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Interpolation with Discontinuous Functions: Application to Calculation of Shocks

Abstract: An interpolation procedure, which uses a step function plus a polynomial correction, is devised and studied for application to the numerical solution of problems having discontinuous solutions. We apply the interpolation procedure to the calculation of shock waves produced by a single convex conservation law. The resulting algorithm does not have the usual undesirable numerical features associated with shock-wave calculations. The stability and convergence of the algorithm is also demonstrated.

1. Introduction

Many numerical methods for solving equations are based on the interpolation of data by a family of functions. The latter is most commonly chosen to be the set of monomials x^n, $n = 0, 1, \cdots$. The resulting numerical methods usually work well when the solution that is sought is reasonably approximated, say, by the monomials. When the problem to be solved does not have a smooth solution (e.g., shock-wave problems and stiff differential equations), these usual numerical procedures frequently fail to give good results.

In this paper we attempt to address this difficulty by devising an interpolation procedure in terms of discontinuous as well as continuous functions. In particular, we use the monomials augmented by a step function. First, we give a study of the interpolation procedure and then we give an application of it to devise a numerical method for the initial value problem for a single convex conservation law, the latter being a well-known problem with discontinuous solutions (shocks and rarefactions.)

In Section 2 we develop the interpolation procedure. In Section 3 we devise a numerical algorithm for the conservation law. This proceeds by exploiting an extremal characterization of the solution of a conservation law due to P. Lax [1].

The data at each mesh point are represented by an interpolant composed of a step function plus a smooth correction. We show that the extremal characterization also decomposes, and the solution may be computed simply in terms of the interpolants at each mesh point. In Section 4 we demonstrate the stability and convergence of our algorithm under appropriate restrictions to the conservation law and to initial data. Finally, in Section 5, we give the results of numerical experiments performed with our algorithm to demonstrate its effectiveness in eliminating the usual numerical difficulties associated with the shock-wave problem.

The initial step of the algorithm frequently corresponds to solving a Riemann problem locally along the mesh. In this respect, the application has a resemblance to methods devised by J. Glimm [2] and S. K. Godunov [3].

There are other numerical schemes that exhibit some of the favorable properties produced by our application. One such scheme, and moreover a fairly simple one, is due to G. W. Hedstrom [4], which in turn is based on some observations of C. M. Dafermos [5].

Although our numerical scheme produces excellent results, our main objective has not been to invent an algorithm for solving shock-wave problems which majorizes features of other existing schemes. Rather, it is to introduce a novel method of interpolation using discontinuous functions with the objective of applying it to numerical problems themselves made difficult because of a lack of smoothness. Our application can be viewed as a feasibility study for this idea. We expect it to be the first of other possible ways of exploiting this interpolation process computationally.

In most cases, proofs of technical results are omitted, and we refer to [7] for these details.

2. Interpolation

Let $\phi(x)$, which represents the data in some problem, be a real-valued function of the real variable x. To sample and then interpolate the data, we begin by laying down a uniform mesh with increment Δx. The ith mesh point is denoted by $x_i = i\Delta x$, $i = 0, \pm 1, \cdots$. Let $\phi_i = \phi(x_i)$, $i = 0, \pm 1, \cdots$. At each mesh point x_i, we interpolate the $\{\phi_j\}$ by a function $\chi_i(x) = \psi_{0,i}(x) + \psi_{1,i}(x)$ which satisfies the interpolatory conditions,

$$\chi_i(x_j) = \psi_{0,i}(x_j) + \psi_{1,i}(x_j) = \phi_j, \qquad j = i, i \pm 1. \quad (2.1)$$

$\psi_{0,i}$ and $\psi_{1,i}$ are each chosen from the set composed of step functions, straight lines, and parabolas. We distinguish two cases which may occur at x_i, depending upon whether the sequence $\{\phi_{i-1}, \phi_i, \phi_{i+1}\}$ is monotone (or constant) or is not monotone. We study these two cases separately. In the remainder of Section 2, we suppose that the x axis is translated to that $x_i = 0$. Since no confusion will result, we also suppress the subscript i.

2.1 The monotone case

In the monotone case we choose the two functions ψ_0 and ψ_1 so that one of them is a step function with at most one discontinuity while the other is a straight line:

$$\chi = \psi_0 + \psi_1 = ax + c + \begin{cases} l, & x \le \lambda, \\ r, & x > \lambda. \end{cases}$$

Here $y = ax + c$ is the equation of the straight line in question and l and r are the values of the step function to the left and to the right, respectively, of its discontinuity, which itself is located at λ.

Set $b = c + l$ and the jump, $\sigma = r - l$. Then

$$\psi_0 + \psi_1 = ax + b + \begin{cases} 0, & x \le \lambda, \\ \sigma, & x > \lambda. \end{cases} \quad (2.2)$$

To determine a, b, σ and λ, we apply the interpolatory conditions (2.1) and (2.2) to get

$$\phi_j = ax_j + b + \begin{cases} 0, & \\ \sigma, & j = 0, \pm 1. \end{cases} \quad (2.3)$$

If $\phi_{-1} - \phi_0 \neq \phi_0 - \phi_1$ then λ lies in the interval $[x_{-1}, x_1)$. Were this not the case we would have $\phi_{-1} - 2\phi_0 + \phi_1 = 0$, a contradiction.

Equation (2.3) has two possible solutions, viz.,

$$\begin{matrix} \text{i)} \\ \\ \text{ii)} \end{matrix} \quad a = \begin{cases} \dfrac{\phi_0 - \phi_{-1}}{\Delta x} \\ \dfrac{\phi_1 - \phi_0}{\Delta x}, \end{cases} \quad b = \begin{cases} \phi_0 \\ \phi_{-1} - \phi_0 + \phi_1, \end{cases} \quad \sigma = \begin{cases} \phi_{-1} - 2\phi_0 + \phi_1, \\ -\phi_{-1} + 2\phi_0 - \phi_1. \end{cases} \quad (2.4)$$

If $\phi_0 - \phi_1 = \phi_{-1} - \phi_0$, so that the points (x_j, ϕ_j), $j = 0, \pm 1$ are colinear, the two solutions (2.4) are equal. In this case, a, b and σ are uniquely determined and, moreover, $\sigma = 0$.

If $\phi_0 - \phi_1 \neq \phi_{-1} - \phi_0$ the two solutions (2.4) are different and the corresponding values of σ are of opposite sign. A choice from among these two solutions is made according to the following criterion.

Criterion 2.1 a, b and σ are chosen so that σ has the sign of $\phi_1 - \phi_0$ or of $\phi_0 - \phi_{-1}$ in the event $\phi_1 = \phi_0$. (Note that in the monotone case, $\mathrm{sig}(\phi_1 - \phi_0) = \mathrm{sig}(\phi_0 - \phi_{-1})$ unless $(\phi_1 - \phi_0)(\phi_0 - \phi_{-1}) = 0$.)

A justification of Criterion 2.1 will be given in Lemmas 2.5 and 3.3.

2.2 The non-monotone case

In the non-montone case we choose the two functions ψ_0 and ψ_1 so that one of them is a step function with at most one discontinuity and the other is a parabola with its axis vertical and with its vertex at $x = 0$:

$$\chi = \psi_0 + \psi_1 = a'x^2 + c + \begin{cases} l, & x \le \lambda, \\ r, & x > \lambda. \end{cases} \quad (2.5)$$

Here $y = a'x^2 + c$ is the equation of the parabola and l, r and λ are the parameters of the step function exactly as in Section 2.1. As in Section 2.1, set $b = c + l$ and $\sigma = r - l$. Then the interpolatory conditions (2.1) and (2.5) become

$$\phi_j = a'x_j^2 + b + \begin{cases} 0, & \\ \sigma, & j = 0, \pm 1. \end{cases} \quad (2.6)$$

If $\phi_{-1} - \phi_0 \neq \phi_1 - \phi_0$, then $\lambda \in [x_{-1}, x_1)$. Were this not the case, we would have $\phi_{-1} = \phi_1$, a contradiction.

Equation (2.6) has two possible solutions, viz.,

$$\begin{matrix} \text{i)} \\ \\ \text{ii)} \end{matrix} \quad a' = \begin{cases} \dfrac{\phi_{-1} - \phi_0}{\Delta x^2} \\ \dfrac{\phi_1 - \phi_0}{\Delta x^2}, \end{cases} \quad b = \begin{cases} \phi_0, \\ \phi_{-1} + \phi_0 - \phi_1, \end{cases} \quad \sigma = \phi_1 - \phi_{-1}. \quad (2.7)$$

These two solutions are equal if $\phi_1 = \phi_{-1}$. In the contrary case, a', b and σ are specified according to the following criterion.

Criterion 2.2 a', b and σ are chosen corresponding to the case giving the smaller value of $|a'|$.

A justification of Criterion 2.2 will be given in Lemmas 2.5 and 3.4.

2.3 Properties of the interpolation

We combine the discussion of the monotone and non-monotone cases by writing

$$\chi = a'x^2 + ax + b + \begin{cases} 0, & x \ge \lambda, \\ \sigma, & x < \lambda, \end{cases} \quad (2.8)$$

with the convention: $a' = 0$ in the monotone case and $a = 0$ in the non-monotone case.

We use I to denote the well-defined transformation $(\phi_{-1}, \phi_0, \phi_1) \to (a', a, b, \sigma)$. We now give several lemmas which describe properties of I.

Lemma 2.1 I is a continuous mapping of R^3 into R^4.

Lemma 2.2 Let $I(\phi_{-1}, \phi_0, \phi_1) = (a', a, b, \sigma)$. If for each $k \geq 0$,

$$I(\phi_{-1} + k, \phi_0 + k, \phi_1 + k) = (a_k', a_k, b_k, \sigma_k),$$

then

$$a_k' = a', a_k = a, b_k = b + k \text{ and } \sigma_k = \sigma.$$

The following lemma shows that the analogue of choosing a' according to the minimality criterion 2.2 is valid for the choice of a as well.

Lemma 2.3 Criterion 2.1 is equivalent to

Criterion 2.3 a, b and σ are determined by (2.4i) or (2.4ii) according to which choice results in the smaller value of $|a|$.

Proof of Lemma 2.3 The proof is divided into the following four cases:

(i) $\{\phi_1 - \phi_0 \geq 0\} \leftrightarrow \{\phi_0 - \phi_{-1} \geq 0\}$

$\leftrightarrow \sigma \geq 0$, from Criterion (2.1).

(a) $\phi_{-1} - 2\phi_0 + \phi_1 \geq 0 \leftrightarrow \phi_1 - \phi_0 \geq \phi_0 - \phi_{-1} \geq 0,$

(b) $\phi_{-1} - 2\phi_0 + \phi_1 \leq 0 \leftrightarrow \phi_0 - \phi_{-1} \geq \phi_1 - \phi_0 \geq 0.$

(ii) $\{\phi_1 - \phi_0 \leq 0\} \leftrightarrow \{\phi_0 - \phi_{-1} \leq 0\} \leftrightarrow \sigma \leq 0,$

(a) $\phi_1 - 2\phi_0 + \phi_1 \leq 0 \leftrightarrow \phi_1 - \phi_0 \leq \phi_0 - \phi_{-1} \leq 0,$

(b) $\phi_1 - 2\phi_0 + \phi_{-1} \geq 0 \leftrightarrow \phi_{-1} - \phi_0 \geq \phi_0 - \phi_1 \geq 0.$

In each case, then, we see that the value of a, chosen according to Criterion 2.1, has a smaller magnitude than the rejected choice.

The following lemma characterizes the extent to which the location λ of the discontinuity of the step function, hitherto unspecified, is constrained by the criteria 2.1, 2.2 and 2.3.

Lemma 2.4 $\lambda \in [x_0, x_1) \leftrightarrow |\phi_{-1} - \phi_0| < |\phi_0 - \phi_1|,$

$\lambda \in [x_{-1}, x_0) \leftrightarrow |\phi_0 - \phi_1| < |\phi_{-1} - \phi_0|.$

Proof To begin with, note that if $|\phi_{-1} - \phi_0| = |\phi_0 - \phi_1|$, then there is no discontinuity. Next from the Criteria 2.2 and 2.3 and if $|\phi_{-1} - \phi_0| < |\phi_0 - \phi_1|$, we have $\phi_0 = \mathrm{b}$. Thus $\lambda \in [x_0, x_1)$. Similarly if $|\phi_{-1} - \phi_0| > |\phi_0 - \phi_1|$, $\phi_0 = b + \sigma$, from which it follows that $\lambda \in [x_{-1}, x_0)$. This proves the assertations of the lemma from right to left. With these the assertions from left to right follow directly.

A monotonicity property of the interpolant χ is the subject of the following lemma.

Lemma 2.5 In each of the intervals $[x_{-1}, x_0]$ and $[x_0, x_1]$, χ is a monotone function.
Note: In the monotone case, χ is monotonic in the entire interval $[x_{-1}, x_1]$.

3. Application to the numerical solution of a conservation law

3.1 Preliminaries

We consider the initial value problem

$$(\partial u / \partial t) + (\partial f(u) / \partial x) = 0, \qquad t > 0, \quad |x| < \infty. \quad (3.1)$$

The solution in the classical sense of the system (3.1) may develop singularities after a finite time, and it is thus not continuable as a regular solution. Nevertheless a continuation as a generalized solution is possible. Such a solution is characterized in the following definition.

Definition 3.1 $u(x, t)$ is a weak solution of (3.1) if u and $f(u)$ are integrable on all bounded subsets of $\mathrm{R} \times [0, \infty)$ and if the integral relation

$$\int_0^\infty \int_{-\infty}^\infty [w_t u + w_x f(u)] dx dt + \int_{-\infty}^\infty w(x, 0)\phi(x) dx = 0 \quad (3.2)$$

is satisfied for all test functions $w(x, t) \in \mathrm{C}^1[\mathrm{R} \times [0, \infty)]$ which vanish for $|x| + t$ sufficiently large.

A regular solution of (3.1) is a weak solution and, conversely, a weak solution with continuous first partial derivatives satisfies (3.1).

The behavior of a weak solution at a discontinuity is characterized by the following lemma.

Lemma 3.1 A piecewise continuous weak solution obeys the Rankine-Hugoniot condition,

$$s[u] = [f], \quad (3.3)$$

at each point of a curve of discontinuity. Here $[\cdot]$ denotes the jump across the curve of discontinuity and s is the speed of propagation of the discontinuity.

Extremal characterization of the solution

We now review an extremal characterization of solutions of (3.1) due to P. Lax and refer to [1] for details and proofs.

Given a strictly convex (concave) function $f(u)$, defined for all $u \in \mathrm{R}$, we introduce the conjugate function $g(s)$ by means of the relation

$$g(s) = \max_u \text{ (min) } \{us - f(u)\}. \quad (3.4)$$

Let $u = b(s)$ denote the value of u where the maximum (minimum) of u in (3.4) is attained, and let $a = f'$. It is then easily shown that

$$b[a(u)] = u, \quad (3.5)$$

$$\frac{d}{ds} g(s) = b(s). \quad (3.6)$$

The functions $b(s)$ and $g(s)$ are defined on the range of $a(u)$, where $g(s)$ is itself convex (concave). $g(s)$ tends to infinity as s approaches the extremities of the domain of definition of g.

We introduce $\Phi(y)$ through

$$\Phi(y) = \int_0^y \phi(\eta) d\eta.$$

Consider the expression

$$J_\phi(y) \equiv J(y) = \Phi(y) + tg(x - y/t). \tag{3.7}$$

J possesses a finite minimum (maximum) in the interior of its domain of definition as characterized in the following lemma.

Lemma 3.2 For each t, with the exception of a denumerable set of x, the function $J(y)$ possesses a unique minimum (maximum) which we denote by $\bar{y}_\phi(x, t)$.

We may now state the following theorem characterizing the solution of (3.1).

Theorem 3.1 The function $u_\phi(x, t) = b(x - \bar{y}_\phi/t)$ is a weak solution of (3.1). Moreover if we denote this solution by $u_\phi(x, t) = S(t)\phi$, then the operator $S(t)$ has the following two properties:

i) $S(t)$ forms a one parameter semigroup:

$$S(t_1 + t_2) = S(t_1)S(t_2), \qquad t_1, t_2 \geq 0,$$

$$S(0) = I.$$

ii) For each t, $S(t)$ is continuous in a certain topology.

Property ii) is made precise in Theorem 3.3.

Remark 3.1 Using (3.5) and Theorem 3.1, we may write

$$\bar{y}_\phi = x - tf'[u_\phi(x, t)]. \tag{3.8}$$

Monotonocity of S and the expansion theorem

A key property of S is its monotonocity.

Theorem 3.2 For each t, the operator $S(t)$ is monotone.

Note: In all that follows we will use $\|\cdot\|$ to denote the L_1 norm.

We now give the expansion theorem.

Theorem 3.3 Let the initial data be denoted by ϕ_ε, where

$$\phi_\varepsilon = \sum_{i=0}^{\infty} \varepsilon^i \psi_i.$$

We suppose that each $\psi_i \in L_1$, $i = 0, 1, \cdots$, and that the series $\sum_{i=0}^{\infty} \|\psi_i\| \varepsilon^i$ has radius of convergence strictly greater than $R > 0$. Let $y_0 \equiv \bar{y}_0 = \bar{y}_{\psi_0}$ and let

$$(1/t) + a'[\psi_0(y_0)]\, \psi_0'(y_0) \neq 0. \tag{3.9}$$

If for some $n \geq 1$, $\psi_i \in C^{n+1-i}$, $i = 0, 1, \cdots, n$ and $f \in C^{n+2}$, then there exist y_i, $i = 1, \cdots, n$ such that

$$\bar{y}_{\phi_\varepsilon} = y_0 + y_1 \varepsilon + \cdots + y_n \varepsilon^n \, [1 + o(1)].$$

Remark 3.2 $a'[\psi_0(y_0)] = f''(\psi_0(y_0)) = f''[b(x - y_0/t)]$. Using this and Theorem 3.1, we may obtain

$$y_1 = -\frac{\psi_1(y_0) f''(u_0)}{(1/t) + f''(u_0)\psi'(y_0)} \tag{3.10}$$

Remark 3.3 The only case of indeterminacy of y_1 and y_2, $\cdots$, y_n as well occurs when

$$(1/t) + a'[\psi_0(y_0)]\psi_0'(y_0) = 0. \tag{3.11}$$

Now we give a proof of Theorem 3.3.

Proof of Theorem 3.3 We consider the case $n = 1$, the proof for other n follows directly. Let $\{\varepsilon_\nu\}$ $\nu = 1, 2, \cdots$, be a sequence of positive numbers converging to zero. Since $J_{\psi_0}(y)$ is a continuous function of y,

$$\begin{aligned} J_{\psi_0}(\bar{y}) &= \lim_{\nu\to\infty} J_{\psi_0}(\bar{y}_{\varepsilon_\nu}) \\ &= \lim_{\nu\to\infty} \left[J_{\phi_{\varepsilon_\nu}}(\bar{y}_{\varepsilon_\nu}) - \int_0^{\bar{y}_{\varepsilon_\nu}} \sum_{i=1}^{\infty} \varepsilon_\nu^i \, \psi_i(\eta) d\eta \right] \\ &\leq \lim_{\nu\to\infty} \left[J_{\phi_{\varepsilon_\nu}}(\bar{y}_0) - \int_0^{\bar{y}_{\varepsilon_\nu}} \sum_{i=1}^{\infty} \varepsilon_\nu^i \, \psi_i(\eta) d\eta \right] \\ &= \lim_{\nu\to\infty} \left[J_{\psi_0}(\bar{y}_0) - \int_{y_0}^{\bar{y}_{\varepsilon_\nu}} \sum_{i=1}^{\infty} \varepsilon_\nu^i \, \psi_i(\eta) d\eta \right] \\ &\leq J_{\psi_0}(\bar{y}_0) + \lim_{\nu\to\infty} \sum_{i=1}^{\infty} \varepsilon_\nu^i \, \|\psi_i\|. \end{aligned}$$

Thus

$$J_{\psi_0}(\bar{y}) \leq J_{\psi_0}(\bar{y}_0).$$

This inequality and the uniqueness of the location of the minimum of J_{ψ_0} gives $\bar{y} = \bar{y}_0$. Thus $\bar{y}_{\phi_\varepsilon} \to \bar{y}_0 \equiv y_0$ as $\varepsilon \to 0$. Thus we have demonstrated the continuity of $\bar{y}_{\phi_\varepsilon}$ in ε for $\varepsilon = 0$.

To continue, note that where J_{ψ_0} attains its minimum, its derivative must vanish. This gives

$$\psi_0(y_0) - g'(x - y_0/t) = 0.$$

Therefore,

$$(x - y_0)/t = a[\psi_0(y_0)].$$

Similarly,

$$(x - \bar{y}_\varepsilon)/t = a[\phi_\varepsilon(\bar{y}_\varepsilon)].$$

Subtracting these two equations, we get

$$(y_0 - \bar{y}_\varepsilon)/t = a[\phi_\varepsilon(\bar{y}_\varepsilon)] - a[\psi_0(\bar{y}_0)].$$

Now

$$\begin{aligned} &a[\phi_\varepsilon(\bar{y}_\varepsilon)] - a[\psi_0(y_0)] \\ &\quad = a[\psi_0(\bar{y}_\varepsilon) + \varepsilon\psi_1(\bar{y}_\varepsilon) + \sum_{i=2}^{\infty} \varepsilon^i \psi_i(\bar{y}_\varepsilon)] - a[\psi_0(\bar{y}_0)] \\ &\quad = a[\psi_0(y_0) + (\bar{y}_\varepsilon - y_0)\psi_0'(y_0) + \varepsilon\psi_1(y_0) \\ &\qquad + \varepsilon\, (\bar{y}_\varepsilon - y_0)\psi_1'(y_{0_\varepsilon}) + O(\bar{y}_\varepsilon - y_0)^2 \\ &\qquad + \sum_{i=2}^{\infty} \varepsilon^i \psi_i(\bar{y}_\varepsilon)] - a[\psi_0(y_0)]. \end{aligned}$$

Here $y_{0_\varepsilon} \in (\bar{y}_0, y_\varepsilon)$. Similarly,

$$a[\phi_\varepsilon(\bar{y}_\varepsilon)] - a[\psi_0(y_0)]$$
$$= a'[\psi_0(y_0)][(\bar{y}_\varepsilon - y_0)\psi_0'(y_0) + \varepsilon\psi_1(y_0)$$
$$+ \varepsilon(\bar{y}_\varepsilon - y_0)\psi_1'(y_{0_\varepsilon}) + O(\bar{y}_\varepsilon - y_0)^2 + \sum_{i=2}^{\infty} \varepsilon^i\psi_i(\bar{y}_\varepsilon)]$$
$$+ C(\varepsilon)[(\bar{y}_\varepsilon - y_0)\psi'(y_0) + \varepsilon\psi_1(y_0)$$
$$+ \varepsilon(\bar{y}_\varepsilon - y_0)\psi_1'(y_{0_\varepsilon}) + O(\bar{y}_\varepsilon - y_0)^2 + \sum_{i=2}^{\infty} \varepsilon^i\psi_i(\bar{y}_\varepsilon)]^2.$$

$C(\varepsilon)$ is bounded since it is equal to $\frac{1}{2}a''$ evaluated at an argument which is bounded, since $\bar{y}_\varepsilon \to y_0$ as $\varepsilon \to 0$.

Inserting this last equation into $(y_0 - \bar{y}_\varepsilon)/t = a[\psi_\varepsilon(y_\varepsilon)] - a[\psi_0(y_0)]$ and rearranging terms we get

$$(y_0 - \bar{y}_\varepsilon)[(1/t) + a'(\psi_0(y_0))\psi_0'(y_0) + O(\bar{y}_\varepsilon - y_0)]$$
$$- \varepsilon\psi_1(y_0)a'(\psi_0(y_0))$$
$$= \varepsilon(\bar{y}_\varepsilon - y_0)[a'(\psi_0(y_0))\psi_1'(y_{0\varepsilon}) + 2C(\varepsilon)\psi_1(y_0)]$$
$$+ a'(\psi_0(y_0))\ \varepsilon \sum_{i=1}^{\infty} \varepsilon^{i-1}\psi_i(\bar{y}_\varepsilon)$$
$$+ \varepsilon^2 C(\varepsilon)[\psi_1(y_0) + (\bar{y}_\varepsilon - y_0)\psi_1'(y_{0_\varepsilon})$$
$$+ \sum_{i=2}^{\infty} \varepsilon^{i-1}\psi_i(\bar{y}_\varepsilon)]^2.$$

Then

$$[(y_0 - \bar{y}_\varepsilon)/\varepsilon][(1/t) + a'(\psi_0(y_0))\psi_0'(y_0) + O(\bar{y}_\varepsilon - y_0)]$$
$$- \psi_1)y_0)a'(\psi_0(y_0)) = O(1).$$

From this and (3.8), we have

$$\lim_{\varepsilon\to 0} \frac{y_0 - \bar{y}_\varepsilon}{\varepsilon} = \frac{\psi_1(y_0)a'[\psi_0(y_0)]}{(1/t) + a'[\psi_0(y_0)]\psi_0'(y_0)}.$$

Thus, setting

$$y_1 = -\frac{\psi_1(y_0)a'[\psi_0(y_0)]}{(1/t) + a'[\psi_0(y_0)]\psi_0'(y_0)}$$

completes the proof of the theorem in the case $n = 1$.

Normalizations

In all that follows we assume that the initial data ϕ take values in $[0, M]$ where $M > 0$ is specified. That this presents no loss of generality is proved for to case $f = \frac{1}{2}u^2$ by means of the following theorem.

Theorem 3.4 If u is a weak solution of

$$u_t + uu_x = 0,\ t > 0,\ |x| < \infty \qquad (3.12)$$

with the initial value $\phi(x)$, then

(i) $u_1(x, t) = u(x - ct, t) + c$ is a weak solution of (3.12) with the initial value $u_1(x, 0) = \phi(x) + c$, and

(ii) $u_2(x, t) = (1/m)\ u(mx, t)$ is a weak solution of (3.12) with the initial value $u_2(x, 0) = (1/m)\ \phi(mx)$.

To demonstrate our claim concerning the normalization, let $\psi(x)$ satisfy $\infty < b \le \psi \le B < \infty$ where $b \neq B$. Let $\phi = M[\psi((B-b)x/M) - b]/(B-b)$, then $0 \le \phi \le M$. Then from (ii) of Theorem 3.4, if u is a weak solution of (3.12) with initial value ϕ, $[(B-b)/M]\ u(Mx/(B-b), t)$ is a weak solution of (3.12) with initial value $\psi(x) - b$. Similarly from (i) of Theorem 3.4, we see that

$$\tilde{u} = \frac{B-b}{M}\, u\Big(\frac{M}{B-b}\, x - bt,\ t\Big) + b$$

is a weak solution of (3.12) with initial values ψ.

Remark 3.4 If u is piecewise continuous then $\tilde{u}$ is piecewise continuous, and by Lemma 3.1, $\tilde{u}$ satisfies the Rankine-Hugoniot condition. We now give a proof of Theorem 3.4.

Proof of Theorem 3.4

(i) By hypothesis,

$$\int_0^\infty \int_{-\infty}^\infty (w_t u + \tfrac{1}{2} w_x u^2)dxdt + \int_{-\infty}^\infty w(x, 0)\phi(x)dx = 0$$

for w a test function.

Setting $Q = (x - ct, t)$, this can be written as

$$\int_0^\infty \int_{-\infty}^\infty (w_t(Q)u(Q) + \tfrac{1}{2} w_x(Q)u^2(Q))dxdt$$
$$+ \int_{-\infty}^\infty w(x, 0)\phi(x)dx = 0.$$

Clearly,

$$\int_0^\infty \int_{-\infty}^\infty (c\ w_t(Q) - \tfrac{1}{2} c^2 w_x(Q))dxdt$$
$$+ \int_{-\infty}^\infty cw(x, 0)dx = 0.$$

Adding these last two equations gives

$$\int_0^\infty \int_{-\infty}^\infty [(w_t(Q) - cw_x(Q))(u(Q) + c)$$
$$+ \tfrac{1}{2} w_x(Q)(u(Q) + c)^2]dxdt$$
$$+ \int_{-\infty}^\infty (\phi(x) + c)w(x, 0)dx = 0,$$

Set $v(x, t) = w(Q)$. As w varies over the set of test functions so also does v. We also have $v(x, 0) = w(x, 0)$, $v_t(x, t) = w_t(Q) - c\ w_x(Q)$ and $v_x(x, t) = w_x(Q)$.

Using these observations, we obtain

$$\int_0^\infty \int_{-\infty}^\infty (v_t u_1 + \tfrac{1}{2} v_x u_1^2)dxdt$$
$$+ \int_{-\infty}^\infty (\phi(x) + c)v(x, 0)dx = 0,$$

completing the proof of part (i).

(ii) Setting $x = my$ and $v(y, t) = w(my, t)$ in (3.2) with $f(u) = \frac{1}{2}u^2$, we obtain

$$\int_0^\infty \int_{-\infty}^\infty (v_t u_2 + \tfrac{1}{2} v_x u_2^2)\,dx\,dt + \int_{-\infty}^\infty v(x, 0)\, \frac{1}{m}\, \phi(mx)\,dx = 0,$$

from which the assertion of part (ii) follows.

3.2 Algorithm

In this section we describe the numerical algorithm as a mapping of mesh valued functions into themselves. The mapping is based on the ideas of the expansion Theorem 3.3 and the interpolation procedure developed in Section 2.

At time t the data is represented by a mesh valued function $\phi_i(t)$, $i = 0, \pm 1, \cdots$. Corresponding to each mesh point x_i, we associate an interpolant $\chi_i(x)$.

To use Theorem 3.3 to the first order, we decompose χ_i into two functions. For clarity we suppress the subscript i and we consider first the monotone and then the non-monotone case.

Decomposition of the interpolant. Monotone case

We have

$$\chi(x) = ax + b + \begin{cases} 0, & x \le \lambda, \\ \sigma, & x > \lambda, \end{cases}$$

where $b = c + l$ and $\sigma = r - l$.

Rule 3.1 Let $\theta \ge 0$ be a prescribed tolerance.

i) If $|a| \le \theta$, choose

$$\psi_0 = \begin{cases} b, & x \le \lambda, \\ b + \sigma, & x > \lambda, \end{cases}$$

$\psi_1 = ax.$

ii) If $|a| > \theta$, choose

$\psi_0 = ax + b,$

$$\psi_1 = \begin{cases} 0, & x \le \lambda, \\ \sigma, & x > \lambda, \text{ when } \sigma \ge 0, \end{cases}$$

or

$\psi_0 = a\,x + b + \sigma$

$$\psi_1 = \begin{cases} -\sigma, & x \le \lambda, \\ 0, & x > \lambda, \text{ when } \sigma < 0. \end{cases}$$

The decomposition afforded by Rule 3.1 is illustrated in Fig. 1 and has the following property.

Lemma 3.3 If $|a| \le \theta$, the resulting ψ_1 is of minimum $L_1(x_{-1}, x_1)$-norm.

Remark 3.4 If $|a| > \theta$, the resulting ψ_1 is positive and of minimum L_∞-norm. (Since λ is unknown, we are unable to make a similar statement for the L_1-norm.)

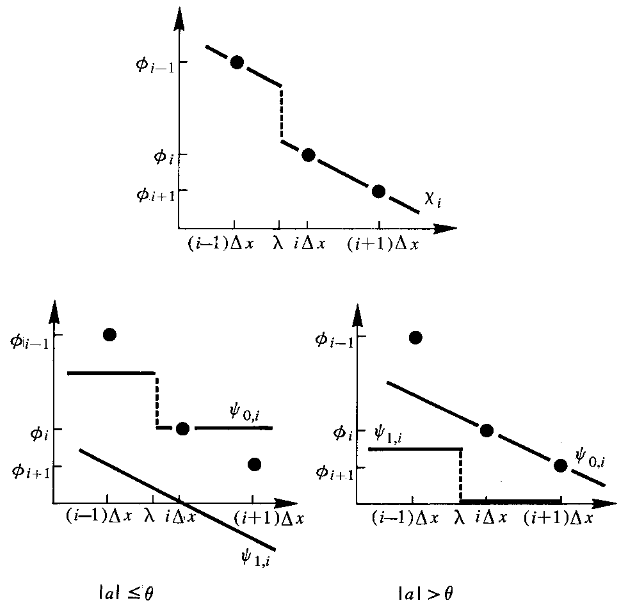


Figure 1 Interpolation and decomposition in the monotone case. The decomposition is afforded by Rule 3.1.

We turn now to the non-monotone case.

Decomposition of the interpolant. Non-monotone case

We have

$$\chi(x) = a'x^2 + b + \begin{cases} 0, & x \le \lambda, \\ \sigma, & x > \lambda, \end{cases}$$

where $b = c + l$ and $\sigma = r - l$.

Rule 3.2 Let $\theta' \ge 0$ be a prescribed tolerance.

i) If $|a'| \le \theta'$, choose

$$\psi_0 = \begin{cases} b + a'\,\dfrac{(\Delta x)^2}{4}, & x \le \lambda, \\ b + \sigma + a'\,\dfrac{(\Delta x)^2}{4}, & x > \lambda, \end{cases}$$

$$\psi_1 = a'\left(x^2 - \frac{(\Delta x)^2}{4}\right).$$

ii) If $|a'| > \theta'$, choose

$\psi_0 = a'x^2 + b,$

$$\psi_1 = \begin{cases} 0, & x \le \lambda, \\ \sigma, & x > \lambda, \text{ when } \sigma \ge 0, \end{cases}$$

or

$\psi_0 = a'x^2 + b + \sigma,$

$$\psi_1 = \begin{cases} -\sigma, & x \le \lambda, \\ 0, & x > \lambda, \text{ when } \sigma < 0. \end{cases}$$

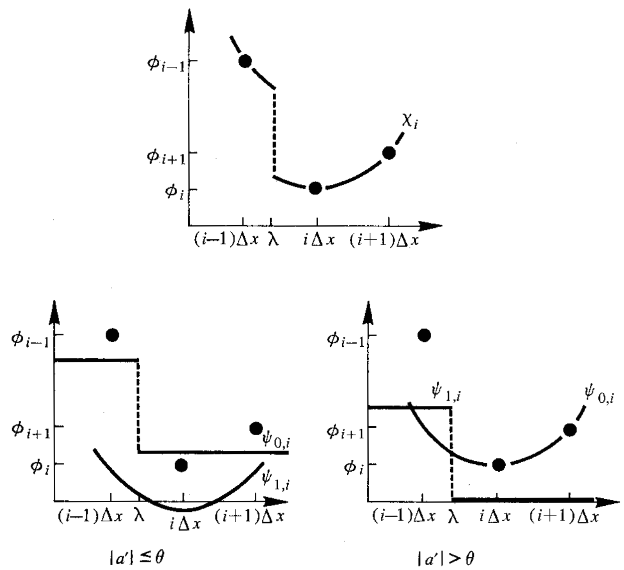


Figure 2 Interpolation and decomposition in the non-monotone case. The decomposition is afforded by Rule 3.2.

The decomposition afforded by Rule 3.2 is illustrated in Fig. 2 and has the following property.

Lemma 3.4 If $|a'| \leq \theta'$, the resulting ψ_1 is of minimum $L_1(x_{-1}, x_1)$-norm.

Remark 3.5 As in the monotone case, and if $|a'| > \theta'$, the resulting ψ_1 is positive and of minimum L_∞-norm.

In Figs. 1 and 2 we illustrate the interpolant and its decomposition in the montone and non-monotone cases, respectively.

Description of the algorithm

The numerical algorithm updates $\phi_i(t)$ to produce $\phi_i(t + \Delta t)$. The calculation for each i depends solely on $\chi_i(x)$. We will require the mesh ratio condition

$$\Delta t \leq \Delta x / \max_i |f'[\phi_i(t)]| \tag{3.13}$$

to be obeyed.

For each i, we solve the problem P_0:

$$P_0 \begin{cases} \dfrac{\partial u_i}{\partial t} + f'(u_i)\dfrac{\partial u_i}{\partial x} = 0. \\ u_i(x, 0) = \psi_{0,\,i}(x). \end{cases}$$

There are three possible problems P_0 corresponding to the three possible $\psi_{0,\,i}(x)$; namely a step function, a straight line, and a parabola. The case of the step function is subdivided into three subcases: a shock corresponding to a monotone nonincreasing step, a rarefaction corresponding to a strictly increasing step, and a special case called a simple shock to be described below. P_0 is a Riemann problem in the case that $\psi_{0,\,i}$ is a step function or is a simple problem in the case that $\psi_{0,\,i}$ is a straight line.

Next, using (3.8), we compute

$y_0 = x_i - \Delta t f'[u_i(x_i, \Delta t)]$.

Then, from (3.10),

$$y_1/\Delta t = -\frac{\psi_{1,\,i}(y_0) f''[u_i(x_i, \Delta t)]}{1 + \Delta t f''[u_i(x_i, \Delta t)]\psi'_{0,\,i}(y_0)}.$$

Then from the expansion theorem we have, neglecting $y_2, \cdots, y_n$,

$$f'[u(x_i, \Delta t)] = (x_i - y_0 - y_1)/\Delta t, \tag{3.14}$$

which we solve to determine $u(x_i, \Delta t)$. It is always possible to solve (3.14) for u if $y_1/\Delta t$ is sufficiently small or if f takes on all values in $(-\infty, \infty)$. Since f' is strictly increasing (decreasing) the solution of (3.14) is unique.

At $t = 0$, we associate a parameter, p_i, with each mesh point x_i where

$$-\Delta x \leq p_i < 0. \tag{3.15}$$

It will suffice to describe the algorithm in the case $t = 0$. The numerical algorithm will update both ϕ_i and p_i to produce $\phi_i(\Delta t)$ and $p_i(\Delta t)$, respectively.

We now consider the various problems P_0.

Solution of the problems P_0

3.2.1 $\psi_{0,\,i}$ is a step function

In this case P_0 is a Riemann problem whose solution is well known. We first compute a type, g_i, of the mesh point x_i determined from the function $\psi_{0,\,i}(x)$.

$$g_i = \begin{cases} L \text{ if } \lambda_i \in [x_{i-1}, x_i) \\ R \text{ if } \lambda_i \in [x_i, x_{i+1}). \end{cases} \tag{3.16}$$

In the case $g_i = R$, $\psi_{0,\,i}(x)$ will be referred to as a pseudojump (pseudoshock or pseudorarefaction, as the case, may be when this emphasis is needed). We now consider the three subcases A, B, and C.

A. $\psi_{0,\,i}$ is a shock

a) Compute the shock speed

$$s = \begin{cases} \dfrac{f(r) - f(l)}{r - l}, & l \neq r \\ f'(r), & l = r. \end{cases} \tag{3.17}$$

b) Set

$$p'_i = p_i + s\Delta t. \tag{3.18}$$

Note (3.13) and (3.17) imply that $p'_i + x_i \in [x_{i-1}, x_{i+1})$.

c) The solution $\psi_{0,\,i}(\Delta t)$ is given by

$$\psi_{0,\,i}(\Delta t) = \begin{cases} r, & g_i = L \text{ and } p'_i < 0 \\ l, & \text{otherwise.} \end{cases} \tag{3.19}$$

This solution may be explained as follows. Consider p_i to be the location of the shock. When $g_i = L$ and $p_i' < 0$, the shock propagates to the right but fails to reach x_i. Thus $\psi_{0,i}(\Delta t) = r$. In all other cases at time Δt, this shock is located to the right of x_i and $\psi_{0,i}(\Delta t) = l$.

d) The choice for the updated value of p_i is computed from $\{p_i', p_{i-1}'\}$ according to the following scheme: Since $p_i' + x_i \in [x_{i-1}, x_{i+1})$ and $p_{i-1}' + x_{i-1} \in [x_{i-2}, x_i)$, the interval $[x_{i-1}, x_i)$ will contain $q = 0$, 1, or 2 of the points $\{p_i' + x_i, p_{i-1}' + x_{i-1}\}$.

If $g = 0$ the shock whose location is associated with p_i has moved off to the right beyond x_i, and a new shock whose location is associated with p_{i-1} has not propagated beyond x_{i-1}. Thus we must supply a candidate shock location in $[x_{i-1}, x_i)$. We take it to be $p_i(\Delta t) = p_i' - \Delta x$.

If $q = 1$ then exactly one of the shocks just referred to is located in $[x_{i-1}, x_i)$. Then we take $p_i(\Delta t)$ as p_i' or $p_{i-1}' - \Delta x$, depending on which of $p_i' + x_i$ or $p_{i-1}' + x_{i-1}$ is in $[x_{i-1}, x_i)$.

If $q = 2$, we have one too many of these shock locations and we choose one of them. If $g_{i-1} = g_i = R$, then both jumps are pseudojumps and we take $p_i(\Delta t) = p_i'$. If g_{i-1} and g_i are of opposite type, we choose $p_i(\Delta t)$ as $p_{i-1}' - \Delta x$ if $g_{i-1} = L$ or as p_i' if $g_i = L$. If $g_{i-1} = g_i = L$, both $p_{i-1}' - \Delta x$ and p_i' are acceptable and we choose the larger of the two.

B. *Simple shock*

The case of simple shock occurs when $\phi_{i-2} = \phi_{i-1}$ and when p_i' as computed in (3.18) is positive. In this case the solution is taken directly to be

$$\phi_i(\Delta t) = \phi_{i-1},$$

$$g_i = L.$$

p_i is updated exactly as in case A.

C. $\psi_{i,0}$ *is a rarefaction*

(a) Compute the speed of the leading edge of the rarefaction

$$s = f'(r). \tag{3.20}$$

(b) Set

$$p_i' = p_i + s\Delta t. \tag{3.21}$$

As above, note that $p_i' + x_i \in [x_{i-1}, x_{i+1})$.

(c) The solution $\psi_{0,i}(\Delta t)$ is given by

$$\psi_{0,i}(\Delta t) = \begin{cases} l & \text{if } g_i = R, \\ \left.\begin{matrix} r \\ k(l,r) \end{matrix}\right\} & \text{if } g_i = L \text{ and } \begin{cases} p_i' < 0, \\ p_i' \geq 0, \end{cases} \end{cases} \tag{3.22}$$

where

$$k(l, r) = r - [f(r) - f(l)]/f'(r). \tag{3.23}$$

To derive (3.22), shift x so that $x_{i-1} = 0$. Now

$$\psi_{0,i} = \begin{cases} l, & x \leq \lambda_i, \\ r, & x > \lambda_i. \end{cases} \tag{3.24}$$

Solving the Riemann problem with these initial data gives

$$\psi_{0,i}(\Delta t) = \begin{cases} l & \text{if } g_i = R, \\ \left.\begin{matrix} r \\ l \\ b\left(\dfrac{x - p_i}{\Delta t}\right) \end{matrix}\right\} & \text{if } g_i = L \text{ and if } \begin{cases} p_i < -f'(r)\Delta t \\ p_i \geq -f'(l)\Delta t \\ -f'(r)\Delta t \leq p_i \\ < -f'(l)\Delta t. \end{cases} \end{cases}$$

In fact, instead of (3.25) we take (3.21), where $k(l, r)$ is the average of the last two cases in the right member of (3.25), viz.,

$$k(l, r) = [f'(r)\Delta t]^{-1}\left[\int_{-f'(r)\Delta t}^{-f'(l)\Delta t} b\left(\frac{x - \sigma}{\Delta t}\right) d\sigma + l \int_{-f'(l)\Delta t}^{0} d\sigma\right]. \tag{3.26}$$

We take this average for the following reason. The third case in (3.25) by itself describes the evolution of a step-rarefaction into a step-rarefaction because of our parametric representation of the data. Since this evolution does not occur in the analytic solution, we wish to eliminate it in the numerical solution. The device of averaging represented in (3.26) accomplishes this. We rejected the alternative way of accomplishing this by means of altering p_i', because altering p_i' would change the location of the leading edge of the rarefaction, an undesirable consequence.

3.2.2 $\psi_{0,i}$ *is a straight line*

Shift x so that $x_i = 0$. If $\psi_{0,i} = ax + c$ and $f(u) = \frac{1}{2}u^2$, then

$$y_0 = -c\Delta t/(1 + a\Delta t)$$

$$u_0 = c(1 - a\Delta t)/(1 + a\Delta t). \tag{3.27}$$

When f is not $\frac{1}{2}u^2$, u_0 and y_0 may be determined in some approximate manner. For example, use some standard numerical scheme to determine u_0 and then (3.8) to determine y_0. Since $\psi_{0,i}$ is a straight line, there will be no significant numerical difficulties associated with this procedure.

Although p_i is not used in this solution procedure, it must be updated for further reference. Corresponding to $\psi_{0,i} = ax + c$ we have $\psi_{1,i}$ a step function with parameters

r and l. Imagine the tolerance θ to be increased indefinitely. We then reverse the roles of straight line and step function. Indeed, in this manner we obtain a step function independent of the tolerance θ and with parameters $r + c$ and $l + c$ respectively. We compute the speed s based on these parameters, viz:

$$s = \begin{cases} \dfrac{f(r+c) - f(l+c)}{r - l}, & l > r \text{ (shock)} \\ f'(r+c), & l \le r \text{ (not a shock)}. \end{cases} \tag{3.28}$$

Then we compute $p_i' = p_i + s\Delta t$, and we update p_i exactly as before.

3.2.3 $\quad \psi_{0,\,i}$ *is a parabola*

In the case of the parabola we do not use the expansion theorem. Instead we compute a numerical approximation directly by means of the finite difference scheme, due to P. Lax [cf. [6]),

$$\phi_i(\Delta t) = (\phi_{i+1} + \phi_{i-1})/2 + [(\Delta t/2\Delta x)][f(\phi_{i+1}) - f(\phi_{i-1})]. \tag{3.29}$$

The quantity p_i, although not used, is updated exactly as is done in Section 4.2.2.

Note: A reason for using (3.29) instead of our expansion technique is that the latter is intended to deal with data which are shock-like (discontinuous). When the dominant part, $\psi_{0,\,i}$ is not shock-like, we expect a standard finite difference scheme to perform satisfactorily.

Evaluation of the corrections

3.3.1 $\quad \psi_{1,\,i}$ *is a step function*

This case occurs only when $\psi_{0,\,i}$ is a straight line. We use this $\psi_{0,\,i}$ and (3.8) to compute y_0. Then y_1 is determined from (3.10), viz:

$$-\frac{y_1}{\Delta t} = \frac{\psi_{1,\,i}(y_0) f''(\psi_{0,\,i}(\Delta t))}{1 + \Delta t f''(\psi_{0,\,i}(\Delta t))\psi_{0,\,i}'(\Delta t)}, \tag{3.30}$$

where

$$\psi_{1,\,i}(y_0) = \begin{cases} l, & y_0 < p_i \text{ or } g_i = R, \\ r, & \text{otherwise.} \end{cases} \tag{3.31}$$

In (3.31) the choice l, should be taken if $y_0 < \lambda_i$. It can be shown that y_0 determined in Section 3.2.2. has the property $y_0 \le x_i$. Thus, if $g_i = R$ (so that $\lambda_i \ge x_i$), then $y_0 < \lambda_i$. When $g_i = L$, we identify λ_i with p_i.

Now $\phi_i(\Delta t)$ is obtained by solving

$$f'[\phi_i(\Delta t)] = (x_i - y_0 - y_1)/\Delta t \tag{3.32}$$

[c.f. (3.14)].

3.3.2 $\quad \psi_{1,\,i}$ *is a straight line*

The case of $\psi_{1,\,i}$ a straight line occurs only when $\psi_{0,\,i}$ is a step function. We proceed as in Section 3.3.1. Here (3.10) for determining y_1 becomes simply

$$-y_1/\Delta t = \psi_{1,\,i}(y_0) f''(\psi_{0,\,i}(\Delta t)) = a\, y_0 f''[\psi_{0,\,i}(\Delta t)]. \tag{3.33}$$

Remark 3.6 It is precisely at this point where the lack of smoothness in the problem (i.e., the step function $\psi_{0,\,i}$) is dealt with. For, although (3.14) requires the derivative $\psi_{0,\,i}'$, it requires it not at the step location. Thus in fact $\psi_{0,\,i}' = 0$.

Now $\phi_i(\Delta t)$ is obtained by solving (3.32).

3.3.3 $\quad \psi_{1,\,i}$ *is a parabola*

This case occurs only when $\psi_{0,\,i}$ is a step function. We proceed as in Section 3.3.2, with (3.33) replaced by

$$-y_1/\Delta t = (a' y_0^2 + c)\, f''[\psi_{0,\,i}(\Delta t)]. \tag{3.34}$$

4. Stability and convergence of the algorithm

For the sake of simplicity we have restricted our arguments on stability and convergence to the case $f(u) = \frac{1}{2}u^2$, although the results in this section are probably true for arbitrary convex (concave) $f(u)$. The convergence is demonstrated only for monotone (decreasing) initial data, but the stability does not require this additional restriction.

In all that follows we will assume a sharp bound M on the data introduced in Section 3.5.

4.1 Stability

Consider the triple (ϕ, p, t), where ϕ is the sequence $\phi = \{\phi_i;\ i = 0, \pm 1, \cdots\}$, p is the sequence $p = \{p_i,\ i = 0, \pm 1, \cdots\}$ and t is a scalar. The numerical algorithm is a mapping of this triple into another denoted

$$Al(\phi, p, t) = (\bar\phi, \bar p, \bar t). \tag{4.1}$$

This mapping has the property

$$Al(\phi, p, \tau) = (\bar\phi, \bar p, \tau + \bar t - t). \tag{4.2}$$

Let us extend the numerical approximation ϕ defined on the mesh to the function $\tilde\phi(x)$ defined for all x, as follows.

$$\tilde\phi(x) = \phi_i, \quad p_i + x_i < x \le p_{i+1} + x_{i+1}. \tag{4.3}$$

Starting at $t = 0$, the repeated application of the algorithm results in the sequence

$$\{(\tilde\phi^k(x), T_k);\ k = 0, 1, 2, \cdots\}, \tag{4.4}$$

where

$$\tilde\phi^0(x) \equiv \tilde\phi(x),$$

$$T_k = \sum_{j=1}^{k} \Delta t_j, \tag{4.5}$$

and where

$$\Delta t_j = \Delta x / \max_x |f'[\tilde\phi^{j-1}(x)]|. \tag{4.6}$$

Note that the values of the third argument in (4.1) and in (4.2) must be chosen from among the elements of the sequence $\{T_k\}$.

We now collect several properties of the algorithm by means of Lemmas 4.1-4.4.

Def 4.0 Let ϕ_i^k denote $\tilde{\phi}^k(x_i)$.

Lemma 4.1 If $\phi_{i-1}^k \geq \phi_i^k \geq \phi_{i+1}^k$, then $\phi_{i-1}^k \geq \phi_i^{k+1} \geq \phi_i^k$.

Lemma 4.2 If $\phi_{i-1}^k \leq \phi_i^k \leq \phi_{i+1}^k$, then $\phi_{i-1}^k \leq \phi_i^{k+1} \leq \phi_i^k$.

Cor 4.1 If $\phi_{i-1}^k = \phi_i^k$, then $\phi_i^{k+1} = \phi_i^k$.

Remark 4.1 Although the value of ϕ_i^{k+1} depends on the tolerance θ, these lemmas themselves are independent of θ. This is not the case for the following two lemmas.

Lemma 4.3 In the non-monotone case, and when $\psi_{0,\,i}$ is a step function, then $\phi_i^{k+1} \in [\phi_{i-1}^k, \phi_i^k]$.

Lemma 4.4 In the non-monotone case, and when $\psi_{0,\,i}$ is not a step function, then $\phi_i^{k+1} \in [\phi_{i-1}^k, \phi_{i+1}^k]$.

A direct consequence of Lemmas 4.1 and 4.2 is the following theorem.

Theorem 4.1 If $\tilde{\phi}(x)$ is monotone, then for each k the sequence $\{\phi_i^k\}$ is monotone.

Remark 4.2 The presence of oscillations accompanying most numerical methods for solving conservation laws is ruled out for the present method by this theorem.

Lemmas 4.1-4.4 imply the following theorem.

Theorem 4.2 For each $k = 0, 1, \cdots$, and for each $i = 0, \pm 1, \cdots$,

$$\min(\phi_{i-1}^k, \phi_i^k, \phi_{i+1}^k) \leq \phi_i^{k+1} \leq \max(\phi_{i-1}^k, \phi_i^k, \phi_{i+1}^k). \quad (4.7)$$

The following corollary which assures the stability of the algorithm is an immediate consequence of this theorem.

Corollary 4.2 (Stability) If $0 \leq m \leq \tilde{\phi}(x) \leq M$, then for each $k = 0, 1, \cdots$, and for each $i = 0, \pm 1, \cdots$, we have $0 \leq m \leq \phi_i^k \leq M$.

From Corollary 4.2 and from (4.6), we have the following theorem, which assures the reachability of any $t \geq 0$ by the algorithm.

Theorem 4.3 $\lim_{k\to\infty} T_k = \infty$.

4.2 Convergence

An estimate of the deviation of the numerical solution from the exact solution is the subject of the following theorem.

Theorem 4.4 If the sequence $\phi^0 = \{\phi_i^0\}$ has N distinct values, then

$$\|S(T_k)\tilde{\phi}^0 - \tilde{\phi}^k\| \leq 3(N-1)(M-m)\Delta x. \quad (4.8)$$

Suppose the initial data are $\Phi(x)$, where

$$\Phi(x) = \begin{cases} M, & \text{if } x \leq 0, \\ m, & \text{if } x \geq A, \end{cases}$$

and is decreasing otherwise. (Here A is some positive constant). To obtain the convergence of the numerical algorithm as $\Delta x \to 0$, we must represent $\Phi(x)$ by a sequence ϕ^0 in which, according to (4.8), the number N of distinct values must be $o(\Delta x^{-1})$. We obtain this representation as follows.

Let

$$B = A / \Delta x^{\frac{1}{2}}$$

and let

$$\phi_i^0 = \Phi(j\Delta x B), \ (j-1)B < i \leq jB, \qquad j = 0, \pm 1, \cdots.$$

Note that

$$\phi_i^0 = \begin{cases} M, & i \leq 0 \\ m, & i \geq B^2, \end{cases}$$

and that the number of distinct values in the sequence $\{\phi_i^0\}$ is not greater than $1 + B$.

The convergence of the algorithm is the subject of the following theorem.

Theorem 4.5 For each $k = 0, 1, \cdots$,

$$\|S(T_k)\Phi - \tilde{\phi}^k\| \leq 5\,[|A/\Delta x|^{\frac{1}{2}}](M-m)\Delta x = O(\Delta x^{\frac{1}{2}}).$$

5. Numerical results

We now assemble the results of calculations performed with our algorithm on a set of illustrative problems. In all cases $f = \frac{1}{2}u^2$.

5.1

We begin with the following simple case of initial data composed of three shock waves.

$$\Phi(x) = \begin{cases} 4, & x \leq 0.75, \\ 2, & 0.75 < x \leq 3.75, \\ 1, & 3.75 < x \leq 7.75, \\ 0, & 7.75 < x. \end{cases}$$

These initial data generate the following exact solution:

$0 \leq t \leq 2$

$$u(x,t) = \begin{cases} 4, & x \leq 0.75 + 3t, \\ 2, & 0.75 + 3t < x \leq 3.75 + 1.5t, \\ 1, & 3.75 + 1.5t < x \leq 7.75 + 0.5t, \\ 0, & 7.75 + 0.5t < x. \end{cases}$$

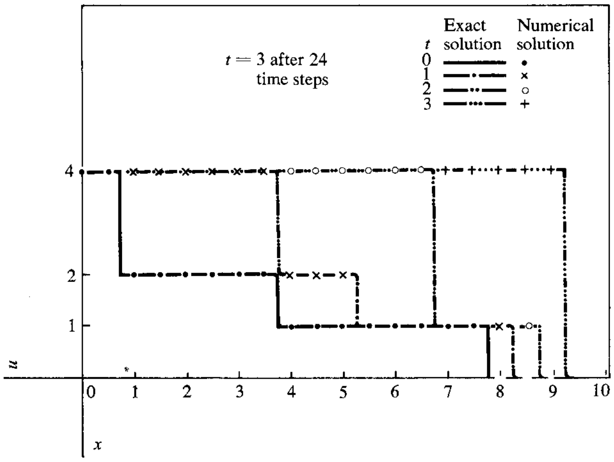


Figure 3 Numerical results for a simple case of initial data composed of three shock waves which overtake one another.

$2 < t \le 3$

$$u(x,t) = \begin{cases} 4, & x \le 1.75 + 2.5t, \\ 1, & 1.75 + 2.5t < x \le 7.75 + 0.5t, \\ 0, & 7.75 + 0.5t < x. \end{cases}$$

$t > 3$

$$u(x,t) = \begin{cases} 4, & x \le 3.25 + 2t, \\ 0, & x > 3.25 + 2t. \end{cases}$$

Set $\Delta x = 0.5$,

$\phi = (4, 4, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 0 \cdots 0)$

and

$p_i = -0.25, \qquad i = 0, \pm 1, \cdots.$

Then

$\tilde{\phi} = \Phi.$

The results displayed in Fig. 3 show that the numerical solution at the times $t = T_8 = 1$, $t = T_{16} = 2$ and $t = T_{24} = 3$, lie on the curve of $u(x, t)$ for the same values of t. It serves no purpose to continue the calculations for times larger than 3, since beyond this time we obtain a step function with a single discontinuity for which the calculation may be shown to be exact. We have verified this for $t = 4$.

5.2.

The second example is a case of two rarefaction waves.

$$\Phi(x) = \begin{cases} 0, & x \le 0.75, \\ 2, & 0.75 < x \le 2.75, \\ 4, & x > 2.75. \end{cases}$$

These initial data generate the following exact solution:

$$u(x,t) = \begin{cases} 0, & x \le 0.75, \\ (1/t)(x - 0.75), & 0.75 < x \le 2t + 0.75, \\ 2, & 2t + 0.75 < x \le 2t + 2.75, \\ (1/t)(x - 2.75), & 2t + 2.75 < x \le 4t + 2.75, \\ 4, & x > 4t + 2.75. \end{cases}$$

This case is calculated with $\Delta x = 0.5$, 0.25 and 0.1, respectively. The number of time steps required to attain $T = 1.5$ are 12, 24 and 60, respectively. The results are displayed in Fig. 4.

The maximum time t in each of the three diagrams of Fig. 4 is the same. Thus these diagrams exhibit an approach by the numerical solution to the exact solution as Δx decreases. This prompts that conjecture that the algorithm converges in the case of monotonic increasing initial data. This conjecture has not as yet been confirmed.

Notice that the location of the leading edge of the rarefaction waves are reproduced exactly in all cases.

5.3

Consider the case where the initial function is

$$\Phi(x) = \begin{cases} 1, & x \le 0, \\ 1 - x, & 0 \le x \le 1, \\ x - 1, & 1 \le x \le 2, \\ 1, & 2 \le x. \end{cases}$$

The corresponding exact solution is for

$0 \le t < 1$

$$u(x,t) = \begin{cases} 1, & x \le t, \\ (1-x)/(1-t), & t \le x \le 1, \\ (x-1)/(t+1), & 1 \le x \le 2 + t, \\ 1, & 2 + t \le x \end{cases}$$

and for

$t \ge 1$, denoting $\tau = (2 + 2t)^{\frac{1}{2}}$,

$$u(x,t) = \begin{cases} 1, & x < 2 + t - \tau, \\ (x-1)/(t+1), & 2 + t - \tau < x \le 2 + t, \\ 1, & 2 + t \le x. \end{cases}$$

This case has been computed up to $t = 15$ with the discretization steps $\Delta x = 0.25$ and 0.05. These correspond to 60 and 300 time steps, respectively. Figure 5 exhibits a comparison between the numerical solutions and the exact solution.

Notice that the position of the shock is determined with good precision while, as in the preceding case, the loca-

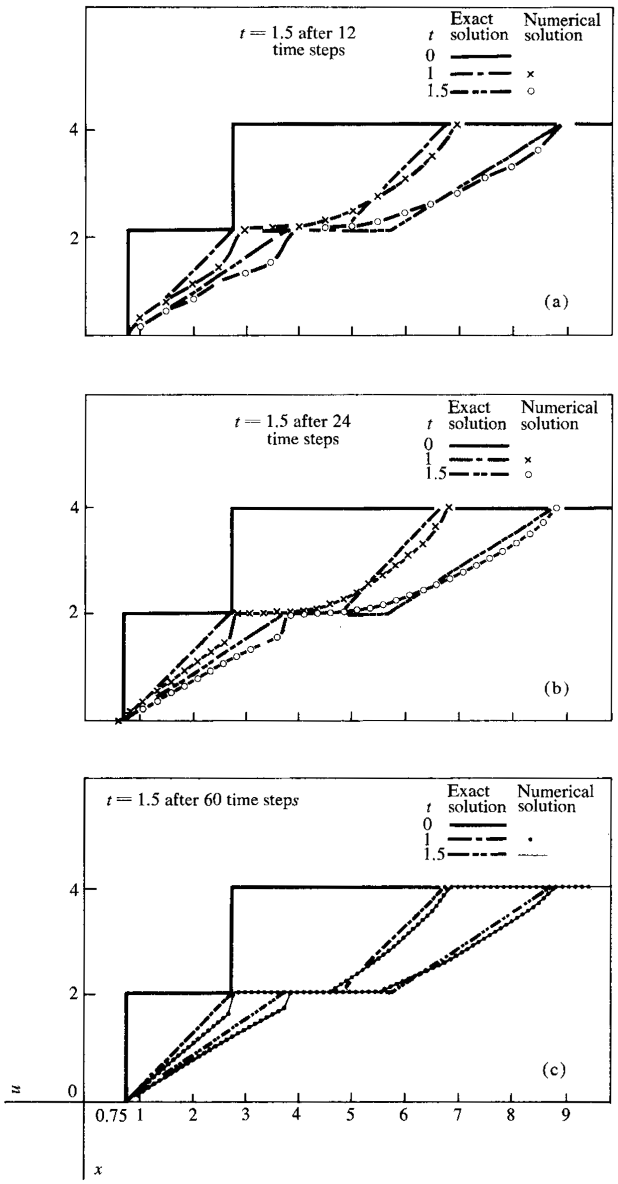


Figure 4 Numerical results for a case of two rarefraction waves for (a) $\Delta x = 0.5$, (b) $\Delta x = 0.25$, and (c) $\Delta x = 0.1$.

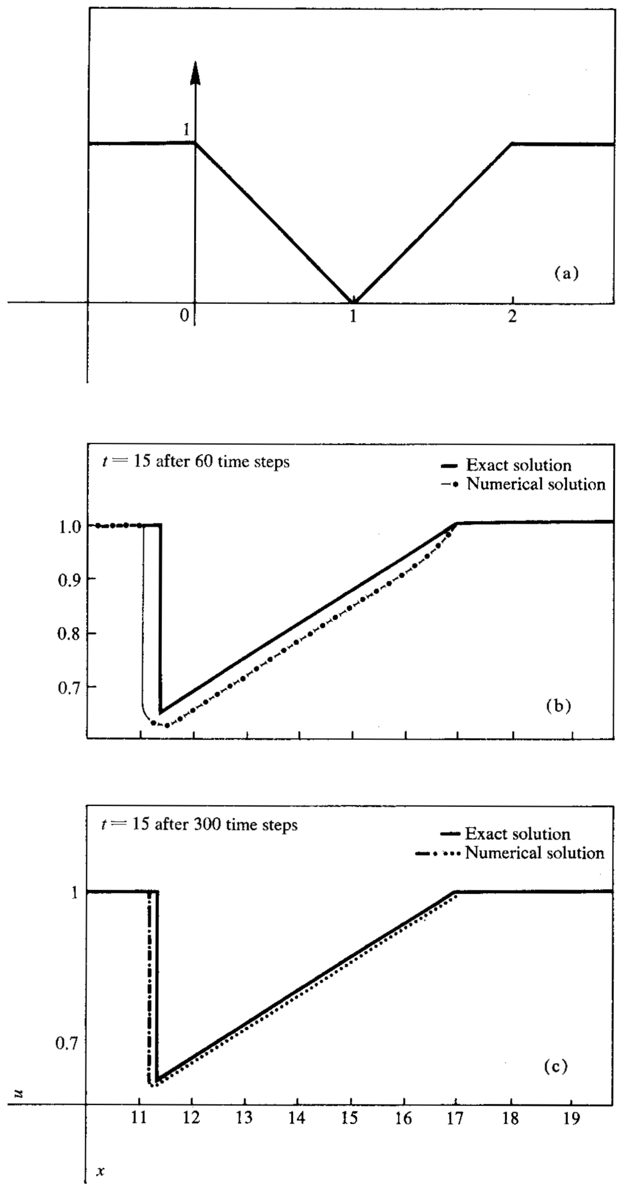


Figure 5 Comparison between numerical and exact solutions for a rarefraction wave: (a) initial function, (b) $\Delta x = 0.25$, and (c) $\Delta x = 0.05$.

tion of the leading edge of the rarefaction wave is reproduced exactly.

5.4

We now discuss the calculation of a case corresponding to initial data with compact support. The following theorem (Lax [1]) contains a relevant characterization of such problems.

Theorem 5.1 Let the initial data $\Phi(x)$ be of compact support. The behavior for large t of the corresponding weak solution $u(x, t)$, of the initial value problem is given by

$$u(x,t) \approx \begin{cases} k[(x/t) - c], & ct - \alpha\sqrt{t} < x < ct + \beta\sqrt{t}, \\ 0, & \text{otherwise.} \end{cases} \qquad (5.1)$$

Here k, c, α and β are appropriate constants. By this theorem is meant that for t large, each point of the graph of $u(x, t)$ is within $o(\sqrt{t})$ of the graph of the function in (5.1). In the case that f is convex (concave), the constants in (5.1) have the values

$$c = f'(0),\ k = 1/f''(0),\ \alpha = \sqrt{2m_1/k},\ \text{and}\ \beta = \sqrt{2m_2/k},$$

where

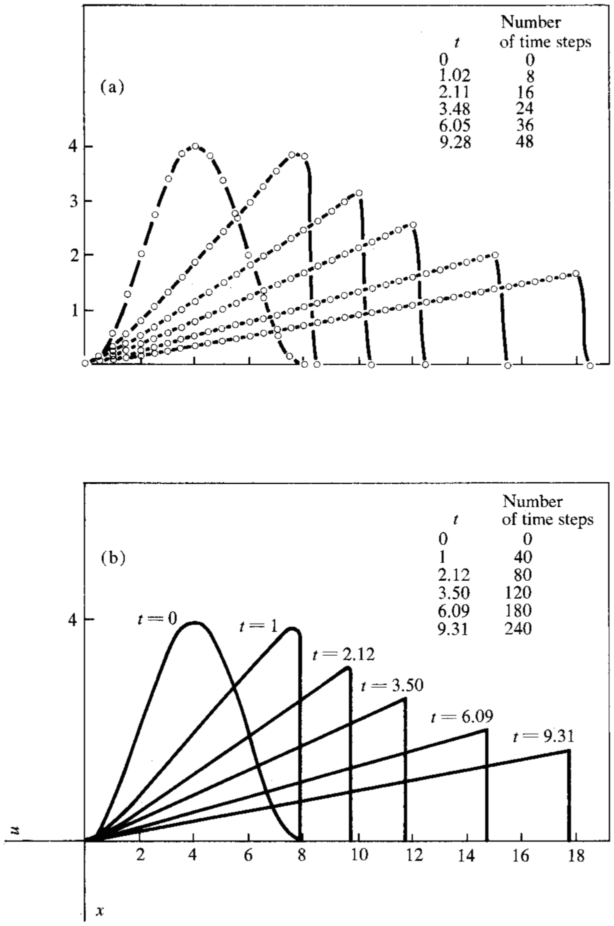
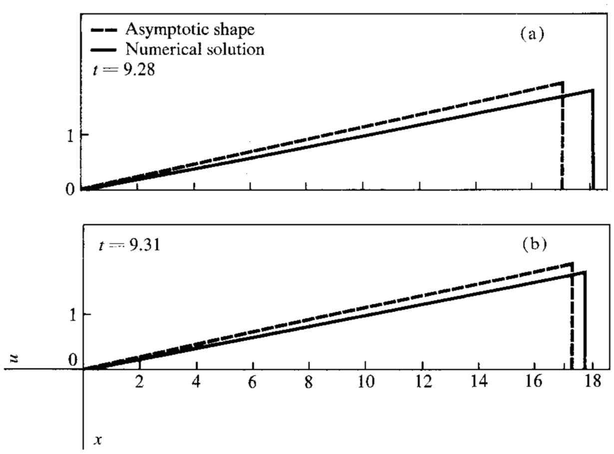


Figure 6 Calculation of a case corresponding to initial data with compact support: (a) $\Delta x = 0.5$, and (b) $\Delta x = 0.1$. The development of a shock is shown.

Figure 7 Calculated solution with asymptotic behavior given in Theorem 6.1: (a) $\Delta x = 0.5$, and (b) $\Delta x = 0.1$. This calculation verifies the long-time behavior of solutions with compact support.



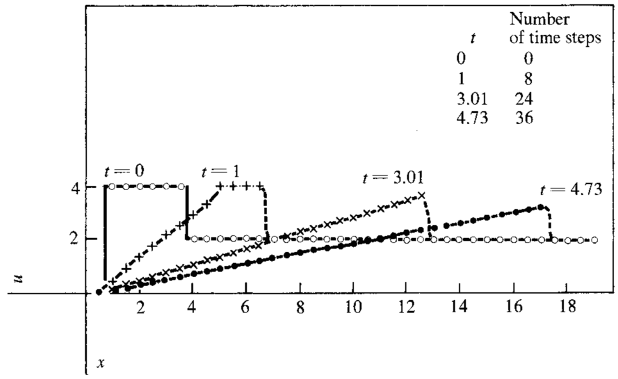


Figure 8 Calculation showing a rarefaction overtaking a shock, with $\Delta x = 0.5$.

$$m_1 = \max_x \ (\min) \int_x^{-\infty} \Phi(\xi)d\xi;$$

$$m_2 = \max_x \ (\min) \int_x^{\infty} \Phi(\xi)d\xi.$$

For $f = \frac{1}{2} u^2$, these constants become

$$c = 0,\ k = 1,\ \alpha = 0, \text{ and } \beta = 4\sqrt{2}.$$

Our calculation treats the case

$$\Phi(x) = \begin{cases} 4 \sin^2 \dfrac{\pi}{8} x, & 0 \leq x \leq 8, \\ 0, & \text{otherwise.} \end{cases}$$

In Fig. 6(a) the results for $\Delta x = 0.5$ are illustrated. The maximal time, $t = 9.28$ is achieved after 48 steps. In Fig. 6(b) we illustrate correspondingly $\Delta x = 0.1$, $t = 9.31$ and 252 steps.

In Fig. 7 we compare the calculated solution with the asymptotic behavior given in Theorem 5.1. Notice that for $\Delta x = 0.1$, the difference between the two is of the order of 0.2 for $\sqrt{t} > 3$.

5.5

Our final example is illustrated in Fig. 8 and deals with a case where the data contain both a shock and a rarefaction.

$$\Phi(x) = \begin{cases} 0, & x \leq 0.75, \\ 4, & 0.75 < x \leq 3.75, \\ 2, & x > 3.75. \end{cases}$$

This calculation is performed with $\Delta x = 0.5$.

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