain [34]

$$Lg^{(m)}(s) = s^m Lg(s) = s^m \int_0^{\infty} e^{-st} g(t)\,dt,$$

$$M[g^{(m)}; -s] = (s+m) \cdots (s+1) M[g; -s-m], \tag{3.12}$$

both of which are entire, since the support of g is compact. If

$$K(s, t) = t^{-1}(s + \log t)^{-1} \tag{3.13}$$

and if $a \leq t \leq b$, then $K(s, t)$ is analytic for $s = \infty$ and all complex s, except perhaps $-\log b \leq s \leq -\log a$. The same holds for $(-\partial/\partial t)^m K(s, t)$ and thus for $G(s)$. However, by (2.1) and (3.12),

$$\begin{aligned} M[g^{(m)}; -s] &= (s+m) \cdots (s+1) M[Lg; 1+s+m] \\ &\quad \div \Gamma(1+s+m) \\ &= \int_0^{\infty} t^{s+m} Lg(t)\,dt/\Gamma(1+s) \\ &= M[f; 1+s]/\Gamma(1+s). \end{aligned} \tag{3.14}$$

Hence by (3.1) and Taylor's theorem

$$n!\, c_n(f) = (d/ds)^n M[g^{(m)}; -s]_{s=0}. \tag{3.15}$$

Moreover, if $|\log t| < |s|$, then

$$K(s, t) = \sum_{n=0}^{\infty} s^{-n-1} t^{-1}(-\log t)^n. \tag{3.16}$$

Indeed, if $|\log a|, |\log b| < |s|$, then the series (3.16), and all its t-derivatives, converge uniformly on $a \leq t \leq b$. By the continuity of $g^{(m)}$ as a linear functional on test functions [35, 36] we can integrate term by term to conclude

$$\begin{aligned} G(s) &= \int_0^{\infty} K(s, t) g^{(m)}(t)\,dt \\ &= \sum_{n=0}^{\infty} M_n(g) s^{-n-1}, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} M_n(g) &= \int_0^{\infty} g^{(m)}(t) \cdot t^{-1}(-\log t)^n dt \\ &= \int_0^{\infty} g^{(m)}(t)[(\partial/\partial z)^n t^{-z-1}] \end{aligned}$$

$$= \int_0^\infty g(t)[(-\partial/\partial t)^m(\partial/\partial z)^n t^{-z-1}]_{z=0} dt$$

$$= [(d/dz)^n \int_0^\infty g(t) \cdot (-\partial/\partial t)^m t^{-z-1} dt]_{z=0}$$

$$= (d/dz)^n M[g^{(m)}; -z]_{z=0}. \quad (3.18)$$

To complete the proof we compare (3.15) and (3.18).

To state our final result we recall some further definitions. An entire function f is said to have *order a* if

$$f(t) = O[\exp(|t|^{a+\varepsilon})] \text{ as } |t| \to \infty \quad (3.19)$$

for all $\varepsilon > 0$. A function of order a is said to have *type b* if

$$f(t) = O[\exp((b+\varepsilon|t|^a)] \text{ as } |t| \to \infty \quad (3.20)$$

for all $\varepsilon > 0$. Functions of this sort have been studied elsewhere in great detail [37, 38].

Theorem 3 Let f be an entire function which is bounded by some polynomial on the imaginary axis, has order 1 and type b for some positive b, and satisfies

$$f(t) = O[\exp(-at)] \text{ as } t \to +\infty \quad (3.21)$$

for some positive a. Then $M[f; z]/\Gamma(z)$ is an entire function of z, and (3.2) holds for $|s| > \max(a, b, a^{-1}, b^{-1})$, where the infinite sum is absolutely convergent.

Proof By our first two assumptions and a result of distribution theory [39], f is the bilateral Laplace transform of some generalized function g with support in $[-b, b]$. By the assumption (3.21) and the inversion integral for such transforms [40], the support of g lies in $[a, b]$. For some finite n and any $\varepsilon > 0$,

$$g(t) = \sum_{m=0}^{n} (d/dt)^m g_m(t) \quad (3.22)$$

by a standard decomposition theorem [41], for some continuous functions g_m with support in $(a-\varepsilon, b+\varepsilon)$. Thus we may let $f = Lg^{(m)}$ without loss of generality, where g is a continuous function with support in $(a-\varepsilon, b+\varepsilon)$.

If $|\log u| \le \max(|\log(a-\varepsilon)|, |\log(b+\varepsilon)|) < R = |v| < |\log|s||$ then

$$(s+u)^{-1}$$
$$= (2\pi i)^{-1} \oint_{|v|=R} dv/u(v+\log u)(1+se^v). \quad (3.23)$$

Thus by Lemmas 2 and 3, partial integration, and Fubini's theorem,

$$Lf(s) = \int_0^\infty e^{-st}t^m dt \int_0^\infty e^{-tu} g(u)du$$

$$= \int_0^\infty g(u)(-\partial/\partial u)^m (s+u)^{-1} du$$

$$= (2\pi i)^{-1} \int_0^\infty g(u)du$$
$$\times \oint_{|v|=R} (-\partial/\partial u)^m K(u, v)P(s, v)dv$$
$$= (2\pi i)^{-1} \oint_{|v|=R} P(s, v)G(v)dv. \quad (3.24)$$

If $\log|s| > R$, then (3.4) and (3.11) are both absolutely convergent, whence

$$\sum_{n=0}^{\infty} n! c_n(f)(s \cdot d/ds)^n (1+s)^{-1}/n! \quad (3.25)$$

is also absolutely convergent. However the sum in (3.25) is obtained from the last integral in (3.24) through evaluation by residues.

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